BOUNDARY-VALUE PROBLEMS IN MICROCONTINUUM FLUID MECHANICS*

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Abstract. This paper examines the existence and uniqueness of solutions of certain boundary-value problems associated with a seemingly complex system of coupled partial differential equations frequently encountered in the field of microcontinuum fluid mechanics. These problems are analyzed using potential theory methods and appealing to results from the theory of singular integral equations. Matrices are introduced to facilitate this analysis.

1. Introduction. The equations of motion characterizing slow, steady incompressible flow of a micropolar fluid are given by [1]

$$\begin{aligned} t_{ji,j} + f_i &= 0, \\ m_{ji,j} + \epsilon_{ijk} t_{jk} + l_i &= 0, \end{aligned} \quad (1.1)$$

while the linear constitutive laws take the form

$$\begin{aligned} t_{ij}(\mathbf{u}, \mathbf{v}) &= -p\delta_{ij} + \frac{1}{2}(2\mu + \kappa)(u_{i,j} + u_{j,i}) + \kappa\epsilon_{ijk}(w_k - v_k), \\ m_{ij}(\mathbf{v}) &= \alpha v_{k,k}\delta_{ij} + \beta v_{i,j} + \gamma v_{j,i}, \end{aligned} \quad (1.2)$$

with the continuity equation being

$$u_{i,i} = 0. \quad (1.3)$$

Here t_{ij} are components of the Cauchy stress; m_{ij} are the components of the couple stress; $\mathbf{u}$ is the velocity vector; $\mathbf{v}$ is the microrotation vector; $\mathbf{f}$ is the body force, $\mathbf{l}$ is the body couple p is the pressure; ϵ_{ijk} is the alternating tensor; δ_{ij} is the Kronecker delta; $(\alpha, \beta, \gamma, \mu, \kappa)$ are material constants and

$$w_i = \frac{1}{2}(\nabla \times \mathbf{u})_i.$$

The above equations reduce to the following vector system of coupled differential equations:

$$\begin{aligned} -(\mu + \kappa)\nabla \times \nabla \times \mathbf{u} + k\nabla \times \mathbf{v} - \nabla p &= -\mathbf{f}, \\ (\alpha + \beta + \gamma)\nabla \nabla \cdot \mathbf{v} - \gamma n\nabla \times \nabla \times \mathbf{v} + \kappa\nabla \times \mathbf{u} - 2\kappa\mathbf{v} &= -\mathbf{l}, \\ \nabla \cdot \mathbf{u} &= 0. \end{aligned} \quad (1.4)$$

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The Stokes flow in a region Ω is defined as a flow whose field parameters $\mathbf{u}$, $\boldsymbol{\nu}$, p satisfy the homogeneous system

$$\begin{aligned} -(\mu + \kappa)\nabla \times \nabla \times \mathbf{u} + \kappa\nabla \times \boldsymbol{\nu} - \nabla p &= 0, \\ (\alpha + \beta + \gamma)\nabla \nabla \cdot \boldsymbol{\nu} - \gamma\nabla \times \nabla \times \boldsymbol{\nu} + \kappa\nabla \times \mathbf{u} - 2\kappa\boldsymbol{\nu} &= 0, \\ \nabla \cdot \mathbf{u} &= 0, \end{aligned} \quad (1.5)$$

in Ω . Our objective is to investigate the existence and uniqueness of solutions of such flows for both an interior domain $\Omega_{(i)}$ and an exterior domain $\Omega_{(e)}$ when certain conditions are prescribed on the boundary surface S . For our purposes, we shall assume that S lies in a finite region of three-dimensional space and is connected.

The Dirichlet-type boundary-value problem where $\mathbf{u}$ and $\boldsymbol{\nu}$ assume prescribed values on S has already been investigated [2]. However, very little work, if any, has been done on other boundary conditions suggested by Eringen [1]. We propose to examine boundary-value problems associated with two such conditions. In the first problem the boundary conditions

$$t_{ji}(\mathbf{u}, \boldsymbol{\nu})n_j|_S = T_{0i}, \quad m_{ji}(\boldsymbol{\nu})n_j|_S = M_{0i}, \quad (1.6)$$

are specified, while in the second problem the mixed conditions

$$[t_{ji}(\mathbf{u}, \boldsymbol{\nu})n_j + hu_i]|_S = a_i, \quad [m_{ji}(\boldsymbol{\nu})n_j + h\nu_i]|_S = b_i \quad (1.7)$$

are specified. Here $h = h(\xi)$, $\xi \in S$, and $\mathbf{n}$ is the unit outward surface normal. It should be noted that condition (1.6) is equivalent to saying that the boundary forces and moments are prescribed. This paper utilizes the methods of potential theory to examine the fundamental problem of existence and uniqueness of solutions of the homogeneous system (1.5) subject to the conditions (1.6) and in turn (1.7).

2. Preliminaries. Let $\mathbf{f}$ be the concentrated point force acting at the point $y(y_1, y_2, y_3)$ in an unbounded medium and given by

$$\mathbf{f} = \delta(x - y)\mathbf{e}^k, \quad (2.1)$$

where $\delta(x - y)$ is the Dirac delta function and $\mathbf{e}^k$ is a unit vector defined along the k th co-ordinate axis. The solution of (1.4) due to this force in the absence of any body couple ($\mathbf{l} = 0$), will be denoted by $({}_F\mathbf{u}^k, {}_F\boldsymbol{\nu}^k, {}_Fp^k)$. In a similar way we represent the solution of (1.4) due to a concentrated point couple in the form of (2.1) and in the absence of any body force ($\mathbf{f} = 0$), as $({}_L\mathbf{u}^k, {}_L\boldsymbol{\nu}^k, {}_Lp^k)$. These are the so-called fundamental singular solutions and explicit expressions have been generated for them [3]. Furthermore, in terms of these, the solution to the system (1.5) under investigation has the following integral representation [3]:

$$\begin{aligned} u_k(x) &= \int_S \{t_{ji}(\mathbf{u}, \boldsymbol{\nu})_F u_i^k(x, y) - {}_F t_{ji}^k(x, y) u_i^k(y)\} n_j dS_y \\ &\quad + \int_S \{m_{ji}(\boldsymbol{\nu})_F \nu_i^k(x, y) - {}_F m_{ij}^k(x, y) \nu_i(y)\} n_j dS_y, \end{aligned} \quad (2.2)$$

$$\begin{aligned} \nu_k(x) = & \int_S \{ t_{ji}(\mathbf{u}, \boldsymbol{\nu})_L u_i^k(x, y) - {}_L t_{ji}^k(x, y) u_i(y) \} n_j dS_y \\ & + \int_S \{ m_{ji}(\boldsymbol{\nu})_L \nu_i^k(x, y) - {}_L m_{ji}^k(x, y) \nu_i(y) \} n_j dS_y, \end{aligned} \quad (2.3)$$

where ${}_F t_{ji}^k$, ${}_F m_{ji}^k$, ${}_L t_{ji}^k$ and ${}_L m_{ji}^k$ are components of stresses associated with the fundamental solutions. That is,

$$\begin{aligned} {}_F t_{ji}^k(x, y) &\equiv t_{ji}({}_F \mathbf{u}^k, {}_F \boldsymbol{\nu}^k), & {}_F m_{ji}^k(x, y) &\equiv m_{ji}({}_F \boldsymbol{\nu}^k), \\ {}_L t_{ji}^k(x, y) &\equiv t_{ji}({}_L \mathbf{u}^k, {}_L \boldsymbol{\nu}^k), & {}_L m_{ji}^k(x, y) &\equiv m_{ji}({}_L \boldsymbol{\nu}^k). \end{aligned}$$

Attention will be focussed on $\mathbf{u}(x)$, $\boldsymbol{\nu}(x)$ as $p(x)$ can always be obtained by substitution. The representations (2.2) and (2.3) can be written in the matrix form

$$U(x) = \int_S \{ A(x, y) \tau(U) - B(x, y) U(y) \} dS_y, \quad (2.4)$$

where we have introduced the matrices

$$\begin{aligned} U &= \begin{bmatrix} u_k \\ \nu_k \end{bmatrix}, & \tau(U) &= \begin{bmatrix} t_{ji} n_j \\ m_{ji} n_j \end{bmatrix} \equiv \begin{bmatrix} T_i \\ M_i \end{bmatrix}, \\ A(x, y) &= \begin{bmatrix} {}_F u_i^k & {}_F \nu_i^k \\ {}_L u_i^k & {}_L \nu_i^k \end{bmatrix}, & B(x, y) &= \begin{bmatrix} {}_F t_{ji}^k n_j & {}_F m_{ji}^k n_j \\ {}_L t_{ji}^k n_j & {}_L m_{ji}^k n_j \end{bmatrix}. \end{aligned}$$

As in ordinary potential theory, the representation formula (2.4) suggests the introduction of the single-layer potential

$$V(x; \psi) = \int_S A(x, y) \psi(y) dS_y, \quad (2.5)$$

and the double-layer potential

$$W(x; \phi) = \int_S B(x, y) \phi(y) dS_y, \quad (2.6)$$

where ψ and ϕ are matrices of the form

$$\psi = \begin{bmatrix} \psi_i^1 \\ \psi_i^2 \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_i^1 \\ \phi_i^2 \end{bmatrix}.$$

The integral representation (2.4) can now be written as

$$U(x) = V(x; \tau(U)) - W(x; U). \quad (2.7)$$

Let us make the following assumptions:

- (i) S is a closed Lyapunov surface.
- (ii) The density functions ϕ , ψ are continuous.
- (iii) ξ , η are points on the surface S .
- (iv) $H(\lambda)$ represents a Holder class of exponent $\lambda \leq 1$.

Under these assumptions, the following results, stated in the form of theorems, have been verified [2]

THEOREM 1. The single-layer potential $V(x; \psi)$ is continuous in infinite space including the surface S .

THEOREM 2. Let $W(x; \phi)_{(i)}$ and $W(x; \phi)_{(e)}$ denote the limiting values of the double-layer potential $W(x; \phi)$ on S as S is approached from $\Omega_{(i)}$ and $\Omega_{(e)}$ respectively. If $\phi \in H(\lambda)$ and $W(\xi; \phi)$ is the directly defined value of $W(x; \phi)$ on S , then

$$W(\xi; \phi)_{(i)} = -\frac{1}{2}\phi(\xi) + W(\xi; \phi) = -\frac{1}{2}\phi(\xi) + \int_S B(\xi, \eta)\phi(\eta) dS_\eta, \quad (2.8)$$

$$W(\xi; \phi)_{(e)} = \frac{1}{2}\phi(\xi) + W(\xi; \phi) = \frac{1}{2}\phi(\xi) + \int_S B(\xi, \eta)\phi(\eta) dS_\eta. \quad (2.9)$$

THEOREM 3. If $V(x; \psi)$ is a single-layer potential with $\psi \in H(\lambda)$, then

$$\tau(V)_{(i)} = \frac{1}{2}\psi(\xi) + \tau(V) = \frac{1}{2}\psi(\xi) + \int_S \{\Gamma_\xi A(\xi, \eta)\}\psi(\eta) dS_\eta, \quad (2.10)$$

$$\tau(V)_{(e)} = -\frac{1}{2}\psi(\xi) + \tau(V) = -\frac{1}{2}\psi(\xi) + \int_S \{\Gamma_\xi A(\xi, \eta)\}\psi(\eta) dS_\eta, \quad (2.11)$$

where

$$\Gamma_y A^*(x, y) = \begin{bmatrix} T_i A_1^{*k} & T_i A_2^{*k} \\ M_i A_1^{*k} & M_i A_2^{*k} \end{bmatrix}$$

with A^* being the transpose of A , A_m^{*k} ($m = 1, 2$) being the m th column of the matrix A^* and the subscript y indicating, as usual, that the operators are applied at that point. The systems (2.8)–(2.9) and (2.10)–(2.11) are pairwise adjoint [2] and the Fredholm alternative will be applied to them to investigate the existence and uniqueness of our boundary-value problem. The use of the Fredholm alternative in this particular case is justified in the Appendix.

As mentioned earlier, the Dirichlet boundary-value problem where

$$\mathbf{u}|_S = \mathbf{u}_0, \quad \boldsymbol{\nu}|_S = \boldsymbol{\nu}_0 \quad (2.12)$$

are specified, has been investigated. The results are summarized in the following theorems [2]:

THEOREM 4. The system (1.5) subject to the Dirichlet boundary conditions (2.12) and the supplementary conditions

$$\mathbf{u}, \boldsymbol{\nu} \text{ and } p \rightarrow 0 \quad \text{as } r \rightarrow \infty, \quad (2.13)$$

has a unique solution in $\Omega_{(e)}$ for any continuous fields $\mathbf{u}_0$ and $\boldsymbol{\nu}_0$.

THEOREM 5. A necessary and sufficient condition for the same problem (1.5), (2.12) to have a unique solution in $\Omega_{(i)}$, is that

$$\int_S \mathbf{u}_0 \cdot \mathbf{n} dS = 0. \quad (2.14)$$

3. First boundary-value problem. Here we wish to investigate the existence and uniqueness of solution of the system (1.5)

$$\begin{aligned} & -(\mu + k)\nabla \times \nabla \times \mathbf{u} + k\nabla \times \mathbf{v} - \nabla p = 0, \\ & (\alpha + \beta + \gamma)\nabla \nabla \cdot \mathbf{v} - \gamma\nabla \times \nabla \times \mathbf{v} + k\nabla \times \mathbf{u} - 2k\mathbf{v} = 0, \\ & \nabla \cdot \mathbf{u} = 0, \end{aligned} \quad (3.1)$$

in both an interior and exterior domain subject to the condition (1.6)

$$t_{ji}(\mathbf{u}, \mathbf{v})n_j|_S = T_{0i}, \quad m_{ji}(\mathbf{v})n_j|_S = M_{0i}. \quad (3.2)$$

For the exterior problem, we again assume the asymptotic conditions given by (2.13). Under this assumption and with the aid of (1.2), it can be shown that

$$\int_S U^* \tau(U) dS \equiv \int_S \{t_{ji}(\mathbf{u}, \mathbf{v})u_i + m_{ji}(\mathbf{v})v_i\}n_j dS = \int_{\Omega_{(e)}} H d\Omega, \quad (3.3)$$

$$\int_S U^* \tau(U) dS \equiv \int_S \{t_{ji}(\mathbf{u}, \mathbf{v})u_i + m_{ji}(\mathbf{v})v_i\}n_j dS = - \int_{\Omega_{(i)}} H d\Omega, \quad (3.4)$$

where

$$\begin{aligned} H = & (2\mu + k)d_{ij}d_{ji} + 2k(w_j - v_j)(w_j - v_j) \\ & + (\alpha v_{j,j}v_{i,i} + \beta v_{j,i}v_{i,j} + \gamma v_{i,j}v_{i,j}) \end{aligned} \quad (3.5)$$

with

$$d_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}).$$

From thermodynamic considerations [1]

$$\alpha v_{j,j}v_{i,i} + \beta v_{j,i}v_{i,j} + \gamma v_{i,j}v_{i,j} \geq 0$$

and hence $H \geq 0$.

The exterior problem. Let $(\mathbf{u}', \mathbf{v}')$ and $(\mathbf{u}'', \mathbf{v}'')$ be two solutions of (3.1) in $\Omega_{(e)}$ which satisfy the same boundary conditions (3.2) and the asymptotic conditions (2.13). If $\bar{\mathbf{u}} = \mathbf{u}' - \mathbf{u}''$ and $\bar{\mathbf{v}} = \mathbf{v}' - \mathbf{v}''$, it follows that $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ satisfies (3.1) in $\Omega_{(e)}$ and has the homogeneous boundary conditions

$$t_{ji}(\bar{\mathbf{u}}, \bar{\mathbf{v}})n_j|_S = 0, \quad m_{ji}(\bar{\mathbf{v}})n_j|_S = 0 \quad \text{or} \quad \tau(\bar{\mathbf{U}})|_S = 0.$$

Substitution into (3.3) shows that

$$\frac{1}{2}(\bar{u}_{i,j} + \bar{u}_{j,i}) = 0, \quad (3.6)$$

$$(\bar{w}_i - \bar{v}_i) = 0, \quad (3.7)$$

$$\alpha \bar{v}_{j,j} \bar{v}_{i,i} + \beta \bar{v}_{j,i} \bar{v}_{i,j} + \gamma \bar{v}_{i,j} \bar{v}_{i,j} = 0. \quad (3.8)$$

Equation (3.6) represents a right body type motion. Since $\bar{\mathbf{u}} \rightarrow 0$ as $r \rightarrow \infty$, it follows that $\bar{\mathbf{u}} \equiv 0$ in $\Omega_{(e)}$. From (3.7) $\bar{\mathbf{v}} \equiv 0$ and (3.8) is then automatically satisfied. The associated pressure $\bar{p}$ also vanishes in $\Omega_{(e)}$ as can be seen from (3.1) and (2.13). This establishes the uniqueness of solution. For future reference, we note that the set of equations (3.6)–(3.8) which is equivalent to

$$\int_S U^* \tau(U) dS = 0,$$

has six linearly independent solutions given by

$$\begin{aligned} U^1 &= \begin{bmatrix} \delta_{1k} \\ 0 \end{bmatrix}, \quad U^2 = \begin{bmatrix} \delta_{2k} \\ 0 \end{bmatrix}, \quad U^3 = \begin{bmatrix} \delta_{3k} \\ 0 \end{bmatrix}, \\ U^4 &= \begin{bmatrix} \epsilon_{1kj} x_j \\ -\delta_{1k} \end{bmatrix}, \quad U^5 = \begin{bmatrix} \epsilon_{2kj} x_j \\ -\delta_{2k} \end{bmatrix}, \quad U^6 = \begin{bmatrix} \epsilon_{3kj} x_j \\ -\delta_{3k} \end{bmatrix}. \end{aligned} \quad (3.9)$$

Recall from (2.7) that in matrix form the solution to (1.5) is given by

$$U(x) = V(x; \tau(U)) - W(x; U). \quad (3.10)$$

To establish the existence of solution to our problem, we seek a solution in the form of a single-layer potential $V(x; \psi)$. From (2.11), we get the singular integral equation

$$F(\xi) = -\frac{1}{2}\psi(\xi) + \int_S \{\Gamma_\xi A(\xi, \eta)\} \psi(\eta) dS_\eta, \quad (3.11)$$

where

$$F(\xi) = \tau(U)|_S = \begin{bmatrix} T_{0i} \\ M_{0i} \end{bmatrix}. \quad (3.12)$$

We need to investigate the corresponding homogeneous equation

$$0 = -\frac{1}{2}\psi(\xi) + \int_S \{\Gamma_\xi A(\xi, \eta)\} \psi(\eta) dS_\eta, \quad (3.13)$$

and its adjoint

$$0 = -\frac{1}{2}\phi(\xi) + \int_S B(\xi, \eta) \phi(\eta) dS_\eta. \quad (3.14)$$

It has been shown [2] that the integral equation (3.13) has a unique solution given by

$$\psi(\xi) = \begin{bmatrix} n_k(\xi) \\ 0 \end{bmatrix}. \quad (3.15)$$

Hence, its adjoint (3.14) will also have a unique solution, say $\Phi(\xi)$, and consequently (3.11) will have a solution provided

$$\int_S F^* \Phi dS = 0. \quad (3.16)$$

We have thus proved the following theorem:

THEOREM 6. A necessary condition for the first boundary-value problem (3.1)–(3.2) together with the supplementary conditions (2.13) to have a unique solution in $\Omega_{(e)}$ is that

$$\int_S F^* \Phi dS = 0$$

where F is given by (3.12) and Φ is the solution of (3.14).

The interior problem. For this problem we seek a solution in the form of a single-layer potential. From (2.10), the unknown density function ψ satisfies the equation

$$F(\xi) = \frac{1}{2}\psi(\xi) + \int_S \{\Gamma_\xi A(\xi, \eta)\} \psi(\eta) dS_\eta. \quad (3.17)$$

As before, we shall investigate the corresponding homogeneous equation

$$0 = \frac{1}{2}\psi(\xi) + \int_S \{\Gamma_\xi A(\xi, \eta)\} \psi(\eta) dS, \quad (3.18)$$

and its adjoint

$$0 = \frac{1}{2}\phi(\xi) + \int_S B(\xi, \eta)\phi(\eta) dS_\eta. \quad (3.19)$$

Consider (U^α, p^α) ($\alpha = 1, 2, \dots, 6$), where $p^\alpha = 0$ and U^α is given by (3.9). They satisfy (3.1) and furthermore, since $\tau(U^\alpha) = 0$, it follows from (3.10) that

$$U^\alpha(x) = -W(x; U^\alpha) \quad (3.20)$$

for $x \in \Omega_{(i)}$. Now taking the limit of (3.20) as $x \rightarrow \xi$, and utilizing (2.8), we obtain

$$\frac{1}{2}U^\alpha(\xi) + \int_S B(\xi, \eta)U^\alpha(\eta) dS_\eta = 0,$$

which shows that U^α is a solution of (3.19). This means that (3.19) has at least six linearly independent solutions and consequently its adjoint (3.18) will also possess at least six linearly independent solutions which we shall denote by Ψ^α . In fact, they form a complete system of linearly independent solutions for (3.18). For if this were not the case, then there would exist a solution, say Ψ , which does not depend linearly on Ψ^α , $\alpha = 1, 2, \dots, 6$. If $V = V(x; \Psi)$ is the corresponding single-layer potential then, from (2.10), $\tau(V)_{(i)} = 0$. Substitution into (3.4) shows that V represents a rigid body motion. The same is true of the potentials $V^\alpha = V(x; \Psi^\alpha)$ which constitute a complete system. Hence,

$$V = \sum_{\alpha=1}^6 C_\alpha V^\alpha$$

or equivalently

$$\int_S A(x, \eta) \left[\Psi(\eta) - \sum_{\alpha=1}^6 C_\alpha \Psi^\alpha(\eta) \right] dS_\eta = 0. \quad (3.21)$$

This represents a single-layer potential which vanishes in $\Omega_{(i)}$ and hence, from Theorem 1, vanishes in infinite space. Consequently

$$\Psi(x) - \sum_{\alpha=1}^6 C_\alpha \Psi^\alpha(x) = 0,$$

which contradicts our assumption. Thus, Ψ^α , $\alpha = 1, 2, \dots, 6$, form a complete system, linearly independent solutions of (3.18) and as a result its adjoint (3.19) will have precisely six linearly independent solutions given by U^α , $\alpha = 1, 2, \dots, 6$. Hence, the non-homogeneous equation (3.17) has a solution only if

$$\int_S F^* U^\alpha dS = 0, \quad \alpha = 1, 2, \dots, 6. \quad (3.22)$$

This establishes the conditional existence of a solution to the interior problem. The equation (3.22) is equivalent to

$$\int_S T_{0i} dS = 0, \quad (3.23)$$

and

$$\int_S (\epsilon_{ijk} x_j T_{0k} + M_{0i}) dS = 0, \quad (3.24)$$

which expresses the vanishing of the hydrodynamic force and torque respectively, on the surface S . It should be noted that with the aid of the first equation of (1.1) (with $\mathbf{f} = 0$), we get

$$0 = \int_{\Omega_{(i)}} t_{ji,j} d\Omega = \int_S t_{ji} n_j dS = \int_S T_{0i} dS,$$

while with the aid of the second equation (with $\mathbf{l} = 0$) as well as the first, we get

$$\begin{aligned} 0 &= \int_{\Omega_{(i)}} (m_{ji,j} + \epsilon_{ijk} t_{jk}) d\Omega = \int_{\Omega_{(i)}} \{m_{ji,j} + \epsilon_{ijk} (x_j t_{lk,l})\} d\Omega \\ &= \int_S (m_{ji} n_j + \epsilon_{ijk} x_j t_{lk} n_l) dS \\ &= \int_S (M_{0i} + \epsilon_{ijk} x_j T_{0k}) dS. \end{aligned}$$

These conditions (3.23), and (3.24) are therefore necessary and sufficient for a solution to exist. As regards uniqueness, we see from (3.4) that for the homogeneous boundary condition $\tau(U)|_S = 0$, $\mathbf{u}$ is determined up to an additive rigid body motion. These results may be summarized as follows:

THEOREM 7. The first boundary-value problem (3.1)–(3.2) has a solution in an interior domain $\Omega_{(i)}$ for continuous fields $\mathbf{T}_0$ and $\mathbf{M}_0$ on S , which satisfy the conditions (3.23)–(3.24). Moreover, this solution is unique up to an additive rigid body motion.

4. Second boundary-value problem. In this problem we again examine the system (1.5)

$$\begin{aligned} -(\mu + k) \nabla \times \nabla \times \mathbf{u} + k \nabla \times \mathbf{v} - \nabla p &= 0, \\ (\alpha + \beta + \gamma) \nabla \nabla \cdot \mathbf{v} - \gamma \nabla \times \nabla \times \mathbf{v} + k \nabla \times \mathbf{u} - 2k\mathbf{v} &= 0, \\ \nabla \cdot \mathbf{u} &= 0, \end{aligned} \quad (4.1)$$

in both an interior and exterior domain but now subject to the mixed boundary conditions (1.7)

$$[t_{ji}(\mathbf{u}, \mathbf{v}) n_j + h u_i] |_S = a_i, \quad [m_{ji}(\mathbf{v}) n_j + h v_i] |_S = b_i \quad (4.2)$$

where h is a continuous scalar function.

The interior problem. In the case of homogeneous boundary conditions

$$[t_{ji}(\mathbf{u}, \mathbf{v}) n_j + h u_i] |_S = 0, \quad [m_{ji}(\mathbf{v}) n_j + h v_i] |_S = 0$$

we get on utilizing (3.4),

$$\int_{\Omega_{(i)}} H d\Omega = \int_S h U * U dS \equiv \int_S h (\mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v}) dS. \quad (4.3)$$

Assuming that h is a negative function, it follows that on S , $\mathbf{u} = 0$, and $\mathbf{v} = 0$ since $H \geq 0$.

The uniqueness of solution then follows from Theorem 5. The condition (2.14) is satisfied since

$$0 = \int_{\Omega_{(i)}} \nabla \cdot \mathbf{u} \, d\Omega = \int_S \mathbf{u} \cdot \mathbf{n} \, dS.$$

The solution to this problem is sought in the form of a single-layer potential. From (2.10) and (2.5), we get

$$\tau(U) + hU = \frac{1}{2}\psi(\xi) + \int_S [\Gamma_\xi A(\xi, \eta) + A(\xi, \eta)] \psi(\eta) \, dS_\eta.$$

Hence,

$$D = \frac{1}{2}\psi(\xi) + \int_S [\Gamma_\xi A(\xi, \eta) + A(\xi, \eta)] \psi(\eta) \, dS_\eta \quad (4.4)$$

where

$$D_i = \begin{bmatrix} a_i \\ b_i \end{bmatrix}.$$

The corresponding homogeneous equation

$$0 = \frac{1}{2}\psi(\xi) + \int_S [\Gamma_\xi A(\xi, \eta) + A(\xi, \eta)] \psi(\eta) \, dS_\eta, \quad (4.5)$$

admits only a trivial solution. The proof of this is by contradiction. Assume Ψ is a nontrivial solution and let

$$\bar{V} = V(x; \Psi)$$

be the corresponding single-layer potential. It satisfies the boundary condition

$$[\tau(\bar{V}) + h\bar{V}]|_S = 0, \quad h < 0,$$

and hence (4.3). That is,

$$\int_{\Omega_{(i)}} H \, d\Omega = \int_S hV * V \, dS.$$

As we have seen before, this implies that $\bar{V} = 0$ on S and so from Theorem 1, $\bar{V} = 0$ everywhere. Consequently, $\Psi = 0$ in contradiction to our assumption. Thus, the homogeneous equation (4.5) admits only trivial solution. Hence,

THEOREM 8. The second boundary-value problem (4.1)–(4.2) has a unique solution in an interior domain $\Omega_{(i)}$ for any negative continuous function h on S .

The exterior problem. Again we seek a solution in the form of a single-layer potential and use the same analysis as above. In place of (4.3) and (4.4), we get respectively

$$\int_{\Omega_{(e)}} H \, d\Omega = \int_S hU * U \, dS = - \int_S h(\mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v}) \, dS, \quad (4.6)$$

and

$$D = -\frac{1}{2}\psi(\xi) + \int_S [\Gamma_\xi A(\xi, \eta) + A(\xi, \eta)] \psi(\eta) \, dS_\eta. \quad (4.7)$$

Taking h to be a positive function and repeating the identical analysis used for the interior problem above, we get

THEOREM 9. The second boundary-value problem (4.1)–(4.2) has a unique solution in an exterior domain $\Omega_{(e)}$ for any positive continuous function h on S .

APPENDIX. Here it will be shown that the Fredholm alternative holds for the systems (2.8)–(2.11) of integral equations on the boundary S which will be assumed to be a closed manifold.

From [2], we have the following expansions for the stresses:

$$\begin{aligned}
 {}^F t_{ji}^k &= \frac{\kappa \delta_{jk}}{4\pi(2\mu + \kappa)} \frac{(x_i - y_i)}{r^3} - \frac{\kappa \delta_{ik}}{4\pi(2\mu + \kappa)} \frac{(x_j - y_j)}{r^3} - \frac{3(x_i - y_i)(x_j - y_j)(x_k - y_k)}{r^5} \\
 &\quad + O(r^{n-1}) \\
 {}^F m_{ji}^k &= O(r^{n-1}), \\
 {}^L t_{ji}^k &= O(r^{n-1}) \\
 {}^L m_{ji}^k &= -\frac{(\beta + \gamma)(\beta - \alpha) - 2\alpha^2}{8\pi\gamma(\alpha + \beta + \gamma)} \delta_{jk} \frac{(x_i - y_i)}{r^3} + \frac{(\alpha + \beta)(\beta - \gamma) - 2\gamma^2}{8\pi\gamma(\alpha + \beta + \gamma)} \delta_{ik} \frac{(x_j - y_j)}{r^3} \\
 &\quad + \frac{(\beta + \gamma)(\beta - \alpha) - 2\alpha^2}{8\pi\gamma(\alpha + \beta + \gamma)} \delta_{ij} \frac{(x_k - y_k)}{r^3} - \frac{3(\alpha + \beta)(\beta + \gamma)}{8\pi\gamma(\alpha + \beta + \gamma)} (x_i - y_i)(x_j - y_j)(x_k - y_k) \\
 &\quad + O(r^{n-1}).
 \end{aligned} \tag{1}$$

Mikhlin's approach [4] shall now be utilized to generate symbols associated with the system of integral equations under consideration. If s is the index of the Lyapunov surface S , then $(x_j - y_j)n_j = O(r^{1+s})$. Hence, from (1)

$$\begin{aligned}
 {}^F t_{ji}^k n_j &= a r_i n_j + O(r^{s-2}), \\
 {}^F m_{ji}^k n_j &= O(r^{s-2}), \\
 {}^L t_{ji}^k n_j &= O(r^{s-2}), \\
 {}^L m_{ji}^k n_j &= b r_k n_i - b r_i n_k + O(r^{s-2}),
 \end{aligned} \tag{2}$$

where

$$a = \frac{\kappa}{4\pi(2\mu + \kappa)}, \quad b = \frac{(\beta + \gamma)(\beta - \alpha) - 2\alpha^2}{8\pi\gamma(\alpha + \beta + \gamma)}$$

and $r_i = x_i - y_i$. Each of the equations (2.8)–(2.11), actually represents two systems each of which has three equations. Let us consider (2.8) for example. If

$$W(\xi; \phi)_{(i)} = \begin{bmatrix} G_i^1 \\ G_i^2 \end{bmatrix},$$

then from (2.8)

$$2G_i^1(\xi) = -\phi_i^1(\xi) + 2 \int_S ({}_F t_{ji}^k \phi_k^1 + {}_F m_{ji}^k \phi_k^2) n_j dS_\eta, \quad (3)$$

$$2G_i^2(\xi) = -\phi_i^2(\xi) + 2 \int_S ({}_L t_{ji}^k \phi_k^1 + {}_L m_{ji}^k \phi_k^2) n_j dS_\eta. \quad (4)$$

Using the identical method advanced by Mikhlin [4], we introduce local coordinates at each point $\xi \in S$, directing the ξ_3 -axis along the normal $\mathbf{n}$ to S and the axes ξ_1, ξ_2 in a tangential plane to S . Hence $n_1 = n_2 = 0, n_3 = 1$. With the aid of (2), we now write (3) and (4) in the local coordinates getting, respectively,

$$\begin{aligned} 2G_1^1(\xi) &= -\phi_1^1(\xi) + 2 \int_S \frac{a\phi_3^1 \xi_1}{r^3} dS_\eta + I_1, \\ 2G_2^1(\xi) &= -\phi_2^1(\xi) + 2 \int_S \frac{a\phi_3^1 \xi_2}{r^3} dS_\eta + I_2, \\ 2G_3^1(\xi) &= -\phi_3^1(\xi) + 2 \int_S \frac{a\phi_3^1 \xi_3}{r^3} dS_\eta + I_3, \end{aligned} \quad (5)$$

and

$$\begin{aligned} 2G_1^2(\xi) &= -\phi_1^2(\xi) - 2 \int_S \frac{b\phi_3^2 \xi_1}{r^3} dS_\eta + I_4, \\ 2G_2^2(\xi) &= -\phi_2^2(\xi) - 2 \int_S \frac{b\phi_3^2 \xi_2}{r^3} dS_\eta + I_5, \\ 2G_3^2(\xi) &= -\phi_3^2(\xi) + 2 \int_S \frac{b}{r^3} (\xi_1 \phi_1^2 + \xi_2 \phi_2^2) dS_\eta + I_6, \end{aligned} \quad (6)$$

where $I_1, I_2, \dots, I_6$ are integrals with weak singularities. The system (5)–(6) is similar to one obtained by Mikhlin [4] and it can be verified [4] that its symbolic matrix M is given by

$$M = \begin{pmatrix} -1 & 0 & ic \cos \theta & 0 & 0 & 0 \\ 0 & -1 & ic \sin \theta & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -id \cos \theta \\ 0 & 0 & 0 & 0 & -1 & -id \sin \theta \\ 0 & 0 & 0 & id \cos \theta & id \sin \theta & -1 \end{pmatrix} \quad (7)$$

where

$$c = 4a\pi = \frac{\kappa}{2\mu + \kappa} \quad \text{and} \quad d = 4b\pi = \frac{(\beta + \gamma)(\beta - \alpha) - 2\alpha^2}{2\gamma(\alpha + \beta + \gamma)}.$$

Garding's inequality [5]

$$\operatorname{Re}(\Theta Mu, u)_{L^2} > \rho \|u\|_{L^2}^2 - \operatorname{Re}(Cu, u)_{L^2} \quad (8)$$

implies Fredholm alternative. Here $\rho > 0$ is a constant independent of u , (Cu, u) is a compact bilinear form and Θ is a 6×6 matrix of functions. Furthermore, positive definiteness of θM implies Garding's inequality. Hence, it suffices to prove that θM is positive definite. That is, we need to show that there exists a k such that for all $\sigma \in C^6$,

$$\operatorname{Re}(\sigma^T \theta M \bar{\sigma}) \geq k |\sigma|^2. \quad (9)$$

Take

$$\Theta = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

and let $\sigma^T = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6)$. It can be easily verified that

$$\begin{aligned} \operatorname{Re}(\sigma^T \Theta M \bar{\sigma}) &= |\sigma_1|^2 + |\sigma_2|^2 + \varepsilon |\sigma_3|^2 \\ &\quad - \operatorname{Re}(i\sigma_1 \bar{\sigma}_3 \cos \theta + i\sigma_2 \bar{\sigma}_3 \sin \theta) \\ &\quad + |\sigma_4|^2 - \operatorname{Re}(i\sigma_6 \bar{\sigma}_4 \cos \theta) \\ &\quad + |\sigma_5|^2 - \operatorname{Re}(i\sigma_6 \bar{\sigma}_5 \sin \theta) \\ &\quad + |\sigma_6|^2 + \operatorname{Re}(i\sigma_4 \bar{\sigma}_6 \cos \theta + i\sigma_5 \bar{\sigma}_6 \sin \theta) \\ &\geq |\sigma_1|^2 + |\sigma_2|^2 + \varepsilon |\sigma_3|^2 - k_1 |\sigma_1 \sigma_3| - k_1 |\sigma_2 \sigma_3| \\ &\quad + |\sigma_4|^2 + \dots \end{aligned} \quad (10)$$

Now

$$\frac{|\sigma_1|^2}{3\delta} + \frac{\delta |\sigma_3|^2}{2} \leq |\sigma_1| |\sigma_3|.$$

Using this in (10) gives

$$\begin{aligned} \operatorname{Re}(\sigma^T \Theta M \bar{\sigma}) &\geq |\sigma_1|^2 + |\sigma_2|^2 + \varepsilon |\sigma_3|^2 - k_1 \delta |\sigma_3|^2 \\ &\quad - \frac{k_1}{2\delta} |\sigma_1|^2 - \frac{k_1}{2\delta} |\sigma_2|^2 + |\sigma_4|^2 + \dots \end{aligned} \quad (11)$$

where δ and ε are arbitrary. First we choose δ such that $k_1 \delta = \frac{1}{2} \varepsilon$ and then take $\varepsilon = 2k_1^2$. Inserting these in turn into (11), we obtain

$$\operatorname{Re}(\sigma^T \Theta M \bar{\sigma}) \geq k_2 (|\sigma_1|^2 + |\sigma_2|^2 + |\sigma_3|^2) + |\sigma_4|^2 + \dots \quad (12)$$

where $k_2 = \min(1, \varepsilon)$. Similarly, by examining the other combinations in (10), it can be seen that for some k ,

$$\operatorname{Re}(\sigma^T \Theta M \bar{\sigma}) \geq k (|\sigma_1|^2 + |\sigma_2|^2 + \dots + |\sigma_6|^2) \equiv k |\sigma|^2.$$

Hence the result. That is, the Fredholm alternative holds for (2.8). In a similar manner, we can show that the same is also true for systems (2.9)–(2.11).

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