ON RADIAL MOTION OF A NON-LINEAR VISCOELASTIC TUBE*

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1. Introduction. Radial motion of a non-linear elastic tube is well understood [1], [2]. In considering the corresponding motion of a non-linear viscoelastic tube an equation describing the behaviour can be derived in a straight forward manner. However the form of the constitutive equation is complicated and it is not clear how one should proceed in general. Of course, given an explicit form of the constitutive equations a purely numerical solution of the resulting integrodifferential equation may be possible. In the present note we specialize the problem by considering a thin-walled tube and investigate a model exhibiting impact and long time relaxed non-linear elastic behaviour. In this case the problem reduces to solving an ordinary differential equation and the results may give some insight into the behaviour of a viscoelastic tube when we are close to the non-linear elastic case.

2. Equations. It is known that there are a number of families of non-homogeneous motions which are dynamically possible in incompressible, isotropic elastic materials [2]. These have been extended to dynamically possible motions of general incompressible, isotropic, simple materials in [3], and [4]; we take incompressible here to mean that the density is constant and that an arbitrary hydrostatic pressure p is included in the constitutive equation. We deal with a particular member of one of these families.

We use standard notation. Assuming an underlying Cartesian coordinate system the spatial and material coordinates of a particle X are denoted by $\mathbf{x}$, $\mathbf{X}$ respectively. The deformation gradient tensor $\underline{F}$ may be written as $\underline{R}\underline{U}$ where $\underline{R}$ is a proper orthogonal tensor and $\underline{U}$ a positive definite symmetric tensor; $\underline{B} = \underline{F}\underline{F}^T$ and $\underline{C} = \underline{F}^T\underline{F}$ are the left and right Cauchy Green strain tensors respectively. The constitutive equation may then be written in properly invariant form as

$$\underline{T} = -p\underline{I} + \underline{R}(t)\mathscr{F}_0^\infty(\underline{C})\underline{R}^T(t). \tag{2.1}$$

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where $\underline{T}$ is the Cauchy stress tensor, $\underline{I}$ the unit tensor and $\mathscr{F}$ is a tensor functional of the history of $\underline{C}$,

$$\mathscr{F}_0^\infty(\underline{C}) = \mathscr{F}_{s=0}^\infty\{\underline{C}(t-s)\}. \tag{2.2}$$

Isotropy then implies that

$$\mathscr{F}_0^\infty(\underline{Q}\underline{C}\underline{Q}^T) = \underline{Q}\mathscr{F}_0^\infty(\underline{C})\underline{Q}^T, \tag{2.3}$$

for all orthogonal tensors $\underline{Q}$.

We are concerned with the motion

$$r^2 = R^2 + \Phi(t), \quad \theta = \Theta, \quad z = Z, \tag{2.4}$$

appropriate for radial motion okf a cylindrical tube, where (r, θ, z), (R, Θ, Z) are polar coordinates at $\mathbf{x}$, $\mathbf{X}$ respectively. The base vectors of these polar systems are parallel relative to the underlying Cartesian system, so that if we orthonormalise we form an orthonormal triad, fixed for all time. We deal exclusively with physical components relative to this triad. Then $\underline{F}$ is symmetric and $\underline{R} = \underline{I}$. $\underline{B}$ and $\underline{C}$ have the same physical components so that

$$\underline{T} = -p\underline{I} + \mathscr{F}_0^\infty(\underline{C}), \tag{2.5}$$

where

$$\underline{C} = \text{diag}\left\{\left(\frac{R}{r}\right)^2, \left(\frac{r}{R}\right)^2, 1\right\}. \tag{2.6}$$

Applying equation (2.3) with

$$\underline{Q} = \begin{pmatrix} 1, & 0, & 0 \\ 0, & -1, & 0 \\ 0, & 0, & 0 \end{pmatrix} \quad \text{and} \quad \underline{Q} = \begin{pmatrix} -1, & 0, & 0 \\ 0, & -1, & 0 \\ 0, & 0, & 1 \end{pmatrix} \tag{2.7}$$

in turn, we see that $\underline{T}$ is also diagonal. If p is independent of θ, z the equations of motion reduce to

$$\frac{\partial T\langle 11 \rangle}{\partial r} + \frac{T\langle 11 \rangle - T\langle 22 \rangle}{r} = \rho\ddot{r}. \tag{2.8}$$

If we write $\ddot{r} = \partial\phi/\partial r$ it now follows that

$$T\langle 11 \rangle = \rho\phi + \Omega(t) - \int\{\mathscr{F}_0^\infty\langle 11 \rangle - \mathscr{F}_0^\infty\langle 22 \rangle\} \, dr/r, \tag{2.9}$$

where $\Omega(t)$ is arbitrary.

3. The cylindrical tube. Consider now a long cylindrical tube, inner radius R_0, outer radius R_1 in the reference configuration, subject to internal pressure $P_0(t)$, external pressure $P_1(t)$, with $\Delta P = P_0(t) - P_1(t)$. Then equation (2.9) may be written as

$$\Delta P = \rho\phi\big|_{r_2}^{r_1} - \int_{r_2}^{r_1}\{\mathscr{F}_0^\infty\langle 11 \rangle - \mathscr{F}_0^\infty\langle 22 \rangle\} \, dr/r, \tag{3.1}$$

with

$$r_i^2 = R_i^2 + \Phi(t), i = 0, 1$$

and

$$\phi\big|_{r_2}^{r_1} = \frac{1}{2}\ddot{\Phi}(t)\ln(r_1/r_0) - \frac{1}{8}\dot{\Phi}^2(t)\big(r_0^{-2} - r_1^{-2}\big). \tag{3.2}$$

Equations (3.1), (3.2) are in agreement with equations derived in [7], and [8], when appropriate changes are made in the notation. A class of motions of incompressible materials with memory is considered in [7] and [8] together with a thorough discussion of the stability of these motions. Since our interest lies in short time behaviour of solutions we proceed in a slightly different manner.

If $\Phi(t)$ is given together with the functional form of $\mathscr{F}$ the pressure difference required to maintain the motion may be found. On the other hand given ΔP and $\mathscr{F}$ it is a more difficult problem to find $\Phi(t)$, and unlike the elastic case [1] it does not seem possible to proceed in general.

As a first simplification we suppose the tube is thin-walled and write

$$R_1 = R_0 + H, \gamma = H/R_0 \ll 1,$$

and set $\lambda = r_0(t)/R_0$. The equation (3.1) becomes

$$\lambda\ddot{\lambda} - \frac{\lambda^2\Delta P}{\rho\gamma R_0^2} = \frac{1}{\rho R_0^2}\big\{\mathscr{F}_0^\infty\langle 11\rangle(\underline{C}) - \mathscr{F}_0^\infty\langle 22\rangle(\underline{C})\big\}, \tag{3.3}$$

where $\underline{C} = \text{diag}\{\lambda^{-2}, \lambda^2, 1\}$. In general the Green, Rivlin, Spencer [5] theory in the form adopted by Lockett [6] gives $\mathscr{F}$ as a multiple integral expansion in terms of $\underline{\dot{C}}$, the linear part of which is given by

$$\mathscr{F}_0^\infty(\underline{C}) = \int_{-\infty}^t \big\{\underline{I}\psi_1(t-s)tr\underline{\dot{C}}(s) + \psi_2(t-s)\dot{C}(s)\big\}\,ds. \tag{3.4}$$

The first term in (3.4) may be combined with the pressure p. We now propose setting

$$\psi_2(t) = \mu\big\{m + (1-m)e^{-t/\tau}\big\} \tag{3.5}$$

where $0 < m \leqslant 1$, and μ, τ are constants. If $m = 1$ it is clear that we are dealing with a Neo-Hookean elastic solid. If $m \neq 1$, this suggests that the model given by (3.5) might be considered as a viscoelastic generalization of the Neo-Hookean solid, and μ should be interpreted as the impact shear modulus for infinitesimal deformation from the ground state and τ as a relaxation time. We note that the model exhibits an impact elastic response and a long term relaxed elastic response. Equation (3.5) is the relaxation function for the so called standard linear material and is the simplest relaxation function for a realistic viscoelastic solid which exhibits finite impact modulus. Uniaxial tests of a viscoelastic polymer, Sorbothane II, which has been developed recently for medical uses, indicate that (3.5) is a good approximation for the relaxation behaviour of this material for a wide range of strains. It is, of course, possible to include higher order terms in the expansion (3.4), and with suitable choice of kernels obtain a constitutive equation modelling a viscoelastic generalization of a Mooney-Rivlin material. An equation similar to (3.7) can then be derived which contains additional powers of λ in the integral. For the moment we consider the simpler case given above.

If we nondimensionalise by setting

$$\bar{t} = \sqrt{\mu/\rho}\, t/R_0, \quad \overline{\Delta P} = \Delta P/\mu, \quad \bar{\tau} = \sqrt{\mu/\rho}\, \tau/R_0, \quad \bar{\lambda} = \lambda, \qquad (3.6)$$

and drop the bars we then have

$$\lambda\ddot{\lambda} - \lambda^2\Delta P/\gamma = \int_{-\infty}^{t} \{ m + (1 - m)\exp\{-(t - s)/\tau\} \} \frac{d}{ds}(\lambda^{-2} - \lambda^2)\, ds. \qquad (3.7)$$

For $m = 1$, the elastic case, (3.7) becomes

$$\lambda\ddot{\lambda} - \lambda^2\Delta P/\gamma = (\lambda^{-2} - \lambda^2), \qquad (3.8)$$

and we note for comparison later that
 (a) for free oscillation

$$\lambda(0) = \lambda_1 > 1, \quad \dot{\lambda}(0) = 0, \quad \Delta P = 0,$$
$$\lambda^2 = (\lambda_1^2 + \lambda_2^2)/2 + (\lambda_1^2 - \lambda_2^2)\cos(2t)/2 \qquad (3.9)$$

where $\lambda_1\lambda_2 = 1$,
 (b) for constant internal pressure

$$\lambda(0) = 1, \quad \dot{\lambda}(0) = 0, \quad \Delta P = P_0 > 0$$

 (i) if $P_0 \geqslant \gamma\mu$ the solution is unstable,
 (ii) $0 < P_0 < \gamma\mu$

$$\lambda^2 = \frac{(2\gamma - P_0)}{2(\gamma - P_0)} - \frac{P_0}{2(\gamma - P_0)}\cos\{2\sqrt{(\gamma - P_0)/\gamma}\, t\} \qquad (3.10)$$

and
 (c) for isothernal expansion of an enclosed mass of gas, at initial internal pressure P_0, and with the tube subject to a constant external pressure P_e, we have

$$\lambda(0) = 1, \quad \dot{\lambda}(0) = 0, \quad \Delta P = P_0/\lambda^2 - P_e,$$

and

$$t = \frac{1}{2}\int_{1}^{\lambda^2} du/\{(P_0 u \ln u)\gamma + (2 + P_e/\gamma)u - 1 - (1 + P_e/\gamma)u^2\} \qquad (3.11)$$

for $1 \leqslant \lambda^2 < \lambda^2\text{max}$, where $\lambda^2\text{max}$ is the positive zero exceeding 1 of the quantity inside the square root sign.

4. The viscoelastic case.

Equation (3.7) may be recast as

$$\left(\frac{d}{dt} + \frac{1}{\tau}\right)\left(\lambda\frac{d^2\lambda}{dt^2} - \frac{\lambda^2\Delta P}{\gamma}\right) = \left(\frac{d}{dt} + \frac{m}{\tau}\right)(\lambda^{-2} - \lambda^2). \qquad (4.1)$$

Consider problem (a) above. If the tube is maintained at rest with $\lambda(0) = \lambda_1 > 1$ and then released it follows from Eq. (3.7) that

$$\Delta P(0^-) = \gamma\lambda_1^{-2}\int_{-\infty}^{0^-} \{ m + (1 - m)e^{s/\tau} \} \frac{d}{ds}(\lambda^{-2} - \lambda^2)\, ds. \qquad (4.2)$$

Assuming that the process has taken place slowly and has been completed at $t = -T$ and then maintained we can take (4.2) as

$$\Delta P(0^-) = \frac{\gamma m}{1}\left(\lambda_1^{-2} - \lambda_1^2\right)/\lambda_1^2. \tag{4.3}$$

We then require a solution of (4.1) with $\Delta P = 0$, for $t > 0$, subject to the conditions.

$$\lambda(0) = \lambda_1 > 1,$$

$$\dot{\lambda}(0) = 0,$$

$$\ddot{\lambda}(0) = m\left(\lambda_1^{-2} - \lambda_1^2\right)/\lambda_1. \tag{4.4}$$

Problem (b) is simpler to formulate. Assuming the tube at rest for $t < 0$ we require a solution of (4.1), with $\Delta P = P_0 > 0$ such that

$$\lambda(0) = 1,$$

$$\dot{\lambda}(0) = 0, \tag{4.5}$$

$$\ddot{\lambda}(0) = P_0/\gamma.$$

Problem (c) is the same as (b) with ΔP in equation (4.1) replaced by P_0/λ^2.

The solutions have been computed numerically, and the results are below. Two series of checks are possible. First as m approaches 1 the solutions approach the solutions for the elastic cases discussed above. It is also possible to make a comparison with the linear case by setting $\lambda = 1 + u/R_0$ for infinitesimal displacement u. Returning to the elastic case for a moment it is easily seen that for problem (a) a linearisation is straight forward and the result agrees with the linear situation. Problem (b) requires a little care in so far as noting that equation (3.8) should first be multiplied by λ^{-2} before linearising. If one considers the static case it is clear that for a given internal pressure P_0, that P_0/γ is of the order of u/R_0 so that $P_0 \ll \gamma$. As a result P_0 will not appear in the cosine term in (3.10). Similar comments apply to the viscoelastic case. In general in the linear viscoelastic case the behaviour will be determined by the roots of the equation

$$\alpha^2(\alpha + 1/\tau) + 4(\alpha + m/\tau) = 0 \tag{4.6}$$

and so depend on τ, m. If say $\tau > 1/2\sqrt{3}$, one root will be real, say α_1, $-1 < \alpha_1\tau < -m$ and two will be complex conjugates, each with negative real part $-(1 + \alpha_1\tau)/2\tau$, so that the solution will be damped oscillatory. One would then expect similar results in the non-linear situation.

Equation (4.1) for problem (a), that is with $\Delta P = 0$ and initial conditions (4.4), and for problem (b), that is with initial conditions (4.5) and $\Delta P = P_0 > 0$, has been solved numerically for $\tau = 1, 10$; $m = 0.8, 0.6, 0.4$ and various values of the parameters in the initial conditions. The results for $\lambda(t)$ are shown graphically in Figs. 1, 2, 3, 6 and 7 along with the results obtained from the elastic solutions (3.9) and (3.10). Phase plane trajectories are shown in Figs. 4 and 5 for problem (a). The results for problem (c) show tendencies which are very similar to those for problem (b) and are not shown.
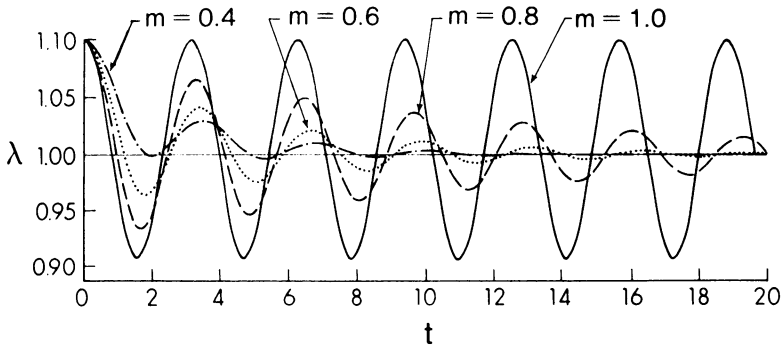
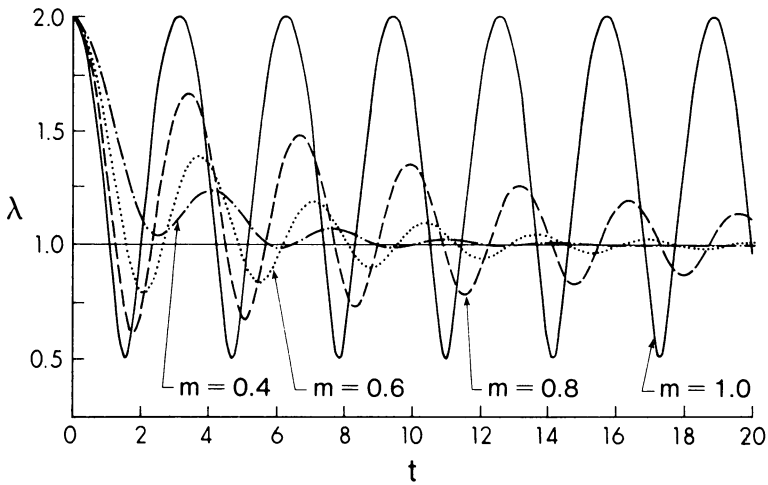FIG. 1. Problem (a) with $\lambda(0) = 1.1$ and nondimensional relaxation time $\tau = 1.0$.

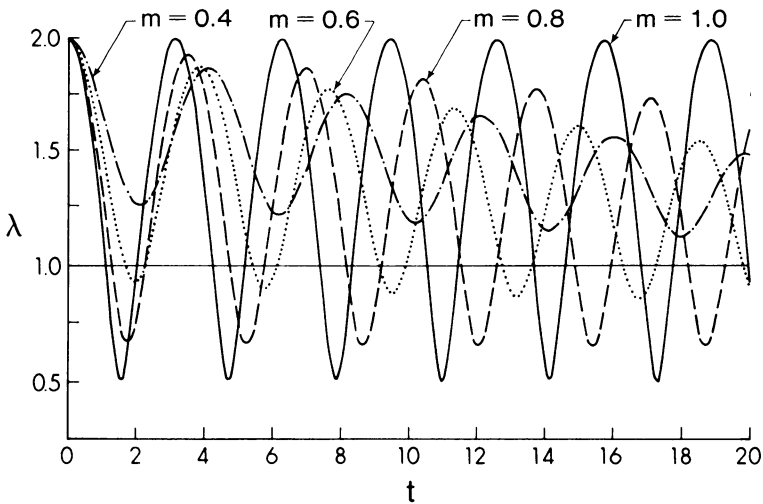FIG. 2. Problem (a) with $\lambda(0) = 2.0$ and nondimensional relaxation time $\tau = 1.0$.

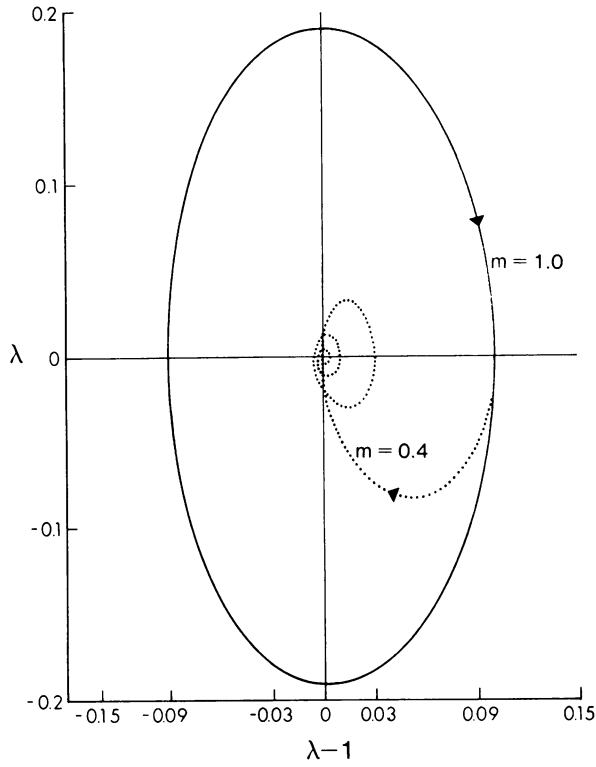FIG. 3. Problem (a) with $\lambda(0) = 2.0$ and nondimensional relaxation time $\tau = 10.0$.

FIG. 4. Phase plane diagrams for problem (a) with $\lambda(0) = 1.1$ and non-dimensional relaxation time $\tau = 1.0$.
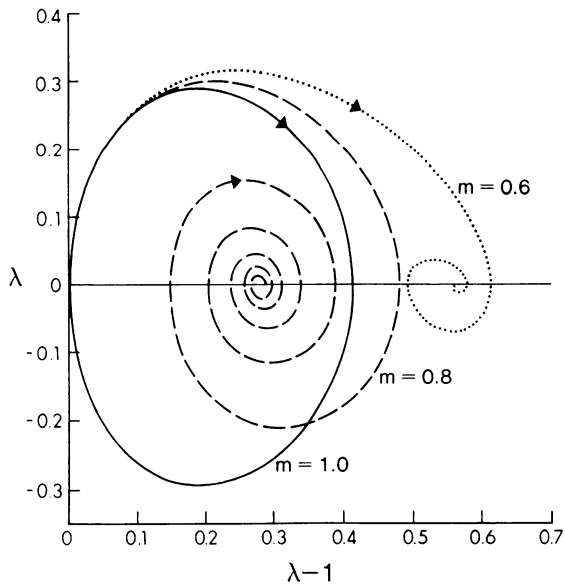

FIG. 5. Phase plane diagrams for problem (b) with $P_0/\gamma = 0.5$ and non-dimensional relaxation time $\tau = 1.0$.
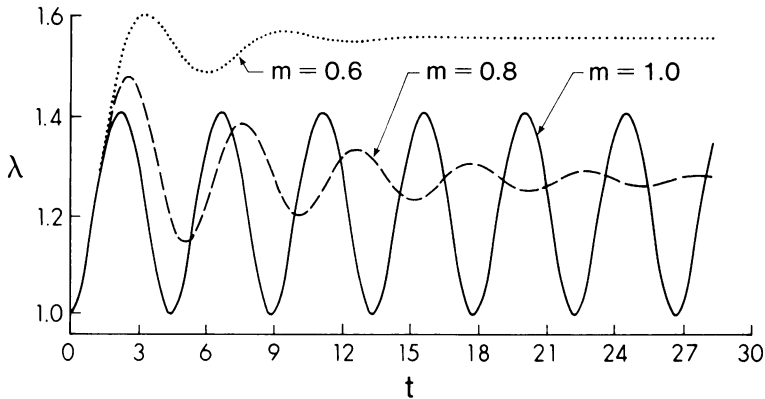
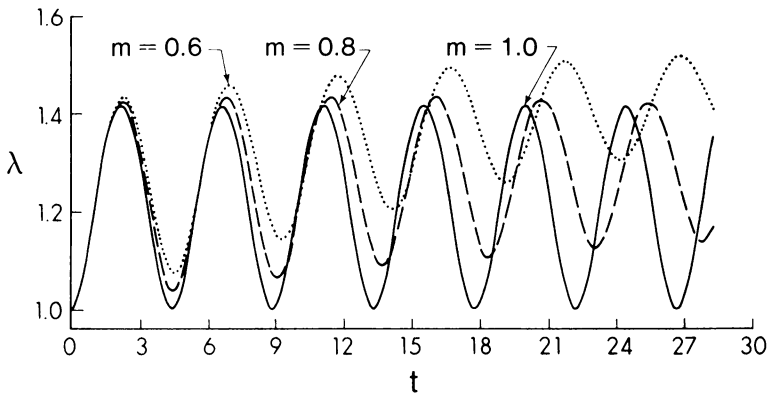FIG. 6. Problem (b) with $P_0/\gamma = 0.5$ and nondimensional relaxation time $\tau = 1.0$.

FIG. 7. Problem (b) with $P_0/\gamma = 0.5$ and nondimensional relaxation time $\tau = 10.0$.

5. Concluding remarks. The results shown in the figures indicate that dissipative effects increase as both m and τ decrease. It follows from equation (4.3) that for problem (b),

$$\lim_{t \to \infty} \lambda(t) = \left\{ m/(m - P_0/\gamma) \right\}^{1/4} \qquad (5.1)$$

and consequently the radius of the tube grows without bound if $P_0 > \gamma m$. Results are presented for $P_0 < \gamma m$. The approach of λ to the limit given by (5.1) is evident in Figs. 6 and 7. Results for problem (a) with $m = 0.4$, shown in Figs. 1 and 2 indicate an almost overdamped situation, leading one to conjecture that there exists a critical value of m for which the motion is overdamped.

In problem (c) the analysis is given for an isothermal expansion, $p\rho^{-1} = $ constant, where ρ is the density of the enclosed gas, for simplicity. The analysis is easily extended to consider a polytropic expansion $p\rho^{-n} = $ constant.

REFERENCES

[1] J. A. Knowles, *Large amplitude oscillations of a tube in incompressible elastic material*, Quart. J. Appl. Math., **18**, 71–77 (1960)

[2] C. Truesdell and W. Noll, *The non-linear field theories of mechanics*, Encyclopedia of Physics Volume 111/3, Springer Verlag, 1965

[3] M. M. Carrol, *Controllable deformations of incompressible simple materials*, Int. J. Eng. Sc., **5**, 515–525 (1967)

[4] R. L. Fosdick, *Dynamically possible motions of incompressible, simple materials*, Arch. Rational Mech. Anal., **29**, 272–288 (1968)

[5] A. E. Green, R. S. Rivlin and A. J. M. Spencer, *The mechanics of non-linear materials with memory*, Arch. Rat. Mech. Anal., **3**, 82–90 (1959)

[6] F. J. Lockett, *Nonlinear viscoelastic solids*, Academic Press, 1972

[7] B. D. Coleman and Ellis H. Dill, *On the stability of certain motions of incompressible materials with memory*, Arch. Rat. Mech. Anal., **30**, 197–224 (1968)

[8] B. D. Coleman and Ellis H. Dill, *On thermodynamics and the stability of motions of materials with memory*, Arch. Rat. Mech. Anal., **51**, 1–53 (1973)