RESTORATION OF DISCRETE "STAR" IMAGES*

BY

WALTER FREIBERGER AND ULF GRENANDER

Brown University

1. General formulation. We shall discuss in this paper the restoration of images from observations under the assumption that the distortions are caused by random elastic fields of a given type. The pure (undistorted) images are taken to be points lying on straight lines. Let these points have coordinates $z = (x, y) \in \mathbb{R}^2$, and let them be taken into the deformed images by the transformation

$$\mathbf{z} \rightarrow \mathbf{z} + \mathbf{w}(\mathbf{z}) \quad \text{or} \quad (x, y) \rightarrow (x, y) + [u(x, y), v(x, y)] \quad (1)$$

where the displacements $u(x, y), v(x, y)$ satisfy Navier's equations for elastic deformations

$$\begin{aligned} \mu \nabla^2 u + (\lambda + \mu) \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - cu + X &= 0, \\ \mu \nabla^2 v + (\lambda + \mu) \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - cv + Y &= 0, \end{aligned} \quad (2)$$

i.e.

$$\begin{aligned} \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + (\lambda + \mu) \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} \right) - cu + X &= 0, \\ \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + (\lambda + \mu) \left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} \right) - cv + Y &= 0, \end{aligned}$$

i.e.

$$\begin{aligned} (\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + (\lambda + \mu) \frac{\partial^2 v}{\partial x \partial y} - cu + X &= 0, \\ (\lambda + 2\mu) \frac{\partial^2 v}{\partial y^2} + \mu \frac{\partial^2 v}{\partial x^2} + (\lambda + \mu) \frac{\partial^2 u}{\partial x \partial y} - cv + Y &= 0, \end{aligned}$$

i.e.

$$\begin{aligned} a_1 u_{xx} + a_2 u_{yy} + a_3 v_{xy} - cu + X &= 0, \\ a_1 v_{yy} + a_2 v_{xx} + a_3 u_{xy} - cv + Y &= 0, \end{aligned} \quad (3)$$

say. Here the terms cu and cv are included to prevent rigid body motions.

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$X(x, y)$ and $Y(x, y)$ represent the components of the body forces per unit volume. We shall regard $\mathbf{K} = (X, Y)$ as a vector-valued, two-dimensional stochastic process (i.e., the process has two components and each is a random function of two space-coordinates; see [1, p. 209, 2, Ch. 3]. The spectral representation of such a process is as follows:

$$X(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(i(\lambda_1 x + \lambda_2 y)) dZ_x(\lambda_1, \lambda_2),$$

$$Y(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(i(\lambda_1 x + \lambda_2 y)) dZ_y(\lambda_1, \lambda_2), \quad (4)$$

or

$$X(\mathbf{z}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(i(\mathbf{z} \cdot \boldsymbol{\lambda})) dZ_x(\boldsymbol{\lambda}), \quad Y(\mathbf{z}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(i(\mathbf{z} \cdot \boldsymbol{\lambda})) dZ_y(\boldsymbol{\lambda}), \quad (5)$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ are points in wave-number space. $Z_x(\boldsymbol{\lambda})$ and $Z_y(\boldsymbol{\lambda})$ are components of a vector-valued, two-dimensional spectral process, which is a doubly orthogonal process in λ_1 and λ_2 (i.e., a process of independent increments), with the properties:

$$E[dZ_x(\boldsymbol{\lambda}) dZ_x(\boldsymbol{\lambda}')] = 0, \quad E[dZ_y(\boldsymbol{\lambda}) dZ_y(\boldsymbol{\lambda}')] = 0, \quad E[dZ_x(\boldsymbol{\lambda}) dZ_y(\boldsymbol{\lambda}')] = 0; \quad (6)$$

i.e., there are no contributions to either of

$$E\left\{\int dZ_x(\lambda_1, \lambda_2) \int dZ_x(\lambda'_1, \lambda'_2)\right\} \quad E\left\{\int dZ_y(\lambda_1, \lambda_2) \int dZ_y(\lambda'_1, \lambda'_2)\right\}$$

except along $\lambda_1 = \lambda'_1, \lambda_2 = \lambda'_2$.

We shall assume the processes X and Y to be two-dimensional stationary processes (or 'spatially homogeneous'); that is, their covariance functions have the properties

$$\begin{aligned} \rho_x(\tau, v) &= E\{X(x, y)X(x + \tau, y + v)\}, & \rho_y(\tau, v) &= E\{Y(x, y)Y(x + \tau, y + v)\}, \\ \rho_{xy}(\tau, v) &= E\{X(x, y)Y(x + \tau, y + v)\}, & \rho_{yx}(\tau, v) &= E\{Y(x, y)X(x + \tau, y + v)\}. \end{aligned} \quad (7)$$

Let $\mathbf{r} = (\tau, v)$.

By a theorem of Cramér, these covariances can be represented in terms of spectral distribution functions:

$$\begin{aligned} \rho_x(\tau, v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[i(\tau\lambda_1 + v\lambda_2)] dF_x(\lambda_1, \lambda_2) = \iint \exp[i(\mathbf{r} \cdot \boldsymbol{\lambda})] dF_x(\boldsymbol{\lambda}), \\ \rho_y(\tau, v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[i(\mathbf{r} \cdot \boldsymbol{\lambda})] dF_y(\boldsymbol{\lambda}), \\ \rho_{xy}(\tau, v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[i(\mathbf{r} \cdot \boldsymbol{\lambda})] dF_{xy}(\boldsymbol{\lambda}), \quad \rho_{yx}(u, v) = \iint \exp[i(\mathbf{r} \cdot \boldsymbol{\lambda})] dF_{yx}(\boldsymbol{\lambda}), \end{aligned} \quad (8)$$

where $dF_x(\boldsymbol{\lambda}) = E|dZ_x(\boldsymbol{\lambda})|^2$, $dF_y(\boldsymbol{\lambda}) = E|dZ_y(\boldsymbol{\lambda})|^2$, and

$$dF_{xy}(\boldsymbol{\lambda}) = E|dZ_x(\boldsymbol{\lambda}) \overline{dZ_y(\boldsymbol{\lambda})}|, \quad dF_{yx}(\boldsymbol{\lambda}) = E|dZ_y(\boldsymbol{\lambda}) \overline{dZ_x(\boldsymbol{\lambda})}|. \quad (9)$$

We define the covariance matrix of the force process $\mathbf{K} = (X, Y)$ by

$$R_f(\tau, v) = \begin{pmatrix} \rho_x(\tau, v) & \rho_{xy}(\tau, v) \\ \rho_{yx}(\tau, v) & \rho_y(\tau, v) \end{pmatrix}. \quad (10)$$

We now represent the random displacements $u(x, y)$, $v(x, y)$ analogously through their spectral processes $Z_u(\lambda)$, $Z_v(\lambda)$:

$$u(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[i(\mathbf{z} \cdot \lambda)] dZ_u(\lambda), \quad v(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[i(\mathbf{z} \cdot \lambda)] dZ_v(\lambda). \quad (11)$$

These spectral processes will also be orthogonal, and we can define the covariance matrix of the displacements as

$$R_d(\tau, v) = \begin{pmatrix} \rho_u(\tau, v) & \rho_{uv}(\tau, v) \\ \rho_{vu}(\tau, v) & \rho_v(\tau, v) \end{pmatrix} \quad (12)$$

where

$$\begin{aligned} \rho_u(\tau, v) &= E\{u(x, y)u(x + \tau, y + v)\}, & \rho_v(\tau, v) &= E\{v(x, y)v(x + \tau, y + v)\}, \\ \rho_{uv}(\tau, v) &= E\{u(x, y)v(x + \tau, y + v)\}, & \rho_{vu}(\tau, v) &= E\{v(x, y)u(x + \tau, y + v)\}. \end{aligned} \quad (13)$$

The displacement covariances thus have the following representation:

$$\begin{aligned} \rho_u(\tau, v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[i(\mathbf{r} \cdot \lambda)] dF_u(\lambda), & \rho_v(\tau, v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[i(\mathbf{r} \cdot \lambda)] dF_v(\lambda), \\ \rho_{uv}(\tau, v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[i(\mathbf{r} \cdot \lambda)] dF_{uv}(\lambda), & \rho_{vu}(\tau, v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[i(\mathbf{r} \cdot \lambda)] dF_{vu}(\lambda). \end{aligned} \quad (14)$$

The orthogonality of the displacement spectral process is expressed by

$$E[dZ_u(\lambda) dZ_u(\lambda')] = 0, \quad E[dZ_v(\lambda) dZ_v(\lambda')] = 0, \quad E[dZ_u(\lambda) dZ_v(\lambda')] = 0; \quad (15)$$

$$dF_u(\lambda) = E|dZ_u(\lambda)|^2 = f_u(\lambda) d\lambda, \quad dF_v(\lambda) = E|dZ_v(\lambda)|^2 = f_v(\lambda) d\lambda,$$

$$dF_{uv}(\lambda) = E[dZ_u(\lambda) \overline{dZ_v(\lambda)}] = f_{uv}(\lambda) d\lambda,$$

$$dF_{vu}(\lambda) = E[dZ_v(\lambda) \overline{dZ_u(\lambda)}] = f_{vu}(\lambda) d\lambda. \quad (16)$$

We now transform the Navier equations into the spectral domain by substituting (5) and (11) into (3) to get

$$(a_1 \lambda_1^2 + a_2 \lambda_2^2 + c) dZ_u(\lambda) + a_3 \lambda_1 \lambda_2 dZ_v(\lambda) = dZ_x(\lambda),$$

$$(a_1 \lambda_1^2 + a_2 \lambda_2^2 + c) dZ_v(\lambda) + a_3 \lambda_1 \lambda_2 dZ_u(\lambda) = dZ_y(\lambda)$$

or

$$\begin{pmatrix} a_1 \lambda_1^2 + a_2 \lambda_2^2 + c & a_3 \lambda_1 \lambda_2 \\ a_3 \lambda_1 \lambda_2 & a_1 \lambda_1^2 + a_2 \lambda_2^2 + c \end{pmatrix} \begin{pmatrix} dZ_u(\lambda) \\ dZ_v(\lambda) \end{pmatrix} = \begin{pmatrix} dZ_x(\lambda) \\ dZ_y(\lambda) \end{pmatrix}, \quad (17)$$

i.e.

$$M(\lambda) dZ_w(\lambda) = dZ_f(\lambda), \quad (18)$$

where Z_w is the displacement vector spectral process and Z_f is the force vector spectral

process. Hence

$$d\mathbf{Z}_w(\lambda) = M^{-1}(\lambda) d\mathbf{Z}_f(\lambda). \quad (19)$$

Note that

$$\begin{aligned} |M| &= (a_1\lambda_1^2 + a_2\lambda_2^2 + c)(a_1\lambda_2^2 + a_2\lambda_1^2 + c) - a_3^2\lambda_1^2\lambda_2^2 \\ &= a_1a_2(\lambda_1^4 + \lambda_2^4) + (a_1^2 + a_2^2 - a_3^2)\lambda_1^2\lambda_2^2 + c(a_1 + a_2)(\lambda_1^2 + \lambda_2^2) + c^2 \\ &= a_1a_2(\lambda_1^2 + \lambda_2^2)^2 + c(a_1 + a_2)(\lambda_1^2 + \lambda_2^2) + c^2 + [(a_1 - a_2)^2 - a_3^2]\lambda_1^2\lambda_2^2, \end{aligned}$$

but $a_1 - a_2 = a_3$, and therefore

$$|M| = a_1a_2\|\lambda\|^4 + c(a_1 + a_2)\|\lambda\|^2 + c^2. \quad (20)$$

Hence M is nonsingular since c , a_1 and a_2 are strictly positive. We have thus

$$\begin{aligned} M^{-1}(\lambda) &= \frac{1}{a_1a_2\|\lambda\|^4 - c(a_1 + a_2)\|\lambda\|^2 + c^2} \begin{pmatrix} a_1\lambda_2^2 + a_2\lambda_1^2 - c & -a_3\lambda_1\lambda_2 \\ -a_3\lambda_1\lambda_2 & a_1\lambda_1^2 + a_2\lambda_2^2 - c \end{pmatrix} \\ &= \frac{1}{(c + a_2\|\lambda\|^2)(c + a_1\|\lambda\|^2)} \begin{pmatrix} a_1\|\lambda\|^2 + c - a_3^2 & -a_3\lambda_1\lambda_2 \\ -a_3\lambda_1\lambda_2 & a_1\|\lambda\|^2 + c - a_3\lambda_2^2 \end{pmatrix} \\ &= \frac{1}{(c + a_2\|\lambda\|^2)} \left\{ I - \frac{a_3}{c + a_1\|\lambda\|^2} \begin{pmatrix} \lambda_1^2 & \lambda_1\lambda_2 \\ \lambda_1\lambda_2 & \lambda_2^2 \end{pmatrix} \right\} \\ &= \frac{1}{(c + a_2\|\lambda\|^2)} \left(I - \frac{a_3}{c + a_1\|\lambda\|^2} \lambda\lambda^T \right) \end{aligned}$$

and

$$\begin{aligned} M^{-2}(\lambda) &= \frac{1}{(c + a_2\|\lambda\|^2)^2} \left(I - \frac{2a_3}{c + a_1\|\lambda\|^2} \lambda\lambda^T + \frac{a_3\|\lambda\|^2}{[c + a_1\|\lambda\|^2]^2} \lambda\lambda^T \right) \\ &= \frac{1}{[c + a_2\|\lambda\|^2]^2} \left(I - a_3 \frac{2c + (a_1 + a_2)\|\lambda\|^2}{[c + a_1\|\lambda\|^2]^2} \lambda\lambda^T \right) \end{aligned} \quad (21)$$

We now assume the covariance function of the force field to be

$$R_f(\mathbf{r}) = \begin{pmatrix} \rho_x(\mathbf{r}) & \rho_{xy}(\mathbf{r}) \\ \rho_{yx}(\mathbf{r}) & \rho_y(\mathbf{r}) \end{pmatrix} = \sigma^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma^2 I. \quad (22)$$

From (19) we get

$$E\{d\mathbf{Z}_w(\lambda)[d\mathbf{Z}_w(\lambda)]^T\} = M^{-2}(\lambda)E\{d\mathbf{Z}_f(\lambda)[d\mathbf{Z}_f(\lambda)]^T\}. \quad (23)$$

From (9) and (16) we thus get the relationship between the spectral density matrices of the displacement and force processes to be

$$\begin{pmatrix} dF_u(\lambda) & dF_{uv}(\lambda) \\ dF_{vu}(\lambda) & dF_v(\lambda) \end{pmatrix} = M^{-2}(\lambda) \begin{pmatrix} dF_x(\lambda) & dF_{xy}(\lambda) \\ dF_{yx}(\lambda) & dF_y(\lambda) \end{pmatrix} \quad (24)$$

and with assumption (22)—which is equivalent to assuming the force spectral process $\mathbf{Z}_f(\lambda) = (Z_x(\lambda), Z_y(\lambda))$ where each of Z_x and Z_y , independent of each other, are derivatives of the Wiener process—we get the spectral density matrix of the velocity field as

$$f(\lambda) = \begin{pmatrix} f_u(\lambda) & f_{uv}(\lambda) \\ f_{vu}(\lambda) & f_v(\lambda) \end{pmatrix} = \begin{pmatrix} \frac{dF_u(\lambda)}{d\lambda} & \frac{dF_{uv}(\lambda)}{d\lambda} \\ \frac{dF_{vu}(\lambda)}{d\lambda} & \frac{dF_v(\lambda)}{d\lambda} \end{pmatrix} = \sigma^2 M^{-2}(\lambda) \quad (25)$$

where $M^{-2}(\lambda)$ is given by (21).

We shall now discuss image restoration when the pure image consists of a single generator, viz. a single straight line.

In Eq. (1) we denoted points on the pure (undeformed) image by $\mathbf{z} = (x, y)$ and points on the deformed (observed) image by $\boldsymbol{\zeta} = \mathbf{z} + \mathbf{w} = (x + u(x, y), y + v(x, y))$. The spectral density matrix of the displacements $u(x, y), v(x, y)$ was given by (25).

2. Single straight line generator. Let us assume that the straight line goes from $P(\beta_1, \beta_2)$ to $Q(\beta_1 + n\beta_3, \beta_2 + n\beta_4)$ and that we observe $n + 1$ points which lie in the *undeformed* image on that straight line, point 0 on P and point n on Q , equidistantly (see Fig. 1). The coordinates of the *deformed* image will be written

$$\begin{aligned} \zeta_t &= x_t + u_t = \beta_1 + t\beta_3 + u_t, \\ \eta_t &= y_t + v_t = \beta_2 + t\beta_4 + v_t, \end{aligned} \quad t = 0, 1, 2, \dots, n, \quad (26)$$

with $Eu_t = Ev_t = 0$. Let $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3, \beta_4)^T$ be the vector of regression coefficients. We want to estimate it and find the asymptotic properties of the estimator when the deformations are such as postulated and when $n \rightarrow \infty$:

$$E \begin{pmatrix} \zeta_t \\ \eta_t \end{pmatrix} = \beta_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \beta_3 \begin{pmatrix} t \\ 0 \end{pmatrix} + \beta_4 \begin{pmatrix} 0 \\ t \end{pmatrix}.$$

FIG. 1.

Let

$$\begin{aligned}
 m_1 &= \beta_1 = x_0 \\
 m_2 &= \beta_1 + \beta_3 = x_1 \\
 &\vdots \\
 m_{n+1} &= \beta_1 + n\beta_3 = x_n \\
 m_{n+2} &= \beta_2 = y_0 \\
 m_{n+3} &= \beta_2 + \beta_4 = y_1 \\
 &\dots\dots\dots \\
 m_{2(n+1)} &= \beta_2 + n\beta_4 = y_n
 \end{aligned}
 \tag{27}$$

and

$$\mathbf{m} = A\boldsymbol{\beta}
 \tag{28}$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & n & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & n \end{pmatrix}.
 \tag{29}$$

Denote

$$\sum_{t=1}^n t = \frac{n(n+1)}{2} = a, \quad \sum_{t=1}^n t^2 = \frac{n}{6} (n+1)(2n+1) = b.
 \tag{30}$$

Hence,

$$B = A^T A = \begin{pmatrix} n+1 & 0 & a & 0 \\ 0 & n+1 & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & b \end{pmatrix}
 \tag{31}$$

so that

$$\begin{aligned}
 B^{-1} &= \frac{1}{(n+1)b - a^2} \begin{pmatrix} b & 0 & -a & 0 \\ 0 & b & 0 & -a \\ -a & 0 & n & 0 \\ 0 & -a & 0 & n+1 \end{pmatrix} \\
 &= \frac{2}{(n+1)(n+2)} \begin{pmatrix} 2n+1 & 0 & -3 & 0 \\ 0 & 2n+1 & 0 & -3 \\ -3 & 0 & \frac{6}{n} & 0 \\ 0 & -3 & 0 & \frac{6}{n} \end{pmatrix} \quad (32)
 \end{aligned}$$

We are interested in the least-squares estimator of the regression coefficients and investigate its asymptotic covariance properties. This estimator is known to be

$$\beta^* = B^{-1} A^T \begin{pmatrix} \zeta_0 \\ \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_n \\ \eta_0 \\ \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}. \quad (33)$$

Hence

$$\beta^* = \frac{2}{(n+1)(n+2)} \begin{pmatrix} 2n+1 \sum_0^n \zeta_i - 3 \sum_0^n i \zeta_i \\ 2n+1 \sum_0^n \eta_i - 3 \sum_0^n i \eta_i \\ -3 \sum_0^n \zeta_i + \frac{6}{n} \sum_0^n i \zeta_i \\ -3 \sum_0^n \eta_i + \frac{6}{n} \sum_0^n i \eta_i \end{pmatrix} = \sum_{i=0}^n \begin{pmatrix} (2n+1-3i)\zeta_i \\ (2n+1-3i)\eta_i \\ \left(-3 + \frac{6i}{n}\right)\zeta_i \\ \left(-3 + \frac{6i}{n}\right)\eta_i \end{pmatrix}. \quad (34)$$

The covariance matrix of β^* is

$$\text{Cov } \beta^* = E[(\beta^* - \beta)(\beta^* - \beta)^T] = B^{-1} A^T R A B^{-1} \quad (35)$$

where

$$R = \left(\begin{array}{ccc|ccc} r_{00}^u & \dots & r_{0n}^u & r_{00}^{uv} & \dots & r_{0n}^{uv} \\ r_{10}^u & \dots & r_{1n}^u & r_{10}^{uv} & \dots & r_{1n}^{uv} \\ r_{20}^u & \dots & r_{2n}^u & r_{20}^{uv} & \dots & r_{2n}^{uv} \\ \vdots & & & & & \\ r_{n0}^u & \dots & r_{nn}^u & r_{n0}^{uv} & \dots & r_{nn}^{uv} \\ \hline & & & r_{00}^v & \dots & r_{0n}^v \\ & & & r_{10}^v & \dots & r_{1n}^v \\ & & & \vdots & & \\ & & & r_{n0}^v & \dots & r_{nn}^v \end{array} \right) \quad (36)$$

with $r_{ij}^u = Eu_i u_j$, $r_{ij}^v = Ev_i v_j$, $r_{ij}^{uv} = Eu_i v_j$. Hence

$$A^T R A = \begin{pmatrix} \sum_{i,j=0}^n r_{ij}^u & \sum r_{ij}^{uv} & \sum j r_{ij}^u & \sum j r_{ij}^{uv} \\ \sum_{i,j=0}^n r_{ij}^{vu} & \sum r_{ij}^v & \sum j r_{ij}^{vu} & \sum j r_{ij}^v \\ \sum i r_{ij}^u & \sum i r_{ij}^{uv} & \sum i j r_{ij}^u & \sum i j r_{ij}^{uv} \\ \sum i r_{ij}^{vu} & \sum i r_{ij}^v & \sum i j r_{ij}^{vu} & \sum i j r_{ij}^v \end{pmatrix}. \quad (37)$$

We have

$$\text{Cov } \beta^* = B^{-1} A^T R A B^{-1}. \quad (38)$$

Let

$$C = \begin{pmatrix} n^{1/2} & 0 & 0 & 0 \\ 0 & n^{1/2} & 0 & 0 \\ 0 & 0 & n^{3/2} & 0 \\ 0 & 0 & 0 & n^{3/2} \end{pmatrix}. \quad (39)$$

We shall evaluate

$$C \text{ cov } \beta^* C = C B^{-1} C C^{-1} A^T R A C^{-1} C B^{-1} C. \quad (40)$$

Note that

$$C^{-1} A^T R A C^{-1} = \begin{pmatrix} \frac{1}{n} \sum r_{ij}^u & \frac{1}{n} \sum r_{ij}^{uv} & \frac{1}{n^2} \sum j r_{ij}^u & \frac{1}{n^2} \sum j r_{ij}^{uv} \\ \frac{1}{n} \sum r_{ij}^{vu} & \frac{1}{n} \sum r_{ij}^v & \frac{1}{n^2} \sum j r_{ij}^{vu} & \frac{1}{n^2} \sum j r_{ij}^v \\ \frac{1}{n^2} \sum i r_{ij}^u & \frac{1}{n^2} \sum i r_{ij}^{uv} & \frac{1}{n^3} \sum i j r_{ij}^u & \frac{1}{n^3} \sum i j r_{ij}^{uv} \\ \frac{1}{n^2} \sum i r_{ij}^{vu} & \frac{1}{n^2} \sum i r_{ij}^v & \frac{1}{n^3} \sum i j r_{ij}^{vu} & \frac{1}{n^3} \sum i j r_{ij}^v \end{pmatrix} \quad (41)$$

and

$$CB^{-1}C = \frac{2}{(n+1)(n+2)} \begin{pmatrix} n(2n+1) & 0 & -3n^2 & 0 \\ 0 & n(2n+1) & 0 & -3n^2 \\ -3n^2 & 0 & 6n^2 & 0 \\ 0 & -3n^2 & 0 & 6n^2 \end{pmatrix} \quad (42)$$

$$\xrightarrow{n \rightarrow \infty} 2 \begin{pmatrix} 2 & 0 & -3 & 0 \\ 0 & 2 & 0 & -3 \\ -3 & 0 & 6 & 0 \\ 0 & -3 & 0 & 6 \end{pmatrix}. \quad (43)$$

Let

$$\begin{aligned} \frac{1}{n} \sum r_{ij}^u &= R^u, \frac{1}{n} \sum r_{ij}^v = R^v, \frac{1}{n} \sum r_{ij}^{uv} = R^{uv} = R^{vu}, \\ \frac{1}{n^2} \sum ir_{ij}^u &= S_i^u, \frac{1}{n^2} \sum ir_{ij}^v = S_i^v, \frac{1}{n^2} \sum ir_{ij}^{uv} = S_i^{uv}; \end{aligned} \quad (44)$$

and similarly for S_j^u, S_j^v, S_j^{uv} ,

$$\frac{1}{n^3} \sum ijr_{ij}^u = T^u, \frac{1}{n^3} \sum ijr_{ij}^v = T^v, \frac{1}{n^3} \sum ijr_{ij}^{uv} = T^{uv}.$$

Hence

$$C^{-1}A^T R A C^{-1} = \begin{pmatrix} R^u & R^{uv} & S_j^u & S_j^{uv} \\ R^{vu} & R^v & S_j^{vu} & S_j^v \\ S_i^u & S_i^v & T^u & T^{uv} \\ S_i^{vu} & S_i^{uv} & T^{vu} & T^v \end{pmatrix} \quad (45)$$

and

$$(CB^{-1}C)(C^{-1}A^T R A C^{-1})(CB^{-1}C) \xrightarrow{n \rightarrow \infty} M =$$

$$4 \begin{pmatrix} 4R^u - 6S_j^u - 6S_i^u + 9T^u & 4R^{uv} - 6S_j^{uv} - 6S_i^{vu} + 9T^{uv} & -6R^u + 9S_j^u + 12S_i^u - 18T^u & -6R^{uv} + 9S_j^{uv} + 12S_i^{vu} - 18T^{uv} \\ 4R^{vu} - 6S_j^{vu} - 6S_i^{uv} + 9T^{vu} & 4R^v - 6S_j^v - 6S_i^v + 9T^v & -6R^{vu} + 9S_j^{vu} + 12S_i^{uv} - 18T^{vu} & -6R^v + 9S_j^v + 12S_i^v - 18T^v \\ -6R^u + 12S_j^u + 9S_i^u - 18T^u & -6R^{uv} + 12S_j^{uv} + 9S_i^{vu} - 18T^{uv} & 9R^u - 18S_j^u - 18S_i^u + 36T^u & 9R^{uv} - 18S_j^{uv} - 18S_i^{vu} + 36T^{uv} \\ -6R^{vu} + 12S_j^{vu} + 9S_i^{uv} - 18T^{vu} & -6R^v + 12S_j^v + 9S_i^v - 18T^v & 9R^{vu} - 18S_j^{vu} - 18S_i^{uv} + 36T^{vu} & 9R^v - 18S_j^v - 18S_i^v + 36T^v \end{pmatrix}. \quad (46)$$

Note: $S_i^u = S_j^u, S_i^v = S_j^v, S_i^{uv} = S_j^{uv}$.

Finally,

$$\text{cov } \beta^* = C^{-1} M C^{-1}$$

with M given by the above matrix.

We now proceed to calculate the elements of this matrix. Let

$$\begin{aligned} X_1 &= \frac{1}{n^{1/2}} \sum_{t=1}^n u_t, & X_2 &= \frac{1}{n^{3/2}} \sum_{t=1}^n t u_t, \\ Y_1 &= \frac{1}{n^{1/2}} \sum_{t=1}^n v_t, & Y_2 &= \frac{1}{n^{3/2}} \sum_{t=1}^n t v_t. \end{aligned} \quad (47)$$

Then we have, with (38), the elements of our covariance matrix:

$$\begin{aligned} R^u &= \text{var } X_1 = \frac{1}{n} \sum_{i,j=1}^n r_{ij}^u, & T^u &= \text{var } X_2 = \frac{1}{n^3} \sum_{i,j=1}^n i j r_{ij}^u, \\ R^v &= \text{var } Y_1 = \frac{1}{n} \sum_{i,j=1}^n r_{ij}^v, & T^v &= \text{var } Y_2 = \frac{1}{n^3} \sum_{i,j=1}^n i j r_{ij}^v, \\ S_j^u &= \text{cov}(X_1, X_2) = \frac{1}{n^2} \sum_{i,j=1}^n j r_{ij}^u, & R^{uv} &= \text{cov}(X_1, Y_1) = \frac{1}{n} \sum_{i,j=1}^n r_{ij}^{uv}, \\ S_j^{uv} &= \text{cov}(X_1, Y_2) = \frac{1}{n^2} \sum_{i,j=1}^n j r_{ij}^{uv}, & S_i^{uv} &= \text{cov}(X_2, Y_1) = \frac{1}{n^2} \sum_{i,j=1}^n i r_{ij}^{uv}, \\ T^{uv} &= \text{cov}(X_2, Y_2) = \frac{1}{n^3} \sum_{i,j=1}^n i j r_{ij}^{uv}, & S_j^v &= \text{cov}(Y_1, Y_2) = \frac{1}{n^2} \sum_{i,j=1}^n j r_{ij}^v. \end{aligned} \quad (48)$$

Thus,

$$\begin{aligned} \text{var}(a_1 X_1 + a_2 X_2 + b_1 Y_1 + b_2 Y_2) &= a_1^2 \text{var } X_1 + a_2^2 \text{var } X_2 + b_1^2 \text{var } Y_1 \\ &\quad + b_2^2 \text{var } Y_2 + 2a_1 a_2 \text{cov } X_1 X_2 \\ &\quad + 2a_1 b_1 \text{cov } X_1 Y_1 + 2a_1 b_2 \text{cov } X_1 Y_2 \\ &\quad + 2a_2 b_1 \text{cov } X_2 Y_1 + 2a_2 b_2 \text{cov } X_2 Y_2 \\ &\quad + 2b_1 b_2 \text{cov } Y_1 Y_2 \\ &= \text{var} \left[\sum_{t=1}^n \left(\frac{a_1}{n^{1/2}} + \frac{a_2 t}{n^{3/2}} \right) u_t + \sum_{t=1}^n \left(\frac{b_1}{n^{1/2}} + \frac{b_2 t}{n^{3/2}} \right) v_t \right] \\ &= \text{var} \left[\sum \alpha_t u_t + \sum \beta_t v_t \right], \end{aligned} \quad (49)$$

say.

We had, in general (see Eq. (14)):

$$\rho_u(\tau, \nu) = E\{u(x, y)u(x + \tau, y + \nu)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[i(\tau \lambda_1 + \nu \lambda_2)] f_u(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2, \text{ etc.,}$$

where

$$f(\lambda) = f(\lambda_1, \lambda_2) = \begin{pmatrix} f_u(\lambda) & f_{uv}(\lambda) \\ f_{vu}(\lambda) & f_v(\lambda) \end{pmatrix} = \sigma^2 M^{-2}(\lambda),$$

(see Eq. (25)). In our special case (26) we have

$$\begin{aligned}\rho_u(\tau, v) &= \rho_u((s-t)\beta_3, (s-t)\beta_4) = \rho_u((s-t)\boldsymbol{\beta}), \quad \text{say} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[i(\tau\lambda_1 + \tau\lambda_2)] f_u(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[i(s-t)(\boldsymbol{\beta}, \boldsymbol{\lambda})] f_u(\boldsymbol{\lambda}) d\boldsymbol{\lambda},\end{aligned}\quad (50)$$

where $\boldsymbol{\beta} = (\beta_3, \beta_4)$. Let us now introduce an orthogonal transformation O in (λ_1, λ_2) space such that

$$\boldsymbol{\mu} = O\boldsymbol{\lambda} \quad \text{with} \quad O = \frac{1}{\|\boldsymbol{\beta}\|^2} \begin{pmatrix} \beta_3 & \beta_4 \\ -\beta_4 & \beta_3 \end{pmatrix}$$

and hence

$$O^{-1} = O^T = \frac{1}{\|\boldsymbol{\beta}\|^2} \begin{pmatrix} \beta_3 & -\beta_4 \\ \beta_4 & \beta_3 \end{pmatrix}. \quad (51)$$

The components μ_1 and μ_2 of $\boldsymbol{\mu}$ have thus been chosen as

$$\mu_1 = \frac{1}{\|\boldsymbol{\beta}\|^2} (\beta_3 \lambda_1 + \beta_4 \lambda_2), \quad \mu_2 = \frac{1}{\|\boldsymbol{\beta}\|^2} (\beta_4 \lambda_1 - \beta_3 \lambda_2), \quad (52)$$

$$\boldsymbol{\lambda} = O^{-1}\boldsymbol{\mu} = \frac{1}{\|\boldsymbol{\beta}\|^2} (\beta_3 \mu_1 - \beta_4 \mu_2, \beta_4 \mu_1 + \beta_3 \mu_2). \quad (53)$$

Also,

$$\begin{aligned}\boldsymbol{\beta} &= \frac{1}{\|\boldsymbol{\beta}\|^2} \begin{pmatrix} \beta_3 & \beta_4 \\ -\beta_4 & \beta_3 \end{pmatrix} \begin{pmatrix} \beta_3 \\ \beta_4 \end{pmatrix} = \frac{1}{\|\boldsymbol{\beta}\|^2} (\beta_3^2 + \beta_4^2, 0) \\ &= (1, 0) \quad \text{and} \quad \|\boldsymbol{\mu}\|^2 = \|\boldsymbol{\beta}\|^2.\end{aligned}\quad (54)$$

Hence, $(O\boldsymbol{\beta}, \boldsymbol{\mu}) = \mu_1$ and

$$(\boldsymbol{\beta}, \boldsymbol{\lambda}) = (\boldsymbol{\beta}, O^{-1}\boldsymbol{\mu}) = (O\boldsymbol{\beta}, \boldsymbol{\mu}) = \mu_1. \quad (55)$$

We then have

$$\begin{aligned}\rho_u(\tau, v) &= \rho_u((s-t)\boldsymbol{\beta}) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[i(s-t)\mu_1] f_u(O^{-1}\boldsymbol{\mu}) d\mu_1 d\mu_2 \\ &= \int_{-\infty}^{\infty} \exp[i(s-t)\mu_1] \left(\int_{-\infty}^{\infty} f_u(O^{-1}\boldsymbol{\mu}) d\mu_2 \right) d\mu_1.\end{aligned}\quad (56)$$

Let

$$g_u(\mu_1) = \int_{-\infty}^{\infty} f_u(O^{-1}\boldsymbol{\mu}) d\mu_2. \quad (57)$$

Then

$$\begin{aligned}
 \rho_u(\tau, v) &= \int_{-\infty}^{\infty} \exp[i(s-t)\mu_1] g_u(\mu_1) d\mu_1 \\
 &= \int_{-\pi}^{\pi} \exp[i(s-t)\mu_1] \left(\sum_{k=-\infty}^{\infty} g_u(\mu_1 + 2\pi k) \right) d\mu_1 \quad (\text{by aliasing}) \\
 &= \int_{-\pi}^{\pi} \exp[i(s-t)\mu_1] G_u(\mu_1) d\mu_1, \quad \text{say,}
 \end{aligned} \tag{58}$$

where

$$G_u(\mu_1) = \sum_{k=-\infty}^{\infty} g_u(\mu_1 + 2\pi k). \tag{59}$$

Let us consider the single terms

$$\int_{-\pi}^{\pi} \exp[i(s-t)\mu_1] g_u(\mu_1 + 2\pi k) d\mu_1. \tag{60}$$

We now have to investigate whether our matrix spectral density

$$g(\mu) = \begin{pmatrix} g_u(\mu) & g_{uv}(\mu) \\ g_{vu}(\mu) & g_v(\mu) \end{pmatrix}$$

can be approximated by a matrix-valued polynomial; that is, whether for any $\varepsilon > 0$ we can find $d \times d$ matrices $g_u(\mu)$ and $g_l(\mu)$, whose entries are trigonometric polynomials, such that

$$g_l(\mu) < g(\mu) < g_u(\mu), \quad \text{for all } \mu$$

where $g(\mu)$ is a continuous $d \times d$ matrix-valued function, $\mu \in [-\pi, \pi]$ with $g(-\pi) = g(\pi)$, and

$$\|g_u - g_l\| \leq \varepsilon.$$

The usual partial ordering of matrices is assumed here, and the norm of a matrix-valued function $a(\mu) = \{a_{ij}(\mu)\}$ can for example be

$$\max_{\mu} \max_{i,j} |a_{ij}(\mu)|$$

which is suitable for our problem.

To construct g_u and g_l , introduce first $g_+(\mu) = g(\mu) + \delta I$, I the unit matrix. Apply Weierstrass' theorem to all entries of g_+ and approximate g_+ by a function g_u , all of whose entries are trigonometric polynomials differing from the corresponding entries of g_+ by at most η . Then, for any d -vector w ,

$$w^T [g_u(\mu) - g(\mu)] w = w^T [g_u(\mu) - g_+(\mu)] w + w^T [g_+(\mu) - g(\mu)] w.$$

Using (54) we see that the left side is equal to

$$w^T [g_u(\mu) - g_+(\mu)] w + \delta \|w\|^2.$$

The first term may be negative but its absolute value is bounded by (Schwarz' inequality)

$$\eta \left(\sum_{k=1}^d |w_k| \right)^2 \leq \eta d \|w\|^2. \tag{61}$$

Hence, if we make $\eta = \delta/d$ it follows that (61) is nonnegative so that

$$g_u(\mu) > g(\mu) \quad \text{for all } \lambda.$$

We construct g_l from below similarly.

It is clear that

$$\|g_u - g_l\| \leq \|g_u - g\| + \|g - g_l\| \leq 2(\delta + \eta) = 2\delta\left(1 + \frac{1}{d}\right)$$

which can be made arbitrarily small by making δ small enough.

In accordance with this result, we now replace the matrix-valued function

$$g(\mu) = \begin{pmatrix} g_u(\mu) & g_{uv}(\mu) \\ g_{vu}(\mu) & g_v(\mu) \end{pmatrix}$$

by a matrix-valued function

$$\tilde{g}(\mu) = \begin{pmatrix} \tilde{g}_u(\mu) & \tilde{g}_{uv}(\mu) \\ \tilde{g}_{vu}(\mu) & \tilde{g}_v(\mu) \end{pmatrix}$$

whose entries are trigonometric polynomials

$$\tilde{g}_u(\mu) = \sum_{v=-m}^m c_v^u \exp(iv\mu), \quad \tilde{g}_v(\mu) = \sum_{v=-m}^m c_v^v \exp(iv\mu), \quad \tilde{g}_{uv}(\mu) = \sum_{v=-m}^m c_v^{uv} \exp(iv\mu).$$

Hence to evaluate

$$\begin{aligned} \text{var}(a_1 X_1 + a_2 X_2 + b_1 Y_1 + b_2 Y_2) &= \text{var}\left(\sum \alpha_t u_t + \sum \beta_t v_t\right) \\ &= \text{var}\left(\sum \alpha_t u_t\right) + \text{var}\left(\sum \beta_t v_t\right) + 2 \text{cov}\left(\sum \alpha_t u_t, \sum \beta_t v_t\right) \end{aligned}$$

we have to find

$$\begin{aligned} &\int_{-\pi}^{\pi} \sum_{s,t=1}^n \alpha_s \alpha_t \exp[i(s-t)\mu] \tilde{g}_u(\mu + 2\pi k) \\ &\quad + \int_{-\pi}^{\pi} \sum_{s,t=1}^n \beta_s \beta_t \exp[i(s-t)\mu] \tilde{g}_v(\mu + 2\pi k) d\mu \\ &\quad + 2 \int_{-\pi}^{\pi} \sum_{s=1}^n \sum_{t=1}^n \alpha_s \beta_t \exp[i(s-t)\mu] \tilde{g}_{uv}(\mu + 2\pi k) d\mu \\ &= \int_{-\pi}^{\pi} \left\{ \left| \sum_1^n \alpha_s \exp(is\mu) \right|^2 \tilde{g}_u(\mu + 2\pi k) + \left| \sum_1^n \beta_s \exp(is\mu) \right|^2 \tilde{g}_v(\mu + 2\pi k) \right. \\ &\quad \left. + 2 \sum_{s,t=1}^n \alpha_s \beta_t \exp[i(s-t)\mu] \tilde{g}_{uv}(\mu + 2\pi k) \right\} d\mu \\ &= \int_{-\pi}^{\pi} \tilde{w}(\mu) \tilde{g}(\mu + 2\pi k) w(\mu) d\mu, \end{aligned} \tag{62}$$

where

$$\tilde{g}(\mu + 2\pi k) = \begin{pmatrix} \tilde{g}_u(\mu + 2\pi k) & \tilde{g}_{uv}(\mu + 2\pi k) \\ \tilde{g}_{vu}(\mu + 2\pi k) & \tilde{g}_v(\mu + 2\pi k) \end{pmatrix} \tag{63}$$

and

$$w(\mu) = \left(\sum_{t=1}^n \left(\frac{a_1}{n^{1/2}} + \frac{a_2 t}{n^{3/2}} \right) \exp(it\mu) \right) \cdot \left(\sum_{t=1}^n \left(\frac{b_1}{n^{1/2}} + \frac{b_2 t}{n^{3/2}} \right) \exp(it\mu) \right). \quad (64)$$

We thus have to evaluate

$$\begin{aligned} & 2 \int_{-\pi}^{\pi} \sum_{s,t=1}^n \left(\frac{a_1}{n^{1/2}} + \frac{a_2 s}{n^{3/2}} \right) \left(\frac{b_1}{n^{1/2}} + \frac{b_2 t}{n^{3/2}} \right) \\ & \quad \times \exp[i(s-t)\mu] \sum_{v=-m}^m c_v^{uv} \exp[iv(\mu + 2\pi k)] d\mu \\ &= 4\pi \underbrace{\sum_{v=-m}^m c_v^{uv} \exp(iv2\pi k)}_{\tilde{g}_{uv}(2\pi k)} \\ & \quad \times \left\{ \frac{a_1 b_1}{n} n + (a_1 b_2 + a_2 b_1) \frac{1}{n^2} \cdot \frac{n(n+1)}{2} + a_2 b_2 \frac{1}{n^3} \cdot \frac{n}{6} (n+1)(2n+1) \right\} \\ & \xrightarrow{n \rightarrow \infty} 4\pi \tilde{g}_{uv}(2\pi k) [a_1 b_1 + \frac{1}{2} a_1 b_2 + \frac{1}{2} a_2 b_1 + \frac{1}{3} a_2 b_2]. \end{aligned} \quad (65)$$

Similarly

$$\begin{aligned} & \int_{-\pi}^{\pi} \sum_{s,t=1}^n \left(\frac{a_1}{n^{1/2}} + \frac{a_2 s}{n^{3/2}} \right) \left(\frac{a_1}{n^{1/2}} + \frac{a_2 t}{n^{3/2}} \right) \exp[i(s-t)\mu] \\ & \quad \times \sum_{v=-m}^m c_v^u \exp[iv(\mu + 2\pi k)] d\mu \xrightarrow{n \rightarrow \infty} 2\pi \tilde{g}_u(2\pi k) \{a_1^2 + a_1 a_2 + \frac{1}{3} a_2^2\} \end{aligned} \quad (66)$$

and

$$\begin{aligned} & \int_{-\pi}^{\pi} \sum_{s,t=1}^n \left(\frac{b_1}{n^{1/2}} + \frac{b_2 s}{n^{3/2}} \right) \left(\frac{b_1}{n^{1/2}} + \frac{b_2 t}{n^{3/2}} \right) \exp[i(s-t)\mu] \sum_{v=-m}^m c_v^v \exp[iv(\mu + 2\pi k)] d\mu \\ & \xrightarrow{n \rightarrow \infty} 2\pi \tilde{g}_v(2\pi k) \{b_1^2 + b_1 b_2 + \frac{1}{3} b_2^2\}. \end{aligned} \quad (67)$$

Hence,

$$\begin{aligned} \text{var}(\sum \alpha_t u_t) &= \sum \sum \alpha_t \alpha_s \rho_u(\tau, \nu) = \sum \alpha_t \alpha_s \rho_u((s-t)\mathbf{t}) \\ &= \sum_{k=-\infty}^{\infty} \sum_{s,t} \alpha_t \alpha_s \exp[i(s-t)\mu_1] \sum_{-\infty}^{\infty} \tilde{g}_u(\mu_1 + 2\pi k) \\ &= 2\pi \sum_{k=-\infty}^{\infty} \tilde{g}_u(2\pi k) \{a_1^2 + a_1 a_2 + \frac{1}{3} a_2^2\} \end{aligned} \quad (68)$$

and similarly for the other terms, so that

$$\begin{aligned} & \frac{1}{2\pi} \text{var}(a_1 X_1 + a_2 X_2 + b_1 Y_1 + b_2 Y_2) \\ &= \sum_{n=-\infty}^{\infty} \{ \tilde{g}_u(2\pi k) (a_1^2 + a_1 a_2 + \frac{1}{3} a_2^2) + \tilde{g}_v(2\pi k) (b_1^2 + b_1 b_2 + \frac{1}{3} b_2^2) \\ & \quad + 2\tilde{g}_{uv}(2\pi k) (a_1 b_1 + \frac{1}{2} a_1 b_2 + \frac{1}{2} a_2 b_1 + \frac{1}{3} a_2 b_2) \} \end{aligned} \quad (69)$$

Hence

$$\begin{aligned}
 R^u &= \text{var } X_1 = \frac{1}{n} \sum r_{ij}^u = \sum_{k=-\infty}^{\infty} \tilde{g}_u(2\pi k), \\
 R^v &= \text{var } Y_1 = \frac{1}{n} \sum r_{ij}^v = \sum \tilde{g}_v(2\pi k), \\
 R^{uv} &= \text{cov}(Y_1, X_1) = \frac{1}{n} \sum r_{ij}^{uv} = 2 \sum \tilde{g}_{uv}(2\pi k) = R^{vu}, \\
 S_i^u &= \text{cov}(X_2, X_1) = \frac{1}{n^2} \sum ir_{ij}^u, \\
 S_j^u &= \text{cov}(X_1, X_2) = \frac{1}{n^2} \sum jr_{ij}^u = \frac{1}{2} \sum_{k=-\infty}^{\infty} \tilde{g}_u(2\pi k) = S_i^u, \\
 S_i^v &= S_j^v = \text{cov}(Y_1, Y_2) = \frac{1}{n^2} \sum ir_{ij}^v = \frac{1}{n^2} \sum jr_{ij}^v = \frac{1}{2} \sum \tilde{g}_v(2\pi k), \\
 S_i^{vu} &= \text{cov}(X_2, Y_1) = \frac{1}{n^2} \sum ir_{ij}^{vu} = \frac{1}{2} \sum \tilde{g}_{vu}(2\pi k), \\
 S_j^{uv} &= \text{cov}(X_1, Y_2) = \frac{1}{n^2} \sum jr_{ij}^{uv} = \frac{1}{2} \sum \tilde{g}_{uv}(2\pi k), \\
 T^{uv} &= \text{cov}(X_2, Y_2) = \frac{1}{n^3} \sum ijr_{ij}^{uv} = \frac{1}{3} \sum \tilde{g}_{uv}(2\pi k), \\
 T^{vu} &= \text{cov}(Y_2, X_2) = \frac{1}{n^3} \sum ijr_{ij}^{vu} = \frac{1}{3} \sum \tilde{g}_{vu}, \\
 T^u &= \text{var } X_2 = \frac{1}{n^3} \sum ijr_{ij}^u = \frac{1}{3} \sum \tilde{g}_u(2\pi k), \\
 T^v &= \text{var } Y_2 = \frac{1}{n^3} \sum ijr_{ij}^v = \frac{1}{3} \sum \tilde{g}_v(2\pi k).
 \end{aligned} \tag{70}$$

Now (omitting the tildes on the $\tilde{g}_u$'s),

$$g_u(\mu_1) = \int_{-\infty}^{\infty} f_u(O^{-1}\mu) d\mu_2, \tag{71}$$

$$g_v(\mu_1) = \int_{-\infty}^{\infty} f_v(O^{-1}\mu) d\mu_2, \quad g_{uv} = \int_{-\infty}^{\infty} f_{uv}(O^{-1}\mu) d\mu_2, \tag{72}$$

where

$$f(\lambda) = \begin{pmatrix} f_u & f_{uv} \\ f_{vu} & f_v \end{pmatrix} = \sigma^2 M^{-2}(\lambda)$$

and $M^{-2}(\lambda)$ is given by (21). We have

$$\begin{aligned}
 O^{-1}\mu &= \frac{1}{\|\beta\|^2} \begin{pmatrix} \beta_3 \mu_1 - \beta_4 \mu_2 \\ \beta_4 \mu_1 - \beta_3 \mu_2 \end{pmatrix}; \\
 \lambda \lambda^T &= \begin{pmatrix} \lambda_1^2 & \lambda_1 \lambda_2 \\ \lambda_1 \lambda_2 & \lambda_2^2 \end{pmatrix}, \quad \lambda_1^2 = \frac{1}{\|\beta\|^4} (\beta_3^2 \mu_1^2 + \beta_4^2 \mu_2^2 - 2\beta_3 \beta_4 \mu_1 \mu_2). \tag{73}
 \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{\sigma^2} f_u(\lambda) &= \frac{1}{[c + a_2 \|\lambda\|^2]^2} \left[1 - a_3 \frac{2c + (a_1 + a_2) \|\lambda\|^2}{[c + a_1 \|\lambda\|^2]} \lambda_1^2 \right], \\ \frac{1}{\sigma^2} f_u(O^{-1}\mu) &= \frac{1}{[c + a_2(\mu_1^2 + \mu_2^2)]} \left[1 - a_3 \frac{2c + (a_1 + a_2)(\mu_1^2 + \mu_2^2)}{[c + a_1(\mu_1^2 + \mu_2^2)]} \frac{1}{\|\beta^4\|} \right. \\ &\quad \left. \cdot (\beta_3^2 \mu_1^2 + \beta_4^2 \mu_2^2 - 2\beta_3 \beta_2 \mu_1 \mu_2) \right]. \end{aligned} \quad (74)$$

These expressions for $f_u(O^{-1}\mu)$, $f_v(O^{-1}\mu)$ and $f_{uv}(O^{-1}\mu)$ must now be integrated with respect to μ_2 from $-\infty$ to $+\infty$ to obtain $g_u(\mu_1)$, $g_v(\mu_1)$ and $g_{uv}(\mu_1)$. This can, in principle, be done by noting that

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dx}{a + x^2} &= \frac{\pi}{\sqrt{a}}, \quad \int_{-\infty}^{\infty} \frac{dx}{(a + x^2)^2} = \frac{1}{2} \pi a^{-3/2}, \\ \int_{-\infty}^{\infty} \frac{dx}{(a + x^2)^2(b + x^2)^2} &= \frac{\pi}{2(b - a)^3} (5b^{-1/2} - 5a^{-1/2} + a^{-3/2}b - ab^{-3/2}), \\ \int_{-\infty}^{\infty} \frac{x^2 dx}{(a + x^2)^2(b + x^2)^2} &= \frac{\pi}{2(b - a)^3} (3a^{1/2} - 3b^{1/2} + ba^{-1/2} - ab^{-1/2}), \\ \int_{-\infty}^{\infty} \frac{x^4 dx}{(a + x^2)^2(b + x^2)^2} &= \frac{\pi}{2(b - a)^3} (3ab^{1/2} - 3ba^{1/2} + b^{3/2} - a^{3/2}). \end{aligned}$$

To find R^u , R^v , etc. and thus the matrix M of Eq. (46), one would now have to put $\mu_1 = 2\pi k$ and sum over all k 's. From M , the asymptotic covariance matrix of the regression coefficients follows as

$$\text{cov } \beta^* = C^{-1} M C^{-1}$$

with C given by Eq. (39).

3. Several straight line generators. The analysis of images of points arising from several generators proceeds in an analogous manner to that presented in the previous section for a single generator. We can, however, distinguish two classes of problems here: those in which a regularity constraint is superimposed on the images, and those where there is no such constraint. We shall briefly look at the situation where we have two generators and the regularity constraint consists of the assumption of a common end point. One would expect that the imposition of a regularity constraint will, in some norm, result in a smaller covariance matrix of the estimated regression coefficients than the absence of such a constraint.

Case (i): *Regularity constraint.* Let there again be $(n + 1)$ points on each generator (as there were in Sec. 2) (Fig. 2). Let $t = 0, 1, 2, \dots, n$; before deformation, these points have the coordinates

$$\begin{aligned} \left. \begin{aligned} x_i^{(1)} &= \beta_1 + t\beta_3^{(1)} \\ y_i^{(1)} &= \beta_2 + t\beta_4^{(1)} \end{aligned} \right\} \text{generator (1),} \\ \left. \begin{aligned} x_i^{(2)} &= \beta_1 + t\beta_3^{(2)} \\ y_i^{(2)} &= \beta_2 + t\beta_4^{(2)} \end{aligned} \right\} \text{generator (2).} \end{aligned} \quad (75)$$

The deformed image will be described by

$$\left. \begin{aligned} \zeta_t^{(1)} &= x_t^{(1)} + u_t^{(1)} \\ \eta_t^{(1)} &= y_t^{(1)} + v_t^{(1)} \end{aligned} \right\} \text{generator (1),}$$

$$\left. \begin{aligned} \zeta_t^{(2)} &= x_t^{(2)} + u_t^{(2)} \\ \eta_t^{(2)} &= y_t^{(2)} + v_t^{(2)} \end{aligned} \right\} \text{generator (2).} \quad (76)$$

Writing out (74) for $t = 0, 1, \dots, n$, we get

$$\begin{aligned} x_0^{(1)} &= \beta_1 &= m_1, \text{ say,} \\ x_1^{(1)} &= \beta_1 + \beta_3^{(1)} &= m_2, \\ x_2^{(1)} &= \beta_1 + 2\beta_3^{(1)} &= m_3, \\ &\vdots \\ x_n^{(1)} &= \beta_1 + n\beta_3^{(1)} &= m_{n+1}; \\ y_0^{(1)} &= \beta_2 &= m_{n+2}, \\ y_1^{(1)} &= \beta_2 + \beta_4^{(1)} &= m_{n+3}, \\ y_2^{(1)} &= \beta_2 + 2\beta_4^{(1)} &= m_{n+4}, \\ &\vdots \\ y_n^{(1)} &= \beta_2 + n\beta_4^{(1)} &= m_{2n+2}; \\ x_0^{(2)} &= \beta_1 &= m_{2n+3} = m_1, \\ x_1^{(2)} &= \beta_1 + \beta_3^{(2)} &= m_{2n+4}, \\ x_2^{(2)} &= \beta_1 + 2\beta_3^{(2)} &= m_{2n+5}, \\ &\vdots \\ x_n^{(2)} &= \beta_1 + n\beta_3^{(2)} &= m_{3n+3}; \\ y_0^{(2)} &= \beta_2 &= m_{3n+4} = m_{n+2}, \\ y_1^{(2)} &= \beta_2 + \beta_4^{(2)} &= m_{3n+5}, \\ y_2^{(2)} &= \beta_2 + 2\beta_4^{(2)} &= m_{3n+6}, \\ &\vdots \\ y_n^{(2)} &= \beta_2 + n\beta_4^{(2)} &= m_{4(n+1)}. \end{aligned} \quad (77)$$

FIG. 2.

Let us denote the vector of regression coefficients to be estimated by

$$\beta = (\beta_1, \beta_2, \beta_3^{(1)}, \beta_4^{(1)}, \beta_3^{(2)}, \beta_4^{(2)})^T \quad (78)$$

and the vector of mean-values of the displacements by

$$\mathbf{m} = (m_1, m_2, \dots, m_{4(n+1)})^T \quad (79)$$

where $m_{2n+3} = m_1$ and $m_{3n+4} = m_{n+2}$ because of the regularity constraint.

We can therefore write $\mathbf{m} = A\beta$, where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 \\ \vdots & & & & & \\ 1 & 0 & n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ \vdots & & & & & \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 \\ \vdots & & & & & \\ 1 & 0 & 0 & 0 & n & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ \vdots & & & & & \\ 0 & 1 & 0 & 0 & 0 & n \end{pmatrix}.$$

Hence, $B = A^T A =$

$$\begin{pmatrix} 2(n+1) & 0 & a & 0 & a & 0 \\ 0 & 2(n+1) & 0 & a & 0 & a \\ a & 0 & b & 0 & 0 & 0 \\ 0 & a & 0 & b & 0 & 0 \\ a & 0 & 0 & 0 & b & 0 \\ 0 & a & 0 & 0 & 0 & b \end{pmatrix} = \begin{pmatrix} N_2 & \alpha & \alpha \\ \alpha & \beta & 0 \\ \alpha & 0 & \beta \end{pmatrix} \quad (80)$$

where

$$N_2 = \begin{pmatrix} 2(n+1) & 0 \\ 0 & 2(n+1) \end{pmatrix}, \quad \alpha = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad \beta = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}; \quad (81)$$

here $a = 1 + 2 + \dots + n = n(n+1)/2$, $b = 1^2 + 2^2 + \dots + n^2 = (n/6)(n+1)(2n+1)$. Hence

$$B^{-1} = \frac{1}{2b^2(n+1) - 2a^2} \times \begin{pmatrix} b & 0 & -a & 0 & -a & 0 \\ 0 & b & 0 & -a & 0 & -a \\ -a & 0 & 2(n+1) - \frac{a^2}{b} & 0 & \frac{a^2}{b} & 0 \\ 0 & -a & 0 & 2(n+1) - \frac{a^2}{b} & 0 & \frac{a^2}{b} \\ -a & 0 & \frac{a^2}{b} & 0 & 2(n+1) - \frac{a^2}{b} & 0 \\ 0 & -a & 0 & \frac{a^2}{b} & 0 & 2(n+1) - \frac{a^2}{b} \end{pmatrix},$$

$$B^{-1} = \frac{1}{(n+1)(n+2)} \times \begin{pmatrix} 2n+1 & 0 & -3 & 0 & -3 & 0 \\ 0 & 2n+1 & 0 & -3 & 0 & -3 \\ -3 & 0 & \frac{3(5n+4)}{n(2n+1)} & 0 & \frac{9}{2n+1} & 0 \\ 0 & -3 & 0 & \frac{3(5n+4)}{n(2n+1)} & 0 & \frac{9}{2n+1} \\ -3 & 0 & \frac{a}{2n+1} & 0 & \frac{3(5n+4)}{n(2n+1)} & 0 \\ 0 & -3 & 0 & \frac{9}{2n+1} & 0 & \frac{3(5n+4)}{n(2n+1)} \end{pmatrix}.$$

Let

$$C = \begin{pmatrix} n^{1/2} & & & & \\ & n^{1/2} & & & \\ & & n^{3/2} & & \\ & & & n^{3/2} & \\ & & & & n^{3/2} \end{pmatrix}$$

FIG. 3.

Then

$$CB^{-1}C \xrightarrow{n \rightarrow \infty} \begin{pmatrix} 2 & 0 & -3 & 0 & -3 & 0 \\ 0 & 2 & 0 & -3 & 0 & -3 \\ -3 & 0 & \frac{15}{2} & 0 & \frac{9}{2} & 0 \\ 0 & -3 & 0 & \frac{15}{2} & 0 & \frac{9}{2} \\ -3 & 0 & \frac{9}{2} & 0 & \frac{15}{2} & 0 \\ 0 & -3 & 0 & \frac{9}{2} & 0 & \frac{15}{2} \end{pmatrix}.$$

The computation of M , and thus of $\text{cov } \beta^* = C^{-1}MC^{-1}$, follows as before.

Case (ii): Let us now turn to the situation when there is *no regularity constraint* (Fig. 3). Let again $t = 0, 1, 2, \dots, n$; before deformation, the points on the generators will now have coordinates

$$\left. \begin{aligned} x_t^{(1)} &= \beta_1^{(1)} + t\beta_3^{(1)} \\ y_t^{(1)} &= \beta_2^{(1)} + t\beta_4^{(1)} \end{aligned} \right\} \text{generator (1),}$$

$$\left. \begin{aligned} x_t^{(2)} &= \beta_1^{(2)} + t\beta_3^{(2)} \\ y_t^{(2)} &= \beta_2^{(2)} + t\beta_4^{(2)} \end{aligned} \right\} \text{generator (2).}$$

The deformed image will now be again described by Eqs. (76), and Eqs. (75) will be the same except that β_1 is replaced by $\beta_1^{(1)}$ and β_2 by $\beta_2^{(1)}$ in the first $2n + 2$ equations, and by $\beta_1^{(2)}$ and $\beta_2^{(2)}$, respectively, in the second $2n + 2$ equations.

Summarizing, again, by

$$\mathbf{m} = A\boldsymbol{\beta},$$

where $\boldsymbol{\beta}$ will now (in the case without regularity constraints) be given by

$$\boldsymbol{\beta} = (\beta_1^{(1)}, \beta_2^{(1)}, \beta_3^{(1)}, \beta_4^{(1)}, \beta_1^{(2)}, \beta_2^{(2)}, \beta_3^{(2)}, \beta_4^{(2)}),$$

the mean value vector is given by Eq. (78) and A by

$$\begin{pmatrix}
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
 \vdots & & & & & & & \\
 1 & 0 & n & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\
 \vdots & & & & & & & \\
 0 & 1 & 0 & n & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
 \vdots & & & & & & & \\
 0 & 0 & 0 & 0 & 1 & 0 & n & 0 \\
 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\
 \vdots & & & & & & & \\
 0 & 0 & 0 & 0 & 0 & 1 & 0 & n
 \end{pmatrix}$$

Hence, $B = A^T A$ becomes

$$B = \begin{pmatrix}
 n+1 & 0 & a & 0 & 0 & 0 & 0 & 0 \\
 0 & n+1 & 0 & a & 0 & 0 & 0 & 0 \\
 a & 0 & b & 0 & 0 & 0 & 0 & 0 \\
 0 & a & 0 & b & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & n+1 & 0 & a & 0 \\
 0 & 0 & 0 & 0 & 0 & n+1 & 0 & a \\
 0 & 0 & 0 & 0 & a & 0 & b & 0 \\
 0 & 0 & 0 & 0 & 0 & a & 0 & b
 \end{pmatrix}$$

Hence

$$B^{-1} = \frac{1}{b(n+1) - a^2}$$

$$\times \begin{pmatrix}
 b & 0 & -a & 0 & 0 & 0 & 0 & 0 \\
 0 & b & 0 & -a & 0 & 0 & 0 & 0 \\
 -a & 0 & n+1 & 0 & 0 & 0 & 0 & 0 \\
 0 & -a & 0 & n+1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & b & 0 & -a & 0 \\
 0 & 0 & 0 & 0 & 0 & b & 0 & -a \\
 0 & 0 & 0 & 0 & -a & 0 & n+1 & 0 \\
 0 & 0 & 0 & 0 & 0 & -a & 0 & n+1
 \end{pmatrix}$$

$$= \frac{1}{n(n+1)(n+2)}$$

$$\times \begin{pmatrix} 2n(2n+1) & 0 & -6n & 0 & 0 & 0 & 0 & 0 \\ 0 & 2n(2n+1) & 0 & -6n & 0 & 0 & 0 & 0 \\ -6n & 0 & 12 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6n & 0 & 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2n(2n+1) & 0 & -6n & 0 \\ 0 & 0 & 0 & 0 & 0 & 2n(2n+1) & 0 & -6n \\ 0 & 0 & 0 & 0 & -6n & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6n & 0 & 12 \end{pmatrix}.$$

Let

$$C = \begin{pmatrix} n^{1/2} & & & & & & & \\ & n^{1/2} & & & & & & \\ & & n^{3/2} & & & & & \\ & & & n^{3/2} & & & & \\ & & & & n^{1/2} & & & \\ & & & & & n^{1/2} & & \\ & \bigcirc & & & & & n^{3/2} & \\ & & & & & & & n^{3/2} \end{pmatrix}$$

Then

$$CB^{-1}C \rightarrow \begin{pmatrix} 4 & 0 & -6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & -6 & 0 & 0 & 0 & 0 \\ -6 & 0 & 12 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6 & 0 & 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & -6 \\ 0 & 0 & 0 & 0 & -6 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6 & 0 & 12 \end{pmatrix}.$$

The computation of M , and thus of $\beta^* = C^{-1}MC^{-1}$, proceeds in an analogous manner to the earlier cases.

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