ON LINEARLY COUPLED RELAXATION OSCILLATIONS*

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Abstract. We study the dynamical behavior of a pair of linearly coupled relaxation oscillators. In such systems vastly different time scales play a crucial rôle, and solutions may be viewed as consisting of portions of slow drift linked by rapid jumps. This feature enables us to reduce the analysis from four dimensional phase space to that of a two dimensional system with discontinuous but well determined behavior at certain points on the phase plane. We determine the existence and stability of periodic motions for identical oscillators and oscillators with an uncoupled frequency ratio of $1:\omega$. We give additional details on nonperiodic motions for the special case of $\omega = 2$.

1. Introduction. Many physical and biological systems exhibit oscillations in which vastly different time scales are evident. As early as 1928, van der Pol and van den Mark [1928] listed some seventeen examples ranging from tetrode multivibrators through economic crises, to the human heart. The oscillator which now bears van der Pol's name is a simple example of such a system:

$$\epsilon \ddot{x} + \phi(x)\dot{x} + x = 0. \quad (1.1)$$

Here ϵ is a small parameter and $(\dot{})$ denotes differentiation with respect to the independent variable, t (time). In applications x is a measure of some physical quantity, such as electrical current.

If $\phi(x)$ is an even function and $\phi(x) > 0$ for $|x| > 1$, $\phi(x) < 0$ for $|x| > 1$, $\phi(x) < 0$ for $|x| < 1$ and $\phi(x)$ remains bounded away from zero as $|x| \rightarrow \infty$, then it is well known that equation (1.1) possesses a unique attracting limit cycle surrounding an unstable fixed point, a source, at $x = \dot{x} = 0$ (Stoker [1950], [1980]). The global behavior is obtained by the use of phase plane techniques, and the Poincaré-Bendixson theorem in particular. Cartwright and Littlewood [1945], Levinson [1949] and more recently Levi [1981] studied the behavior of such oscillators in the presence of periodic forcing terms $p(t) = p(t + T)$

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of relatively high frequency ($T \ll 1$). Here the behavior is much more complex and infinite sets of unstable periodic orbits can coexist with up to two stable periodic orbits of different periods $(2k+1)T$, $(2k-1)T$. Originally van der Pol [1927] had proposed a quadratic form $\phi(x) = (x^2 - 1)$ for the nonlinear term, but Levinson and Levi were able to simplify the analysis without significant qualitative changes by taking the piecewise constant function

$$\phi(x) = \begin{cases} 1, & |x| > 1 \\ -1, & |x| < 1 \end{cases}. \quad (1.2)$$

(In a moment we shall see how a simple integral transformation yields a differential equation whose right-hand side is C^0 -Lipshitz even if ϕ takes the form (1.2).) Additional studies, both numerical and analytical, of forced relaxation oscillations can be found in Flaherty and Hoppensteadt [1978], Grasman, Velig and Willems [1976], Grasman and Jansen [1979] Grasman, Nijmeijer and Velig [1982] and the references therein.

In this paper we shall study a pair of coupled oscillators of the type (1.1) governed by the following system of equations

$$\epsilon \ddot{x} + \phi(x)\dot{x} + x = \alpha(\dot{y} - \dot{x}), \quad \epsilon \ddot{y} + \phi(y)\dot{y} + \omega y = \alpha(\dot{x} - \dot{y}), \quad (1.3)$$

where $\phi(s)$ is the function defined in (1.2). Many of our results can be obtained for more general functions $\phi(s)$, but the piecewise constant case is simplest. Transforming to the Liénard variables:

$$u_1 = \epsilon \dot{x} + \int_0^x \phi(s) ds + \alpha(x - y), \quad u_2 = \epsilon \dot{y} + \int_0^y \phi(s) ds + \alpha(y - x), \quad (1.4)$$

we can write (1.3) as the four dimensional system

$$\begin{aligned} \dot{u}_1 &= -x, & \dot{u}_2 &= -\omega y, \\ \epsilon \dot{x} &= u_1 - \Phi(x) + \alpha(y - x), & \epsilon \dot{y} &= u_2 - \Phi(y) + \alpha(x - y). \end{aligned} \quad (1.5)$$

We note that $\Phi(w) = \int_0^w \phi(s) ds$ and

$$\Phi(w) = \begin{cases} 2 + w; & w < -1 \\ -w; & |w| \leq 1 \\ -2 + w; & w > 1 \end{cases} \quad (1.6)$$

is a Lipschitz function and hence, for $\epsilon \neq 0$, solutions of (1.5) exist (at least locally) and are unique (Coddington and Levinson [1955]).

In equations (1.3) and (1.5) α is the *coupling coefficient*. For $\alpha = 0$ the uncoupled systems have stable limit cycles whose periods approach $T_x = 2 \ln 3$ and $T_y = (2/\omega) \ln 3$ respectively (Stoker [1950]); for this reason we call ω the *frequency ratio*. Both α and ω are taken to be order one quantities, in contrast to the small parameter ϵ . Without loss of generality we take $\omega \geq 1$.

In spite of the relatively large literature on the single oscillator, very little has been done on coupled, mutually interacting, relaxation systems. Gollub, Brunner and Danly [1978] reported interesting experiments on a system of two electrical oscillators in which apparently chaotic and periodic motions were observed and their observations prompted

the present study. Although we have not been able to study the apparent ‘strange attractor’ found by these authors, we do find chaotic solutions of a certain type in oscillators with unequal natural frequencies.

This paper is organized as follows. After a brief review of the single oscillator, in Sec. 3 we develop our method of analysis, using a ‘fast equation’ to determine the jump dynamics, and a ‘slow equation’ to determine the drift dynamics. We see that the frequency ratio ω does not enter the fast equation, but that α plays an important rôle in both systems. In section 4 we perform a complete analysis of the case of identical oscillators ($\omega = 1$). We find that, for all $\alpha > 0$, there is a unique stable in-phase periodic motion, with $x(t) \equiv y(t)$ and $u_1(t) \equiv u_2(t)$, in addition to an unstable out of phase motion. In Sec. 5 we turn to the unequal frequency case. Here the analysis is more awkward, and we only study the case $\omega = 2$ in detail. We find that, for sufficiently large α , a unique stable periodic motion exists, while for small α coexistence of complicated sets of periodic motions can occur. All this analysis is formal, in that it is based on the singular limit $\varepsilon \rightarrow 0^+$. In Sec. 6 we briefly indicate how the techniques of nonstandard analysis (Robinson [1974]), can be used to prove that the limiting behavior persists for finite $\varepsilon \in (0, \varepsilon_0]$. The use of nonstandard methods for such singular perturbation problems was first suggested in the work of Reeb [1977] and has been pursued with some success by Benoit et. al. [1980]. More details on the nonstandard proofs will be published in a second paper (Bélair [1983b]), or can be found in the thesis of Bélair [1983a].

2. A single oscillator; Formal analysis. Here we briefly review the single oscillator, mainly as a means of developing our methods of analysis. Using the transformation

$$u = \varepsilon \dot{x} + \int_0^x \phi(s) ds = \varepsilon \dot{x} + \Phi(x) \quad (2.1)$$

Eq. (1.1) becomes

$$\dot{u} = -x, \quad \varepsilon \dot{x} = u - \Phi(x). \quad (2.2)$$

For $\varepsilon = 0$, (2.2) becomes the *constrained differential equation* (Takens [1976])

$$\dot{u} = -x, \quad u = \Phi(x). \quad (2.3)$$

Now (2.3) will only have solutions if we take initial conditions (u_0, x_0) such that $u_0 = \Phi(x_0)$. In such a situation the behavior is governed by the equation

$$\dot{x} = -x/\phi(x), \quad (2.4)$$

obtained by differentiating the constraint $u = \Phi(x)$ with respect to t and substituting in $\dot{u} = -x$ (alternatively, set $\varepsilon = 0$ in (1.1)). Solutions of (2.4) are well defined as long as $\phi(x)$ is defined and $\phi(x) \neq 0$. Thus, in our case, in which $\phi(x)$ is given by (1.2), solutions exist everywhere except at $x = \pm 1$. Equation (2.4) will be referred to as the *slow* or *drift equation*.

From (2.2) it is easy to see that, except in an ε -neighborhood of the set

$$\mathcal{C} = \{(x, u) | u = \Phi(x)\}, \quad (2.5)$$

the vectorfield is almost horizontal, since

$$\frac{du}{dx} = \frac{-\epsilon x}{u - \Phi(x)} = \mathcal{O}(\epsilon). \quad (2.6)$$

We call this set $\mathcal{C}$ the *slow manifold* (cf. Zeeman [1973]). To determine the behavior of solutions away from $\mathcal{C}$, we rescale time with $\tau = t/\epsilon$, so that (2.2) becomes

$$u' = -\epsilon x, \quad x' = u - \Phi(x), \quad (2.7)$$

where $(\cdot)'$ denotes $d/d\tau(\cdot)$. Letting $\epsilon \rightarrow 0$ in (2.7) we obtain the *fast equation*

$$u' = 0, \quad x' = u - \Phi(x). \quad (2.8)$$

The global behavior of the singular ($\epsilon = 0$) system is then determined by successive solutions of the fast and slow equations (2.8) and (2.4).

To illustrate the method, we take initial conditions $(x_0, u_0) \notin \mathcal{C}$. Eq. (2.8) may be partially solved immediately to give $u(t) \equiv u_0$, and we simply have to solve the scalar equation

$$x' = u_0 - \Phi(x), \quad (2.9)$$

in which u_0 plays the rôle of a parameter. We can represent the one parameter family of systems (2.9) in the form of a bifurcation diagram in the (u, x) plane as in Fig. 1(a). Taking the function $\Phi(x)$ defined in equation (1.6), we see that, for $u_0 > 1$ and $u_0 < -1$ (2.9) has a unique attracting fixed point at $x = u_0 + 2$ or $x = u_0 - 2$ respectively, while for $|u_0| < 1$ there are fixed points at $x = u_0 + 2$, $-u_0$ and $u_0 - 2$, the outer ones being sinks and the inner a source. At $u_0 = \pm 1$ there are degenerate fixed points, the 'smooth' analogues of which would be saddle-nodes (Guckenheimer [1980]). Here it is easy to see that all solutions starting off $\mathcal{C}$ approach a stable equilibrium on one of the two outer branches of $\mathcal{C}$ as $\tau \rightarrow \infty$. In the original time scale t , therefore, solutions jump straight to a *landing point* $(\Phi^{-1}(u_0), u_0)$ on $\mathcal{C}$ determined uniquely by the u -component of their initial condition. The family $\mathcal{F}$ of lines $u = u_0 = \text{constant}$ on which solutions of (2.8) lie is called the *fast foliation* (Zeeman [1973]).

On the outer branches, we have $\phi(x) = \Phi'(x) = \pm 1$, and thus the slow equation (2.4) is simply the linear system

$$\dot{x} = -x, \quad x(0) = x_0 = \Phi^{-1}(u_0), \quad (2.10)$$

with solution

$$x(t) = x_0 e^{-t}. \quad (2.11)$$

Once the solution starts drifting on $\mathcal{C}$, then u begins to change according to the constraint $u = \Phi(x)$ of (2.3), giving

$$u(t) = \Phi(x_0 e^{-t}) = \pm 2 + x_0 e^{-t}. \quad (2.12)$$

The solution is governed by (2.11–12) until it reaches a *jump point* at which the vector field of (2.4) is undefined, i.e. until $(x, u) = (\pm, +1)$. To continue solutions through such a point, we shall assume that $u(t)$ continues to change (slowly) in the same fashion as it did before reaching the point; i.e. at $(x, u) = (1, -1)$ u is decreasing, while at $(x, u) = (-1, 1)$ u is increasing. This may be justified by examining the vectorfield of (2.7) for

(a) The family of fast equations: $-$ = stable fixed points; $---$ X = unstable fixed points.

(b) The flow of the slow equation. Jump points shown as $\odot$, fixed points as $\bullet$.

(c) The global flow, showing the limit cycle.

Fig. 1. The dynamics of a single oscillator.

$\varepsilon \neq 0$. Thus, at the jump point the control of solutions reverts to the fast equation, which has just undergone a bifurcation. The solution then jumps from $(1, -1)$ or $(-1, 1)$ to the corresponding landing point at $(-3, -1)$ or $(3, 1)$ respectively. Continuing in this fashion we see that, after an initial jump and drift, all solutions are attracted to a unique limit cycle Γ given by the parallelogram with vertices at $(3, 1)$, $(1, -1)$, $(-3, -1)$ and $(-1, 1)$ (cf. Fig. 1(c)).

In this discussion we have neglected solutions starting on the middle branch of $\mathcal{C}$, at point $(-u_0, u_0)$, $|u_0| < 1$. However, an examination of the vector field (cf. Fig. 1) shows

that such orbits move ‘slowly’ along this branch until they reach one of the two jump points $(\pm 1, \mp 1)$, after which they circulate on Γ . Of course, solutions based at $(x_0, u_0) = (0, 0)$ remain there, since it is an (unstable) fixed point.

The period of this ‘singular’ oscillation can easily be computed from the time spent in the two drifts from $x = 3$ to $x = 1$ and $x = -3$ to $x = -1$, since the jumps occur instantaneously in t . We therefore have $T = 2t_d$, where t_d is obtained from (2.11):

$$x(t_d) = 1 = 3e^{-t_d} \Rightarrow t_d = \ln 3,$$

so that

$$T = 2 \ln 3, \quad (2.13)$$

as claimed in Sec. 1.

The phase plane and asymptotic analyses of Stoker [1950] and Haag [1943] show that these formal results constitute the zeroth order approximation to the solution of (2.2) for small ϵ . In Bélaïr [1983] this formal analysis is justified by matching fast and slow solutions using techniques of nonstandard analysis, which we outline in Sec. 6.

Before tackling the coupled oscillator system of Eq. (1.5), we summarize the method we are to use in a more general context. Given the singularly perturbed system

$$\begin{cases} \dot{\mathbf{u}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \\ \epsilon \dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, \mathbf{u}) \end{cases}, \quad (\mathbf{x}, \mathbf{u}) \in \mathbf{R}^n \times \mathbf{R}^m, \quad (2.14)$$

we define the *fast system*

$$u' = 0, \quad \mathbf{x}' = \mathbf{g}(\mathbf{x}, \mathbf{u}) \quad (2.15)$$

and the *reduced fast system*

$$\mathbf{x}' = \mathbf{g}(\mathbf{x}, \mathbf{u}_0) \quad (\mathbf{u}(t) \equiv \mathbf{u}_0). \quad (2.16)$$

The m -parameter family of equations (2.16) determines the *jump dynamics* on the n -dimensional slices $\mathbf{u} = \text{constant}$. If we assume that $\mathbf{g}(\mathbf{x}, \mathbf{u}) = 0$, obtained by setting $\epsilon = 0$ in (2.4) can be solved to yield

$$\mathbf{u} = \mathbf{h}(\mathbf{x}) \quad (2.17)$$

where $\mathbf{h}$ is a map from $\mathbf{R}^n$ to $\mathbf{R}^m$, then the flow on the *slow manifold* $\mathcal{C} = \{(\mathbf{x}, \mathbf{u}) | \mathbf{u} = \mathbf{h}(\mathbf{x})\}$ is obtained from the slow system:

$$\dot{\mathbf{u}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) = \mathbf{Dh}(\mathbf{x})\dot{\mathbf{x}}$$

or

$$\dot{\mathbf{x}} = [\mathbf{Dh}(\mathbf{x})]^{-1} \mathbf{f}(\mathbf{x}, \mathbf{h}(\mathbf{x})). \quad (2.18)$$

The *jump set* for this equation is the set of points $J = \{\mathbf{x} | \mathbf{Dh}(\mathbf{x}) \text{ is singular or undefined}\}$, and the *landing set* is obtained by computing the $\mathbf{u}$ component of points in $J \cap \mathcal{C}$ from (2.17) and solving the global bifurcation problem for the fast system (2.16) at those points: the landing point is the ω limit set for solutions leaving the bifurcating fixed point of (2.16). A solution of (2.14) is then approximated, to zeroth order in ϵ , by a succession of a *drift* controlled by the slow system followed by a *jump* controlled by the fast system, ad infinitum (or ad nauseam).

3. Coupled oscillators: Drifts and jumps. With this sketch of the method behind us, we now turn to the coupled system (1.5). In this case the fast equations are given by

$$\dot{u}_1 = 0, \quad \dot{u}_2 = 0, \quad (3.1)$$

$$x' = u_1 - \Phi(x) + \alpha(y - x), \quad y' = u_2 - \Phi(y) + \alpha(x - y),$$

obtained by letting $\tau = t/\varepsilon$ in (1.5) and taking the limit $\varepsilon \rightarrow 0$. Note that the frequency ratio ω vanishes in this limit and hence the *jump dynamics are independent of frequency*. As in the single oscillator, the u_i components of solutions of (3.1) do not change, and thus we can reduce our study to that of the two parameter family

$$x' = u_{10} - \Phi(x) + \alpha(y - x), \quad y' = u_{20} - \Phi(y) + \alpha(x - y) \quad (3.2)$$

The zero set of this equation is the two dimensional *slow manifold*

$$\mathcal{C} = \{(u_1, u_2, x, y) | u_1 = \Phi(x) + \alpha(x - y), u_2 = \Phi(y) + \alpha(y - x)\}$$

embedded in the full four dimensional phase space, and the fast foliation is the family of planes $u_1 = u_{10}, u_2 = u_{20}$. The phase plane of (3.2) for each pair (u_{10}, u_{20}) represents the dynamics on one of these planes.

Our first task, then, is to find the equilibria of (3.2) and determine their stability types. We note that the linearized equation

$$\begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{bmatrix} -\phi(\bar{x}) - \alpha & \alpha \\ \alpha & -\phi(\bar{y}) - \alpha \end{bmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad (3.3)$$

will determine stability at an equilibrium $(\bar{x}, \bar{y})$ given by a root of the pair of equations:

$$u_{10} - \Phi(x) + \alpha(y - x) = 0, \quad u_{20} - \Phi(y) + \alpha(x - y) = 0. \quad (3.4)$$

Since $\phi(s) = \Phi'(s)$ is piecewise constant, taking the value $+1$ for $|s| > 1$ and -1 for $|s| < 1$, analysis of (3.3) is easy. The eigenvalues are given by the roots of

$$\lambda^2 + (2\alpha + \phi(x) + \phi(y))\lambda + \alpha(\phi(x) + \phi(y)) + \phi(x)\phi(y) = 0$$

and the corresponding fixed point is a sink if $\phi(x) = \phi(y) = 1$, and a saddle or a source (depending on the magnitude of α) if $\phi(x)$ and/or $\phi(y) = -1$. The stable (= attracting) pieces of $\mathcal{C}$ and therefore those for which $|x|, |y| > 1$.

The fixed points are themselves most easily located graphically, by intersecting the two piecewise linear 'curves' given by (3.4). Depending upon the values of α, u_{10} and u_{20} , from one to nine equilibria can exist, at most four of them being stable. In Fig. 2 we show the case for $u_{10} = u_{20} = 0$ and $0 \leq \alpha < \frac{1}{2}$, for which nine symmetrically placed equilibria exist: a source at $(0, 0)$, saddles at $(\pm(2\alpha - 2), \pm 2\alpha)$, $(\pm 2\alpha, \pm(2\alpha - 2))$ and sinks at $(\pm 2, \pm 2)$, $(\pm 2/(1 + 2\alpha), \pm 2/(1 + 2\alpha))$. For $\alpha > \frac{1}{2}$ there are only three fixed points: a saddle at $(0, 0)$ and sinks at $(\pm 2, \pm 2)$. For arbitrary u_{10} and α , the fixed points are most easily found by rewriting (3.4) as

$$y = \frac{1}{\alpha}(\Phi(x) + \alpha x - u_{10}), \quad x = \frac{1}{\alpha}(\Phi(y) + \alpha y - u_{20}) \quad (3.5)$$

Fig. 2. The flow of the fast equations (3.2) for $u_{10} = u_{20} = 0$.

and seeking intersection points of these two piecewise linear sets. We note that the 'parameters' u_{i0} simply act to translate the sets vertically and horizontally. As this occurs, some of the fixed points can disappear by coalescence in pairs, triplets or quadruplets, but such coalescences can only occur on the lines $|x| = 1$, $|y| = 1$. In a moment we shall see that these are the *jump lines* for the system.

In Fig. 2 we also show the global structure of the flow of (3.2), which is easily computed since the system is linear in each of the nine regions $|x|, |y| \leq 1$. In particular we see that no closed orbits can exist, since (3.2) is a gradient dynamical system (Hirsch and Smale [1974]), which can be written as

$$\begin{aligned} \dot{x} &= -\partial V / \partial x, \quad \dot{y} = -\partial V / \partial y; \\ V &= -u_{10}x - u_{20}y + \int_0^x \Phi(s) ds + \int_0^y \Phi(s) ds + \frac{\alpha}{2}(x - y)^2, \end{aligned} \quad (3.6)$$

and such systems cannot possess closed orbits or homoclinic loops. Thus, apart from a set of initial conditions $(u_{10}, u_{20}, x_0, y_0)$ of measure zero, lying on the stable separatrices of the saddle points of (3.2), all solutions of (1.5) commence with a jump to a stable equilibrium in one of the four quadrants $|x|, |y| > 1$.

Once at such an equilibrium, our solution will drift along the stable manifold. To obtain the slow equation we return to (1.5), set $\varepsilon = 0$ and differentiate the constraint to obtain

$$\begin{aligned}\dot{u}_1 &= \phi(x)\dot{x} - \alpha(\dot{y} - \dot{x}) & (= -x), \\ \dot{u}_2 &= \phi(y)\dot{y} - \alpha(\dot{x} - \dot{y}) & (= -\omega y).\end{aligned}\quad (3.7)$$

Solving (3.7) for $\dot{x}$ and $\dot{y}$, we obtain the slow equation

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{\Delta} \begin{bmatrix} -(\phi(y) + \alpha) & -\alpha\omega \\ -\alpha & -\omega(\phi(x) + \alpha) \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix};$$

$$\Delta = (\phi(x) + \alpha)(\phi(y) + \alpha) - \alpha^2, \quad (3.8)$$

which is defined everywhere except on the *jump set* $\{|x| = 1\} \cup \{|y| = 1\}$ and is linear.

We will concentrate of those regions of the slow manifold which are attractors for the fast equations. For such regions $|x|, |y| > 1$ and $\phi(x) = \phi(y) = 1$. In this case the matrix of (3.8) simplifies to

$$\frac{1}{1 + 2\alpha} \begin{bmatrix} -(1 + \alpha) & -\alpha\omega \\ -\alpha & -(1 + \alpha)\omega \end{bmatrix}, \quad (3.9)$$

with eigenvalues $\lambda_2 < \lambda_1 < 0$, given by

$$\lambda_{1,2} = \frac{1}{2(1 + 2\alpha)} \left\{ -(1 + \alpha)(1 + \omega) \pm \sqrt{(1 + \alpha)^2(1 + \omega)^2 - 4\omega(1 + 2\alpha)} \right\} \quad (3.10)$$

and eigenvectors

$$\mathbf{e}_{1,2} = \left(1, \frac{-(1 + \alpha) - (1 + 2\alpha)\lambda_{1,2}}{\alpha\omega} \right)^T \stackrel{\text{def}}{=} (1, a_{1,2})^T. \quad (3.11)$$

The solution curves of the slow equation are most easily expressed in the basis determined by the eigenvectors of (3.11). Letting v_1 and v_2 be coordinates along the directions of $\mathbf{e}_1$ and $\mathbf{e}_2$ respectively, (3.8) becomes

$$\dot{v}_1 = \lambda_1 v_1, \quad \dot{v}_2 = \lambda_2 v_2 \quad (3.12)$$

and the solutions can be written as

$$v_2(t) = v_{20} \left(\frac{v_1(t)}{v_{10}} \right)^{\lambda_2/\lambda_1}. \quad (3.13)$$

Since all solutions decay toward the ‘ghost’ stable equilibrium of (3.8) at $x = y = 0$, it follows that all solutions reach one of the jump lines in finite time. Our task now is to determine the landing point corresponding to each jump point in this set. To do this we return to the fast equation and examine its phase portraits with u_{10}, u_{20} values selected so that equilibria lie on the jump set. The symmetry of the fast equations allows us to reduce our study to the two jump lines ($x = 1, y \geq 1$) and ($x = 1, y \leq -1$). On these lines we compute u_{10} and u_{20} directly from (3.4):

$$\begin{aligned}u_{10} &= \Phi(1) + \alpha(1 - y) = \alpha - 1 - \alpha y, \\ u_{20} &= \Phi(y) + \alpha(y - 1) = -(2 + \alpha) + (1 + \alpha)y \quad \text{for } y \geq 1 \\ &= (2 - \alpha) + (1 + \alpha)y \quad \text{for } y \leq -1.\end{aligned}\quad (3.14)$$

(a) Jump from $(1, y), y > \frac{1+6\alpha}{1+2\alpha}$

(b) Double jump from $(1, y), y = \frac{1+6\alpha}{1+2\alpha}$

Fig. 3. Jump Dynamics.

We must now find the landing point(s) associated with each jump point $(1, y)$. In the interiors of the jump lines ($|y| > 1$) only one component ($\phi(x)$) of the matrix of (3.3) is undefined and the system therefore bifurcates in a well determined direction. (In the smooth case, one eigenvalue of (3.3) would be zero and a one dimensional stable manifold would coexist with the one dimensional center manifold (Guckenheimer [1980])). There is consequently a unique trajectory in the fast phase-plane for which the α -limit set is the jump point; the ω limit set for this trajectory is the landing point we seek.

First we take $y > 1$. If $y \geq (4(1 + \alpha) - \alpha^2)/(1 + 2\alpha)$ the system (3.5) has only two roots, one of which is the jump point $(1, y)$, and the landing point is clearly the stable sink at $(-(3 + 2\alpha)/(1 + 2\alpha), y - 4\alpha/(1 + 2\alpha))$ in the second quadrant. For

$$y < (4(1 + \alpha) - \alpha^2)/(1 + 2\alpha)$$

there are three roots in addition to the jump point, but global analysis of the solutions reveals that the landing point is still the sink at $(-(3 + 2\alpha)/(1 + 2\alpha), y - 4\alpha/(1 + 2\alpha))$. This behavior persists until $y = (1 + 6\alpha)/(1 + 2\alpha)$, for which the landing point $(-(3 + 2\alpha)/(1 + 2\alpha), 1)$ is itself a jump point. For this value, and for all $1 < y < (1 + 6\alpha)/(1 + 2\alpha)$ there is a unique stable sink at $(-3, y - 4)$. In Fig. 3(a)–(b) we indicate the fast flow for jumps at $y > (1 + 6\alpha)/(1 + 2\alpha)$ and $y = (1 + 6\alpha)/(1 + 2\alpha)$.

For $y < -1$ the analysis is much simpler, since the local direction of the vector field near $x = 1$ drives all solutions leaving points on $\{x = 1, y < -1\}$ into the third quadrant, where they limit at the unique sink $(-(3 + 2\alpha)/(1 + 2\alpha), y - 4\alpha/(1 + 2\alpha))$.

We can summarize this information in a jump map:

$$(1, y) \rightarrow \begin{cases} (-(3 + 2\alpha)/(1 + 2\alpha), y - 4\alpha/(1 + 2\alpha)); \\ y > (1 + 6\alpha)/(1 + 2\alpha), y < -1 \\ (-3, y - 4); \end{cases} \quad 1 < y \leq (1 + 6\alpha)/(1 + 2\alpha). \quad (3.15)$$

To obtain landing segments for the other six jump segments $\{x = -1, |y| > 1\}$ and $\{y = \pm 1, |x| > 1\}$, we simply rotate the map of (3.15) about the origin by $\pi, -\pi/2$ and $\pi/2$ respectively.

The global jump behavior is illustrated in Fig. 4, in which each point on the jump segment X has a unique landing point on the segment X' . We note that the jump map is linear in y and hence preserves distances.

Fig. 4. The jump segments and landing segments.

Fig. 4(continued). The jump segments and landing segments.

We have not yet considered jumps from the points $(1, \pm 1)$, which are doubly degenerate, since both $\phi(x)$ and $\phi(y)$ are undefined. We shall deal with these points in the next two sections, in which we discuss particular cases of equal and unequal frequency oscillators.

4. Identical oscillators, $\omega = 1$. When $\omega = 1$ in equation (1.5) a useful symmetry exists. The hyperplanes

$$\begin{aligned} H_+ &= \{(u_1, u_2, x, y) | x = y, u_1 = u_2\}, \\ H_- &= \{(u_1, u_2, x, y) | x = -y, u_1 = -u_2\}, \end{aligned} \quad (4.1)$$

are both *invariant* in the sense that solutions based on either of these planes remain on the same plane for all time. On each plane the behavior is governed by that of a single relaxation oscillator, as can be seen by applying the transformation

$$\left. \begin{aligned} w &= \frac{x+y}{\sqrt{2}}, & v_1 &= \frac{u_1+u_2}{\sqrt{2}}, \\ z &= \frac{x-y}{\sqrt{2}}, & v_2 &= \frac{u_1-u_2}{\sqrt{2}} \end{aligned} \right\} \quad (4.2)$$

to yield

$$\left. \begin{aligned} \dot{v}_1 &= -w, & \dot{v}_2 &= -z, \\ \varepsilon \dot{w} &= v_1 - \frac{1}{\sqrt{2}} \left\{ \Phi\left(\frac{w-z}{\sqrt{2}}\right) + \Phi\left(\frac{w+z}{\sqrt{2}}\right) \right\}, \\ \varepsilon \dot{z} &= v_2 + \frac{1}{\sqrt{2}} \left\{ \Phi\left(\frac{w-z}{\sqrt{2}}\right) - \Phi\left(\frac{w+z}{\sqrt{2}}\right) \right\} - 2\alpha z. \end{aligned} \right\} \quad (4.3)$$

If the initial point $(u_{10}, u_{20}, x_0, y_0) \in H_+$, then $(v_{10}, v_{20}, w_0, z_0) = (v_{10}, 0, w_0, 0)$ and it is clear from (4.3) that $\dot{v}_2 = \dot{z} \equiv 0$, on H_+ . Therefore the dynamics is just that of the single oscillator

$$\left. \begin{aligned} \dot{v}_1 &= -w \\ \varepsilon \dot{w} &= v_1 - \frac{2}{\sqrt{2}} \Phi\left(\frac{w}{\sqrt{2}}\right) \stackrel{\text{def}}{=} v_1 - \Psi(w) \end{aligned} \right\}; \quad \Psi(w) = \begin{cases} w + 2\sqrt{2}, & w < -\sqrt{2} \\ -w, & |w| \leq \sqrt{2} \\ w - 2\sqrt{2}, & w > \sqrt{2} \end{cases}, \quad (4.4)$$

so that there is always a unique periodic solution lying in this plane. We shall refer to this as the *in-phase mode*.

A similar analysis reveals that the dynamics on H_- is governed by the system

$$\dot{v}_2 = -z, \quad \varepsilon \dot{z} = v_2 - \sqrt{2} \Phi\left(\frac{z}{\sqrt{2}}\right) - 2\alpha z \stackrel{\text{def}}{=} v_2 - \Psi_\alpha(z), \quad (4.5)$$

where the function

$$\Psi_{\alpha}(z) = \begin{cases} (2\alpha + 1)z + 2\sqrt{2}, & z < -\sqrt{2}, \\ (2\alpha - 1)z, & |z| \leq \sqrt{2}, \\ (2\alpha + 1)z - 2\sqrt{2}, & z > \sqrt{2} \end{cases} \quad (4.6)$$

is shown in Fig. 5 for representative values of α . An analysis similar to that of Sec. 2 shows that $\alpha = \frac{1}{2}$ is a bifurcation point for Eq. (4.5), in that it possesses a periodic solution for all $\alpha < \frac{1}{2}$ and ε sufficiently small (depending on α) while for $\alpha > \frac{1}{2}$ no periodic solution exists, and all orbits approach the equilibrium at $(v_2, z) = (0, 0)$. We refer to both the periodic orbit (for $\alpha < \frac{1}{2}$) and the fixed point (for $\alpha > \frac{1}{2}$) as the *out-of-phase mode*.

Fig. 5. The dynamics on the plane H_- in the case $\omega = 1$.

We now use this information, together with the drift and jump analysis of Sec. 3, to establish the global behavior of solutions of (1.5) with $\omega = 1$ and ε sufficiently small. When $\omega = 1$ the slow equation (3.8) on the attracting pieces of the slow manifold ($|x|, |y| > 1$) simplifies to

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{1 + 2\alpha} \begin{bmatrix} -(1 + \alpha) & -\alpha \\ -\alpha & -(1 + \alpha) \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (4.7)$$

and we compute the eigenvalues

$$\lambda_{1,2} = -\frac{1}{1 + 2\alpha}, -1 \quad (4.8)$$

with eigenvectors $\mathbf{e}_{1,2} = (1, -1)^T$ and $(1, 1)^T$ respectively. For $\alpha = 0$ the matrix is diagonal and any vector (a, b) is an eigenvector. In Fig. 6 we show the phase portraits of the slow system for $\alpha = 0$ and $\alpha > 0$, recalling that this system only describes behavior in the four quadrants $|x|, |y| > 1$. A global solution is now assembled from a sequence of drifts along solution curves of Fig. 6 together with the jumps given by equation (3.15). We will

Fig. 6. The flow of the slow equation (4.7) ($\omega = 1$).

describe our results in terms of reduced (x, y) phase plane of the slow system. Our main result for the identical oscillator case is

THEOREM 4.1. Except for a set of initial conditions of measure zero in the reduced (x, y) system, the in-phase mode is globally asymptotically stable for all $\alpha \in (0, \infty)$.

This theorem is proved by several lemmas:

LEMMA 4.2. For all $\alpha \geq 0$, a solution based at an initial point (x_0, y_0) with $x_0 \neq -y_0$ will eventually hit a jump line $x = 1, y \in [1, 3]$ or $y = 1, x \in [1, 3]$.

PROOF. The jump dynamics of Fig. 4 show that all solutions (except those on H_- with $x = -y$) eventually enter the first quadrant $x > 1, y > 1$. Since the drifts and jumps are symmetric under reflection about $x = y$, we suppose, without loss of generality, that our solution jumps at the point $(x_1, 1)$ in the first quadrant, with $x_1 > 3$. From (3.15) (suitably modified) the landing point is

$$\left(x_1 - \frac{4\alpha}{1+2\alpha}, -\left(\frac{3+2\alpha}{1+2\alpha} \right) \right) \stackrel{\text{def}}{=} (x'_1, y'_1)$$

with $0 > y'_1 > -x'_1$ (otherwise $x_1 < 3$). The next jump point $(x_2, -1)$ can be bounded above (in x) by a line parallel to the eigenvector $(1, -1)^T$ passing through (x'_1, y'_1) : this gives

$$x_2 \leq x'_1 + (1 + y'_1) = x_1 - \frac{4\alpha}{1+2\alpha} + 1 - \frac{3+2\alpha}{1+2\alpha}$$

or

$$x_2 \leq x_1 - 2. \quad (4.9)$$

A second jump now carries the solution back to the first quadrant at

$$(x'_2, y'_2) = \left(x_2 - \frac{4\alpha}{1+2\alpha}, \left(\frac{3+2\alpha}{1+2\alpha} \right) \right) \quad (4.10)$$

and a drift analysis like that just carried out yields the estimate

$$x_3 \leq x_2 - 2,$$

or

$$x_3 \leq x_1 - 4, \quad (4.11)$$

for the next jump point $(x_3, 1)$ on the line $y = 1$, $x > 1$. Thus, after a finite sequence of $2n$ drifts and $2n$ jumps we necessarily have $x_{2n+1} < 3$. This completes the proof, since the analysis for jumps from $x = 1$, $y > 1$ is identical. ■

This lemma shows that it suffices to consider the behavior of solutions in the square $|x|, |y| \leq 3$.

LEMMA 4.3. For all $\alpha \geq 0$ the solutions define a map h_α from the jump lines $\{x = 1, y \in (1, 3)\} \cup \{y = 1, x \in (1, 3)\}$ of the first quadrant to the jump lines $\{x = -1, y \in (-3, -1)\} \cup \{y = -1, x \in (-3, -1)\}$ of the third quadrant.

PROOF. This is a direct consequence of the jump dynamics derived in Sec. 3 and the drift illustrated in Fig. 6. ■

We parametrise the jump lines in the first quadrant by an arc length $s \in [0, 4]$ measured from $(x, y) = (3, 1)$ ($s = 0$) through $(x, y) = (1, 1)$ ($s = 2$) to $(x, y) = (1, 3)$, ($s = 4$). This is achieved by the identifications $s \Leftrightarrow (3 - s, 1)$ for $s \in [0, 2]$ and $s \Leftrightarrow (1, s - 1)$ for $s \in [2, 4]$. A similar parameterization $t \Leftrightarrow (t - 3, -1)$, $t \in [0, 2]$ and $t \Leftrightarrow (-1, 1 - t)$, $t \in [2, 4]$ is applied to the jump lines in the third quadrant. The map of Lemma (4.3) may now be written as $t = h_\alpha(s)$, but we shall identify t and s in the following analysis.

LEMMA 4.4. $h_\alpha(s)$ is continuous for all $\alpha \geq 0$ and symmetric about $s = 2$, in that $h_\alpha(4 - s) = 4 - h_\alpha(s)$ for $s \in (0, 4)$; thus $s = 2$ is a fixed point. Moreover, for $\alpha = 0$, $h_0(s)$ is the identity.

PROOF. The continuity of h follows from consideration of the jump and drift dynamics in the neighborhood of the points

$$(x, y) = (1, (1 + 6\alpha)/(1 + 2\alpha)) \quad \text{and} \quad (1 + 6\alpha)/(1 + 2\alpha), 1).$$

By symmetry, it suffices to consider one of these potentially singular points. Consider jumps from points $((1 + 6\alpha)/(1 + 2\alpha) + \delta, 1)$. From (3.15), for $\delta > 0$ the landing point is $(1 + \delta, -(3 + 2\alpha)/(1 + 2\alpha))$ and the next jump point is $(1, \gamma(\delta) - ((3 + 2\alpha)/(1 + 2\alpha)))$, where $\gamma \rightarrow 0$ as $\delta \rightarrow 0$. The corresponding landing point in the third quadrant is therefore $(-(3 + 2\alpha)/(1 + 2\alpha), \gamma - 3)$. In contrast, for $\delta < 0$ the solution jumps directly to $(-((3 + 2\alpha)/(1 + 2\alpha)) + \delta, -3)$. Since $\gamma(\delta) \rightarrow 0$ as $\delta \rightarrow 0$, these two sets of landing points are connected at $(-((3 + 2\alpha)/(1 + 2\alpha)), 3)$, and we conclude that h is continuous, since the contribution of the drifts is clearly continuous.

The fact that $h(2) = 2$ for all α follows immediately from the existence of the invariant plane H_+ , described earlier: the solution based at $(1, 1)$ jumps to $(-3, -3)$ and then drifts to $(-1, -1)$.

The symmetry about the fixed point $s = 2$ follows immediately from the symmetry of the flow, and the behavior for $\alpha = 0$ is obtained by a simple geometrical construction. In this case the drift occurs along straight lines $y = cx$. Again without loss of generality we

take a point $(x, 1)$, $x \in [1, 3]$ ($s = 3 - x$). This jumps to $(x, -3)$ and then drifts to $(1, -3/x)$. The solution then jumps to $(-3, -3/x)$ and drifts to $(-x, -1)$, or $t = 3 - x$. Then we have $t = h_0(s) = s$. ■

We note that the map h_α is not defined on the endpoints $s = 0, 4$ of the interval, since such points lie on orbits which jump directly to the hyperplane H_- and hence never return to the first or third quadrants.

LEMMA 4.5. $h'_\alpha(s) \in (-1, 0)$ on the segment $s \in [2/(1 + 2\alpha), 2(1 + 4\alpha)/(1 + 2\alpha)]$, so that the fixed point $s = 2$ is locally asymptotically stable.

PROOF. Contraction for solutions based at points lying in the segments $\{x = 1, y \in [1, (1 + 6\alpha)/(1 + 2\alpha)]\} \cup \{y = 1, x \in [1, (1 + 6\alpha)/(1 + 2\alpha)]\}$ follows from the concavity of trajectories of the slow equations, together with the preservation of distances in the (linear) jumps from these segments to the segments

$$\left\{ x = -3, y \in \left[-3, -\left(\frac{3 + 2\alpha}{1 + 2\alpha} \right) \right] \right\} \cup \left\{ y = -3, x \in \left[-3, -\left(\frac{3 + 2\alpha}{1 + 2\alpha} \right) \right] \right\}.$$

To show contraction, we transform to the coordinate system of equation (4.2) based on the eigenvectors $(1, \pm 1)^T$ of the matrix of equation (4.7). In the (w, z) coordinates, the slow trajectories are curves of the form $w = \kappa |z|^{1+2\alpha}$ and the landing and jump points lie along the lines $w = \pm 3\sqrt{2} \pm z$ and $w = \pm \sqrt{2} \pm z$ respectively. Without loss of generality, we take the z coordinate of the landing point $z_0 \in (0, \sqrt{2})$ and that of the next jump point at $z_1 \in (0, \sqrt{2})$ and take the corresponding w coordinates positive; see Fig. 7. Fixing the landing point $(z_0, 3\sqrt{2} - z_0)$, we easily compute the constant κ :

$$w_0 = 3\sqrt{2} - z_0 = \kappa z_0^{1+2\alpha} \Rightarrow \kappa = \left(\frac{3\sqrt{2} - z_0}{z_0^{1+2\alpha}} \right). \quad (4.12)$$

The jump point $(z_1, w_1) = (z_1, \sqrt{2} + z_1)$ is then computed from the transcendental equation

$$w_1 = \sqrt{2} + z_1 = \kappa z_1^{1+2\alpha}. \quad (4.13)$$

However, to prove the lemma it is sufficient to bound z_1 above by $\tilde{z}_1$, where $(\tilde{z}_1, \tilde{w}_1)$ is the intersection of the tangent to $w = \kappa z^{1+2\alpha}$ at (z_0, w_0) with the jump line $w = \sqrt{2} + z$. The tangent is easily found to be given by

$$w = (1 + 2\alpha)(3\sqrt{2} - z_0) \frac{z}{z_0} - 2\alpha(3\sqrt{2} - z_0) \quad (4.14)$$

and, setting $\tilde{w}_1 = \sqrt{2} + \tilde{z}_1$, and solving simultaneously with (4.14), we obtain

$$\tilde{z}_1 = \frac{\sqrt{2} + 2\alpha(3\sqrt{2} - z_0)}{(3\sqrt{2}/z_0 - 1)(1 + 2\alpha) - 1}. \quad (4.15)$$

It is easy to see that $\tilde{z}_1 < z_0$ for $z_0 \in (0, \sqrt{2})$; in fact, from (4.15) we have

$$\frac{z_1}{z_0} < \frac{\tilde{z}_1}{z_0} = \frac{\sqrt{2}(1 + 6\alpha) - 2\alpha z_0}{3\sqrt{2}(1 + 2\alpha) - 2(1 + \alpha)z_0}, \quad (4.16)$$

finite $\alpha > 0$. For if these conditions hold, then all points are eventually mapped under h_α into the interval $[2/(1 + 2\alpha), 2(1 + 4\alpha)/(1 + 2\alpha)]$. Unfortunately, simple estimates using tangent line bounds, as in the proof of Lemma 4.5, do not suffice in this case and the full transcendental equations must be solved to determine h_α as the composition of two maps arising from the slow drift in the fourth and third quadrants, together with the jumps of (3.15).

The computations are performed most easily in the (w, z) coordinate system of equation (4.2). Using the symmetries of the problem, and referring to Fig. 8, we see that the computation of $h_\alpha(s_0)$ given $s_0 \in (1, 2/(1 + 2\alpha))$ reduces to two stages, so that $h_\alpha(s_0) = s_2$ is the solution of

$$4 - s_2 = (4 + s_1) \left[\frac{(2 - s_2)}{(\gamma - s_1)} \right]^{(1+2\alpha)}, \quad (4.18)$$

where s_1 is the solution of

$$s_1 = s_0 \left[\frac{(2 + s_1)}{(\delta - s_0)} \right]^{1+2\alpha}; \quad (4.19)$$

where $\gamma = (2 - 4\alpha)/(1 + 2\alpha)$ and $\delta = (6 + 4\alpha)/(1 + 2\alpha)$.

This pair of equations can easily be solved numerically and graphs of $h_\alpha(s)$ plotted as in Fig. 9. The results clearly indicate that $h_\alpha(s) > s$ for $s \in (0, 2)$ and $\alpha > 0$. We note that it is easy to establish that $h_\alpha(0^+) > 8\alpha/(1 + 2\alpha)$ by bounding solution curves leaving the second landing point by lines parallel to the w axis ($x = -y$).

We are now in a position to prove the main theorem.

PROOF OF THEOREM 4.1. The global stability of the in phase mode $\{x = y, u_1 = u_2\}$ follows directly from the global stability of the unique fixed point $s = 2$ for the map $h_a(s)$, $\alpha > 0$. For $\alpha \geq \frac{1}{4}$ this was proven analytically in Lemma 4.2–4.6; for $\alpha \in (0, \frac{1}{4})$ we rely on the numerical solutions of equations (4.18–19). To complete the proof, we note that initial conditions lying on a discrete set of curves corresponding to drifts which eventually land at points $(\pm 3, \pm 1)$ lead to solutions which are asymptotic to the out of phase mode $\{x = -y; u_1 = -u_2\}$. Apart from this set, the in-phase mode attracts all solutions. ■

Fig. 8. Determining $h_\alpha(s)$: $\delta = (6 + 4\alpha)/(1 + 2\alpha)$.

Fig. 9. Graphs of the map $h_\alpha(s)$ for various α .

We close with the remark that, as $\alpha \rightarrow +\infty$, the map $h_\alpha(s)$ approaches the limit $h_\alpha(s) \rightarrow -s$, as can be seen by reference to the estimate (4.15) used in the proof of Lemma 4.5 (cf. Fig. 9). Thus, in the limit of very large coupling, the in phase mode becomes *neutrally* stable rather than asymptotically stable.

5. Oscillators with different frequencies. For the analysis of oscillators with different frequencies, we must rely almost entirely on numerical results. However, certain limiting cases can be dealt with analytically. We consider these first.

As pointed out in Sec. 3, the jump dynamics are independent of the frequency ratio parameter ω ; however, the slow flow does change with ω as equations (3.8–9) demonstrate. Specifically, for fixed $\alpha > 0$, as ω increases from 1, the eigenvectors swing from $(1, \pm 1)^T$ towards their limits $(1, 0)^T$ and $(1, (1 + \alpha)/\alpha)^T$ corresponding to the limiting eigenvalues $\lim_{\omega \rightarrow \infty} \lambda_{1,2} = -1/(1 + \alpha)$, $-(1 + \alpha)\omega/(1 + 2\alpha)$ (cf. (3.10–11)). Similarly, for fixed $\omega > 1$, as α increases from 0, the eigenvectors swing from $(1, 0)^T$, $(0, 1)^T$ (corresponding to eigenvalues $\lambda_{1,2} = -1, -\omega$) towards their limits $(1, -1/\omega)^T$, $(1, 1)^T$ corresponding to the limiting eigenvalues $\lim_{\alpha \rightarrow \infty} \lambda_{1,2} = 0, -(1 + \omega)/2$. This asymptotic behavior allows us to obtain the following result.

THEOREM 5.1. For $\omega > 1$ fixed there exists a constant $\alpha_c < \infty$ such that, for $\alpha > \alpha_c$ there exists at least one stable ‘almost in phase’ periodic orbit in which two jumps occur in each period, and the drifts occur in the first and third quadrants. Moreover α_c is less than or equal to the smallest positive root of $8(3\omega + 1)\alpha^3 + 24\alpha^2 + 24(1 - \omega)\alpha + 9(1 - \omega) = 0$.

PROOF. Fix $\omega > 1$ and choose α sufficiently large such that the eigenvector $\mathbf{e}_2$ of (3.11) with slope

$$\frac{-(1 + \alpha) - (1 + 2\alpha)\lambda_2}{\alpha\omega} = \frac{(1 + \alpha)(\omega - 1) + \sqrt{(1 + \alpha)^2(1 + \omega)^2 - 4\omega(1 + 2\alpha)}}{2\alpha\omega} \stackrel{\text{def}}{=} \mu \quad (5.1)$$

intersects the line $y = 3, x \geq 0$ with x coordinate x_1 satisfying

$$3 - x_1 < \frac{4\alpha}{1 + 2\alpha}. \quad (5.2)$$

This is guaranteed if

$$x_1 = \frac{3}{\mu} > 3 - \frac{4\alpha}{1 + 2\alpha} = \frac{3 + 2\alpha}{1 + 2\alpha}. \quad (5.3)$$

From (5.3) and (5.1) we find that the intersection occurs as required provided that

$$8(3\omega + 1)\alpha^3 + 24\alpha^2 + 24(1 - \omega)\alpha + 9(1 - \omega) > 0. \quad (5.4)$$

This will yield the bound in the last part of the theorem.

We now show that, under this assumption, the drifts and jumps of the singular system define a one dimensional contraction mapping analogous to the map h_α of Fig. 9. The following description refers to Fig. 10.

We will describe the construction only in the first quadrant, since that in the third is obtained by rotation through π about the origin. We first construct a line segment AC parallel to $\mathbf{e}_2$ and below it, intersecting the jump line $\{y = 1, x \geq 1\}$ and landing line $\{x = 3, y \in [((3 + 2\alpha)/(1 + 2\alpha)), 3]\}$ as shown at distances b and a respectively from the points $(1, 1)$ and $(3, 3)$. Here $b = 3 - 3/\mu$ and $a = \mu - 1$ are determined by the slope μ of the eigenvector. We note that $a < b$ is satisfied for all $\mu \in [1, (3 + \sqrt{13})/2]$, and hence for all α satisfying (5.4).

Fig. 10. The contraction map for Theorem 5.1.

We now consider the flow of points based on the segment AOB on the jump set $\{x = 1, y \geq 1\} \cup \{y = 1, x \geq 1\}$. Our assumption on α guarantees that all the solutions based on these segments jump to the landing set

$$\left\{x = -3, y \in \left[-3, -\left(\frac{3 + 2\alpha}{1 + 2\alpha}\right)\right]\right\} \cup \left\{y = -3, x \in \left[-3, -\left(\frac{3 + 2\alpha}{1 + 2\alpha}\right)\right]\right\};$$

in particular, the extremities A and B are mapped to points A^1 and B^1 ; A^1 lying on the eigenvector $y = \mu x$ and B^1 lying at the end of a segment B^1D parallel to the eigenvector (B^1D is the segment AC rotated through π). The solutions based at A^1 and B^1 then drift to points A^2 and B^2 : A^2 is still on the eigenvector but B^2 lies *strictly inside* the segment DO^2A^2 due to the convexity of the solution curves. The following jump and drift are obtained from the first by rotation through π . B^2 and A^2 jump to B^3 and A^3 and the succeeding drift carries the solutions to B^4 , strictly below B , and A^4 , strictly to the left of A . Therefore, after two jumps and two drifts, the segment AOB is mapped back *into* itself: $A^4OB^4 \subset AOB$. The theorem now follows from the contraction mapping principle. ■

REMARK. It seems quite difficult to obtain a *uniform* contraction estimate for this map, and hence to prove uniqueness of the periodic orbit, but we conjecture that, for (possibly higher) values of α , the orbit is unique and attracts almost all initial conditions (cf. Theorem 4.1). Numerical evidence, described below, supports this conjecture.

We now illustrate some of the interesting dynamics which occur for lower values of α in the unequal frequency case. Most of the results outlined here are obtained from numerical solutions of the drift equations for the specific case $\omega = 2$ (Bélair [1983a]). In this case the bound on α_c obtained from (5.4) is $\alpha_c \leq 0.6830 \pm 10^{-4}$; in fact a unique globally attracting orbit appears to exist for all $\alpha \geq 0.25$.

Our major interest is to investigate the sequence of bifurcations occurring as α increases from zero to above α_c and we move from an $\omega:1$ frequency ratio for the uncoupled oscillators to the $1:1$ phase locked orbit of Theorem 5.1. In the following we present a family of one dimensional maps g_α analogous to the maps h_α of Fig. 9, but with both domain and range chosen on the jump set

$$\{x = 1, y \in [1, 3]\} \cup \{y = 1, x \in [1, 3]\}$$

of the first quadrant (cf. the proof of Theorem 5.1). The parameterisation $s \in [0, 4]$ is the same as for the map $h_\alpha(s)$ in the equal frequency case.

In Fig. 11(a) we show the map for $\alpha = 0$: uncoupled oscillators. To each branch of this discontinuous function we assign a sequence of the symbols 1, 2, 3, 4 which indicates the quadrants through which the solution drifts after leaving 1 and before returning to 1. Note that, although all orbits are periodic with four jumps in the y direction and two jumps in the x direction, such orbits can appear either as points of period 1 or 2 for g_α , depending upon their itineraries. In the table below we list the symbol pairs and note the jumps in x and/or y to which they correspond:

Symbol pairs	Jumps
13, 31,	24, 42 x and y simultaneously, written $\begin{pmatrix} x \\ y \end{pmatrix}$
12, 21,	34, 43 x
23, 32,	14, 41 y

 (a) $\alpha = 0$, uncoupled oscillators

 (b) various α

 Fig. 11. The maps g_α for the unequal frequency case, $\omega = 2$.

Thus, orbits corresponding to points $s \in (3 - \sqrt{3}, 2)$ experience the jump sequence $xyxyxy$, while those corresponding to the point $s = 2$ experiences the jump sequence $(x_y^x)y(x_y^x)y$. Points lying in the interval $s \in (2, 4)$ have the sequence $xyxyxy$, obtained by joining the sequences 12321 and 141: these points all lie on orbits of period two for g_0 ; similarly the sequence 1412321 occurs for points $s \in (0, 3 - \sqrt{3})$. Thus the map g_α reflects the dynamics of periodic orbits somewhat indirectly, in that, while fixed and periodic points of g_α correspond to periodic motions, the symbol sequences must be taken into account to determine the precise phase locking relationships.

In the case $\alpha = 0$, the drift solution are simply given by parabolae $y = \kappa x^2$ and g_0 can be computed analytically. For $\alpha > 0$ this is more difficult, and the graphs shown in Fig. 11(b) were obtained numerically. In all cases $\alpha \geq .25$ there is a unique stable periodic orbit corresponding to a jump sequence 131 $\sim xy$: a 1 : 1 phase locked motion.

Even for $\alpha = 0$, a new problem appears which did not occur in the equal frequency case.: the map g_0 is discontinuous due to the fact that orbits starting on opposite sides of the points $s = 3 - \sqrt{3}$ and $s = 2$ have different itineraries. These orbits all eventually land on the degenerate jump points $(x, y) = (\pm 1, \pm 1)$. In the equal frequency case, the existence of the invariant hyperplanes $H_{\pm}$ enabled us to assign unique landing points to these orbits and to deal with the associated drifts. Here no such symmetries exist, and a careful examination of the orbits nearby shows that, for $\varepsilon = 0$, the discontinuities in g_{α} are genuine (in contrast to the continuity of h , established in Lemma 4.4: in that case the composition of jumps and drifts eliminated potential discontinuities). However, if we let $\varepsilon \neq 0$, small, then the continuous dependence of solutions upon initial conditions clearly rules out discontinuities in g_{α} , since (1.5) is Lipschitz continuous for $\varepsilon \neq 0$. Careful boundary layer analyses (cf. Kevorkian and Cole [1981]) would yield asymptotic expressions for g_{α} showing precisely how the discontinuity should be smoothed, but for our purposes we need merely note that such discontinuities can be bridged with a segment of arbitrarily large slope. However, we cannot assign a symbol sequence to such a segment without boundary layer analyses.

We now end by describing a particular case, $\alpha = 0.15$, in which orbits of different periods coexist. Roughly speaking, as α is increased from zero, the continuum of fixed and 2-periodic points of g_{α} bifurcates to a finite set of discrete points, some of which correspond to 2:1 phase locked motions. As α increases, 1:1 phase locked motions appear, apparently in saddle-node bifurcations (Guckenheimer [1980]). For $\alpha = 0.15$ we compute the map shown in Fig. 12. There are two stable fixed points, at which the slope $|g'_{\alpha}| < 1$, and one unstable fixed point. The stable ones lie on segments in which g_{α} is continuous for $\varepsilon = 0$ and hence have uniquely determined sequences corresponding to 1:1 and 2:1 phase locking respectively. The unstable point lies on or very close to a point of discontinuity and hence we cannot assign a sequence uniquely—it may either correspond to 1:1 or 2:1.

The coexistence of stable periodic orbits with different frequency relationships implies that there must be other, complicated invariant sets for g_{α} , just as in studies of periodically forced relaxation oscillators (Cartwright and Littlewood [1945], Levinson [1949], Levi [1981]). using the method of Markov partitions and *symbolic dynamics* (Bowen [1975]), we now demonstrate the existence of such a set. In Fig. 12(b) we choose two subintervals $I_1, I_2 \subset [0, 4]$ such that $g_{\alpha}(I_1) \supset I_1 \cup I_2$ and $g_{\alpha}(I_2) \supset I_1$. We note that, due to the discontinuities in the singular map in I_1 and I_2 , we can select ε such that $|g'_{\alpha}(s)| > 1$ for $s \in I_1 \cup I_2$. We therefore have the basis of a Markov partition and the general theory allows us to conclude that there is an invariant set $\Lambda \subset I_1 \cup I_2$ on which g_{α} is conjugate to a subshift on an alphabet of the symbols 1, 2. The action of g_{α} is symbolically represented by the transition matrix

$$A = [a_{ij}] = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad (5.5)$$

where $a_{ij} = 1$ if $I_j \subset f(I_i)$ and $a_{ij} = 0$ otherwise. This matrix reflects the fact that orbits passing from I_i and I_j can be found for all ij except the pair 22, since $f(I_2)$ does not contain I_2 .

(a) The map and sequences for its fixed points

(b) An enlargement near the fixed points

Fig. 12. The map g_{15} , showing coexistence of stable periodic motions of frequency ratio 2 : 1 and 1 : 1, and the intervals I_1 and I_2 which contain a chaotic invariant set.

The matrix A can be used to count the number of periodic orbits of period k contained in Λ , since the number of fixed points of g_a^k is equal to $\text{trace } A^k$. From (5.5) we find that

$$A^k = \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix}, \quad k \geq 2, \quad (5.6)$$

where F_k is the k th Fibonacci number. Thus, subtracting the number of periodic points whose periods divide k from $(F_{k+1} + F_{k-1})$ and dividing the result by k we obtain the

number $N(k)$ of periodic orbits of period k . In the following table we give the numbers of orbits for $k \leq 12$:

Period k	1	2	3	4	5	6	7	8	9	10	11	12
number of orbits $N(k)$	1	1	1	1	2	2	4	5	8	11	18	25

It is clear that $N(k) \rightarrow \infty$ as $k \rightarrow \infty$, since $N(k) \sim (F_{k+1} + F_{k-1})/k$. Thus we conclude that A has a countable infinity of periodic orbits, including orbits of arbitrarily long periods. All such orbits are unstable, since $|g'_\alpha| > 1$ on $I_1 \cup I_2$. It is also possible to find an uncountable infinity of nonperiodic orbits, corresponding to nonperiodic symbol sequences which do not contain the forbidden pair 22. The existence of such unstable subshifts in maps gives rise to transient chaotic behavior and sensitive dependence upon initial conditions (Guckenheimer [1980]).

6. From the formal analysis to rigorous results. In this section we briefly outline how the formal ($\varepsilon = 0$) analysis of the previous sections carries over to the behavior of the solutions of system (1.5), when ε is some *positive* real number. Details appear in the thesis of Bélaïr [1983a], and will be published elsewhere (Bélaïr [1983b]).

The essential idea consists in allowing ε to become an *infinitesimal* number: that is, a positive number smaller in absolute value than the reciprocal of all (standard) integers. The existence of such numbers follows from the axiom of choice (Robinson [1974], Davis [1977]). Their use in the study of singularly perturbed differential equations has been proposed by Reeb [1974], and promoted by Benoît et al. [1980], and Lutz and Goze [1981]. (The easiest construction of the hyperreal numbers, as the elements of this "extended" real number system are called, is by way of equivalence classes of sequences of real numbers (Stroyan and Luxemburg [1975]); although intuitively appealing, this approach quickly becomes cumbersome, and an axiomatic approach is seen to be more efficient (Lutz and Goze [1981])).

System (1.5) is then considered with ε a *fixed* infinitesimal number, and the formal analysis in phase space carries essentially over, mutatis mutandis: for example, horizontal vector fields become quasi-horizontal: that is, their slope is now infinitesimal rather than zero, and the slow flow, instead of taking place *on* the slow manifolds, takes place in an ε -(infinitesimal) neighborhood of the slow manifold. There are problems of a technical nature that must be dealt with, for example, the piecewise linear function Φ must be suitably smoothed, but they do not present major problems (cf. Bélaïr [1983a, b]).

Once a certain behavior has been obtained for the solutions of system (1.5), uniformly with respect to infinitesimal ε and with proper technical restrictions on just which behaviors are allowed, it is a direct consequence of the axiomatic theory that this behavior also holds for some sufficiently small ε positive, and *standard*. The real system of interest is thus covered by the analysis performed in the non standard framework, in the limit $\varepsilon \rightarrow 0^+$.

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