NONUNIFORM MOTION OF AN EDGE DISLOCATION IN AN ANISOTROPIC SOLID. II

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Abstract. The nonuniform motion of an edge dislocation in a singularly hyperbolic cubic (or hexagonal) crystal is analyzed, both for general nonuniform motion and for a motion starting from rest with constant velocity. In the latter case the solution is obtained in closed form from which the stress behaviour near the wavefront is also derived.

Introduction. In the first part of this research we studied the two-dimensional problem of the nonuniform motion of an edge dislocation in a material of cubic or hexagonal symmetry and restricted attention to the regular hyperbolic case. Here we consider the singular hyperbolic case for cubic symmetry, which occurs when two sheets of the slowness surface intersect.

According to Duff [1] the slowness cone with vertex at the origin satisfies the equation

$$S(v, \xi) = \det\left[v^2 \delta pq - Cpqrs \xi_p \xi_s\right] \tag{1}$$

where

$$v^2 \equiv v_N^2(\xi), \qquad N = 1, 2, 3.$$

For a hyperbolic system all the roots t of (1) are real for any real ξ_p, and if they are all distinct the system is called strictly (or regularly) hyperbolic. If there are repeated roots for some ξ_p then the system is called non-strictly (singular) hyperbolic.

In the two dimensional case of cubic (or hexagonal) symmetry under consideration the equation (1) takes the form

$$\begin{aligned} S(v, \xi) = {} & \rho^2 v^4 - \rho(C_{66} + C_{11}) v^2 \xi_1^2 - \rho(C_{22} + C_{66}) v^2 \xi_2^2 \\ & + \left\{ C_{11} C_{66} \xi_1^4 + \left[C_{11} C_{22} + C_{66}^2 - (C_{66} + C_{12})^2 \right] \xi_1^2 \xi_2^2 + C_{22} C_{66} \xi_2^4 \right\} \\ = {} & 0 \end{aligned} \tag{2}$$

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which may be solved for v^2 to yield

$$v^2 = \frac{1}{2\rho}\left[(C_{11} + C_{66})\xi_1^2 + (C_{22} + C_{66})\xi_2^2 \pm \frac{1}{\rho}\sqrt{\Delta}\right] \tag{3}$$

where the discriminant Δ is given by

$$\Delta = \rho^2\left\{\left[(C_{11} - C_{66})\xi_1^2 - (C_{22} - C_{66})\xi_2^2\right]^2 + 4(C_{66} + C_{12})^2\xi_1^2\xi_2^2\right\}. \tag{4}$$

The system is singular hyperbolic iff $\Delta = 0$, which occurs iff

$$C_{12} + C_{66} = 0 \tag{5}$$

since we have assumed $C_{11} \neq C_{66}$ and $C_{22} \neq C_{66}$.

In the cubic case for which $C_{11} = C_{22}$, $\Delta = 0$ iff $\xi_1 = \pm\xi_2$, which means that sheet I (the innermost sheet) intersects sheet II (the outer one) at $\xi_1 = \pm\xi_2$. In the hexagonal case for which $C_{11} \neq C_{22}$, $\Delta = 0$ iff $\xi_1 = \pm(C_{22} - C_{66})\xi^2/(C_{11} - C_{66})$ and the treatment will be similar to the cubic symmetry case which follows below.

For a singular hyperbolic cubic material the slowness surface is obtained from (2) by setting $v = 1$ and $C_{11} = C_{22}$, $C_{12} + C_{66} = 0$. Then equation (2) reduces to

$$S(1, \xi) = (\rho - C_{11}\xi_1^2 - C_{66}\xi_2^2)(\rho - C_{66}\xi_1^2 - C_{11}\xi_2^2) \tag{6}$$

and the slowness surface consists of two ellipses (see Fig. 1):

$$\begin{aligned} &\text{(i)} \quad C_{11}\xi_1^2 + C_{66}\xi_2^2 = \rho \\ &\text{(ii)} \quad C_{66}\xi_1^2 + C_{11}\xi_2^2 = \rho, \end{aligned} \tag{7}$$

which intersect for $\xi_1 = \xi_2 = \sqrt{\rho/(C_{11} + C_{66})}$.

Then for the outermost sheet S_1 of the slowness surface (see Fig. 1) we have

$$\begin{aligned} C_{66}\xi_1^2 + C_{11}\xi_2^2 = \rho \quad &\text{for } \xi_1 \geqslant \xi_2, \\ C_{11}\xi_1^2 + C_{66}\xi_2^2 = \rho \quad &\text{for } \xi_1 \leqslant \xi_2, \end{aligned}$$

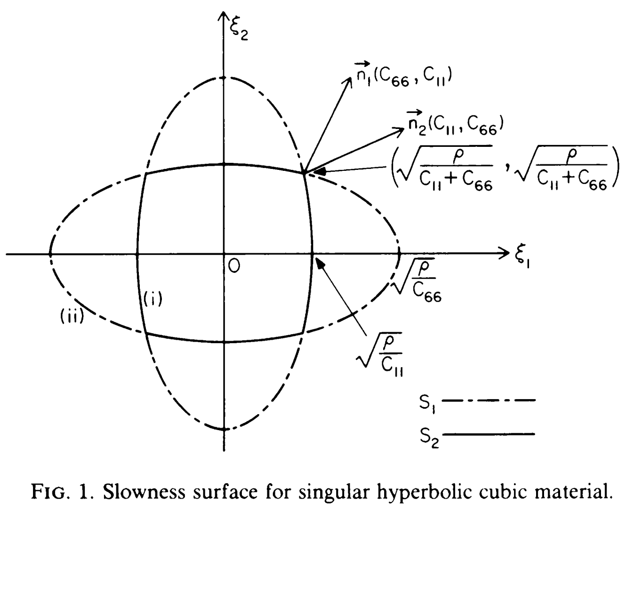

FIG. 1. Slowness surface for singular hyperbolic cubic material.

with corresponding wave surface parts W_1 given by

$$\frac{1}{C_{66}}\left(\frac{x_1}{t}\right)^2 + \frac{1}{C_{11}}\left(\frac{x_2}{t}\right) = \frac{1}{\rho},$$

and

$$\frac{1}{C_{11}}\left(\frac{x_1}{t}\right)^2 + \frac{1}{C_{66}}\left(\frac{x_2}{t}\right)^2 = \frac{1}{\rho}$$

respectively (see Fig. 2).

The inner sheet S_2 of the slowness surface (see Fig. 1) is given by

$$C_{11}\xi_1^2 + C_{66}\xi_2^2 = \rho \quad \text{for } \xi_1 \geqslant \xi_2,$$
$$C_{66}\xi_1^2 + C_{11}\xi_2^2 = \rho \quad \text{for } \xi_1 \leqslant \xi_2,$$

with corresponding wave surface parts W_2 described by

$$\frac{1}{C_{11}}\left(\frac{x_1}{t}\right)^2 + \frac{1}{C_{66}}\left(\frac{x_2}{t}\right)^2 = \frac{1}{\rho}$$

and

$$\frac{1}{C_{66}}\left(\frac{x_1}{t}\right)^2 + \frac{1}{C_{11}}\left(\frac{x_2}{t}\right)^2 = \frac{1}{\rho}$$

respectively (see Fig. 2).

Nonuniform motion of an edge dislocation in the singular hyperbolic case. Let us consider an edge dislocation in a cubic or hexagonal crystal for which $C_{11} + C_{66} = 0$ (i.e. singular hyperbolic case) being at rest at the origin and starting moving at $t = 0$ along the x_1 axis

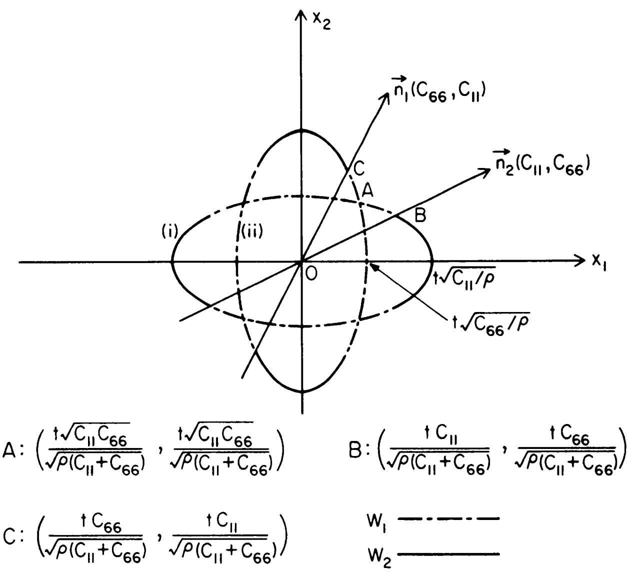

FIG. 2. Wave-surface for singular hyperbolic cubic material.

according to $x_1 = l(t)$ or equivalently $t = \eta(x_1)$. Then, as in part I, the radiated field will satisfy for $t \geqslant 0$ the differential equations

$$C_{11}\frac{\partial^2 u_1}{\partial x_1^2} + C_{66}\frac{\partial^2 u_1}{\partial x_2^2} = \rho\frac{\partial^2 u_1}{\partial t^2},$$
$$C_{66}\frac{\partial^2 u_2}{\partial x_1^2} + C_{22}\frac{\partial^2 u_2}{\partial x_2^2} = \rho\frac{\partial^2 u_2}{\partial t^2} \tag{8}$$

with boundary conditions at $x_2 = 0$:

$$u_1(x_1, x_2, t) = \frac{\Delta u}{2}(H(x_1 - l(t)) - H(x_1)),$$
$$C_{11}\frac{\partial u_1}{\partial x_1} + C_{22}\frac{\partial u_2}{\partial x_2} = 0 \tag{9}$$

superposed to the static solution for an edge dislocation at the origin for all t [2].

The system of differential equations (8) with boundary conditions (9) is equivalent to the superposition of the following two problems:

Problem I.

$$\frac{\partial^2 u_1}{\partial x_1^2} + \frac{C_{66}}{C_{11}}\frac{\partial^2 u_1}{\partial x_2^2} = \frac{\rho}{C_{11}}\frac{\partial^2 u_1}{\partial t^2} \tag{10}$$

with boundary condition at $x_2 = 0$:

$$u_1 = \frac{\Delta u}{2}[H(x_1 - l(t)) - H(x_1)]. \tag{11}$$

Problem II.

$$\frac{\partial^2 u_2}{\partial x_1^2} + \frac{C_{22}}{C_{66}}\frac{\partial^2 u_2}{\partial x_2^2} = \frac{\rho}{C_{66}}\frac{\partial^2 u_2}{\partial t^2} \tag{12}$$

with boundary condition at $x_2 = 0$:

$$C_{12}\frac{\partial u_1}{\partial x_1} + C_{22}\frac{\partial u_2}{\partial x_2} = 0 \tag{13}$$

where u_1 is the solution to Problem I.

For Problem I we set $\tilde{x}_2 = \sqrt{C_{11}x_2/C_{66}}$ and $b = \sqrt{\rho/C_{11}}$ so that equations (10) and (11) become

$$\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial \tilde{x}_2^2} = b^2\frac{\partial^2 u_1}{\partial t^2} \tag{14}$$

with

$$u_1(x_1, 0, t) = \frac{\Delta u}{2}[H(x_1 - l(t)) - H(x_1)] \quad \text{for } t \geqslant 0. \tag{15}$$

The solution to the differential equation (14) with boundary condition (15) has been obtained by Markenscoff [3]. For materials of cubic symmetry the slowness surface is the ellipse (i) in Fig. 1 with corresponding wave surface the ellipse (i) in Fig. 2.

For constant velocity motion with dislocation velocity $v_d = 1/\alpha < 1/b$ (subsonic)

$$\frac{\partial u_1}{\partial x_2}(x_1, x_2, t) = \frac{\partial u_1}{\partial \tilde{x}_2}\sqrt{\frac{C_{11}}{C_{66}}} = \frac{\Delta u}{2\pi}\sqrt{\frac{C_{11}}{C_{66}}}$$

$$\cdot \frac{\left(b^2 - \frac{t^2 x_1^2}{r^4} + \frac{C_{11} x_2^2}{C_{66} r^4}(t^2 - r^2 b^2)\right)\left(\alpha - \frac{t x_1}{r^2}\right) + \frac{2 C_{11} t x_1 x_2}{C_{66} r^6}(t^2 - r^2 b^2)}{\sqrt{t^2 - r^2 b^2}\left[\left(\alpha - \frac{t x_1}{r^2}\right)^2 + \frac{C_{11} x_2^2}{C_{66} r^4}(t^2 - r^2 b^2)\right]} H(t - rb) \tag{16}$$

and

$$\frac{\partial u_1}{\partial x_1} = -\frac{\Delta u}{2\pi}\sqrt{\frac{C_{11}}{C_{66}}}$$

$$\cdot \frac{-t^2 x_1 x_2 (z\alpha r^2 - t x_1) + \frac{C_{11}}{C_{66}} x_2^3 t (t^2 - r^2 b^2) + \alpha x_1 x_2 r^4 b^2}{r^2\sqrt{t^2 - r^2 b^2}\left[(\alpha r^2 - t x_1)^2 + \frac{C_{11}}{C_{66}} x_2^2 (t^2 - r^2 b^2)\right]} H(t - rb) \tag{17}$$

with $r^2 = x_1^2 + C_{11} x_2^2 / C_{66}$. For the non-constant velocity case the solution is also given in [3]. For supersonic motion (i.e. $\alpha < b$) there will be additional Mach wavefronts on which the stress is delta-function [4]. For constant dislocation velocity these fronts are straight lines in the $x_1 - x_2$ plane given by

$$\sqrt{\frac{C_{11}}{C_{66}}}\, x_2 = \frac{-\alpha x_1}{\sqrt{b^2 - \alpha^2}} + \frac{1}{\sqrt{b^2 - \alpha^2}}.$$

The solution to Problem II may be obtained by using the solution to the problem of the regular hyperbolic case developed in part I [2]. From equation (12) of part I, if we set $C_{12} + C_{66} = 0$, we have in the transformed space

$$U_2(\lambda, x_2, s) = -\frac{C_{66}}{C_{22}} \frac{U_1(\lambda, 0, s)\lambda}{\sqrt{\frac{\rho}{C_{22}} - \frac{C_{66}}{C_{22}}\lambda^2}} e^{-s x_2 \sqrt{\rho/C_{22} - C_{66}\lambda^2/C_{22}}}$$

from which it follows that

$$\frac{\partial U_2}{\partial x_2}(\lambda, x_2, s) = \frac{C_{66}}{C_{22}} U_1(\lambda, 0, s)\lambda s e^{-s x_2 \sqrt{\rho/C_{22} - C_{66}\lambda^2/C_{66}}} \tag{18}$$

where

$$U_1(\lambda, 0, s) = -\frac{\Delta u}{2s}\int_0^\infty e^{-s\eta(\xi) - s\lambda\xi}\, d\xi.$$

We notice here that the exponent of e has only one radical so that with change of variables the integrand is reduced to a form analogous to the isotropic case, for which the ususal Cagniard-de Hoop technique applies directly.

For materials of cubic symmetry the slowness surface is the ellipse (ii) in Fig. 1 with corresponding wave surface the ellipse (ii) in Fig. 2.

Applying the change of variables $d = \sqrt{\rho/C_{66}}$, $\hat{x}_2 = \sqrt{C_{66}/C_{22}}\,x_2$ and inverting (18) we have

$$\frac{\partial \hat{u}_2}{\partial x_2}(x_1, x_2, s) = \frac{-\Delta u}{4\pi i} s \frac{C_{66}}{C_{22}} \int_{Br} \int_0^{\infty} \lambda e^{-s\eta(\xi) - s\lambda\xi} e^{+s\lambda x_1 - s\hat{x}_2\sqrt{d^2-\lambda^2}}\, d\xi\, d\lambda$$

the solution to which can be obtained by using the method developed in [3]. For the constant velocity subsonic motion ($\alpha > d$), the above reduces to

$$\frac{\partial u_2}{\partial x_2}(x_1, x_2, t) = \frac{-\Delta u}{2\pi}\left(\frac{C_{66}}{C_{22}}\right)^{3/2}$$
$$\cdot\left\{\frac{-t^2x_1x_2\left(2\alpha\hat{r}^2 - tx_1\right) + t\dfrac{C_{66}}{C_{22}}x_2\left(t^2 - \hat{r}^2d^2\right) + \alpha x_2x_2\hat{r}^4d^2}{\hat{r}^2\sqrt{t^2 - \hat{r}^2d^2}\left[\left(\alpha\hat{r}^2 - tx_1\right)^2 + \dfrac{C_{66}}{C_{22}}\hat{x}_2^2\left(t^2 - \hat{r}^2d^2\right)\right]}\right\} \cdot H(t - rd) \tag{19}$$

where $\hat{r}^2 = x_1^2 + C_{66}x_2^2/C_{22}$. Furthermore we have

$$\frac{\partial \hat{u}_2}{\partial x_1}(x_1, x_2, s) = \frac{\Delta u \cdot s}{4\pi i}\sqrt{\frac{C_{66}}{C_{22}}} \int_{Br}\int_0^{\infty} \frac{\lambda^2}{\sqrt{d^2 - \lambda^2}} e^{-s\eta(\xi) - s\lambda\xi} e^{s\lambda x_1 - s\hat{x}_2\sqrt{d^2-\lambda^2}}\, d\xi\, g\lambda$$

and

$$\frac{\partial u_2}{\partial x_1}(x_1, x_2, t) = \frac{\Delta u}{2\pi}\sqrt{\frac{C_{66}}{C_{22}}}\, H(t - \hat{r}d)$$
$$\cdot\left\{\frac{\dfrac{C_{66}}{C_{22}}tx_2^2\left[\left(\alpha\hat{r}^2 - tx_1\right)t - x_1\left(t^2 - \hat{r}^2d^2\right)\right]\left[t^2x_1^2 + \left(2x_1^2 - \dfrac{C_{66}}{C_{22}}x_2^2\right)\left(t^2 - \hat{r}^2d^2\right)\right]}{\hat{r}^2\sqrt{t^2 - \hat{r}^2d^2}\,\dfrac{C_{66}}{C_{22}}\left[t^2x_2^2 + x_1^2\left(t^2 - \hat{r}^2d^2\right)\right]\left[\left(\alpha\hat{r}^2 - tx_1\right)^2 + \dfrac{C_{66}}{C_{22}}x_2^2\left(t^2 - \hat{r}^2d^2\right)\right]}\right.$$
$$\left. - \frac{x_1\left(t^2 - \hat{r}^2d^2\right)\left[x_1\left(\alpha\hat{r}^2 - tx_1\right) + \dfrac{C_{66}}{C_{22}}tx_1^2\right]\left[t^2\left(\dfrac{C_{66}}{C_{22}}x_2^2 - x_1^2\right) + \dfrac{C_{66}}{C_{22}}x_2^2\left(t^2 - \hat{r}^2d^2\right)\right]}{\hat{r}^2\sqrt{t^2 - \hat{r}^2d^2}\left[\dfrac{C_{66}}{C_{22}}t^2x_2^2 + x_1^2\left(t^2 - \hat{r}^2d^2\right)\right]\left[\left(\alpha\hat{r}^2 - tx_1\right)^2 + \dfrac{C_{66}}{C_{22}}x_2^2\left(t^2 - \hat{r}^2d^2\right)\right]}\right\} \tag{20}$$

with

$$\hat{r}^2 = x_1^2 + \frac{C_{66}}{C_{22}}x_2^2.$$

For supersonic motion (i.e. $\alpha < d$) there will be additional Mach wavefronts as in Problem I. for constant dislocation velocity they are straight lines:

$$\sqrt{\frac{C_{65}}{C_{22}}}\, x_2 = \frac{-\alpha x_1}{\sqrt{d^2 - \alpha^2}} + \frac{t}{\sqrt{d^2 - \alpha^2}}.$$

The above expressions may be used to derive the explicit wavefronts behavior for a dislocation starting from rest and moving with constant velocity. For an isotropic material the wavefront behavior has been obtained in [5].

For to the displacements u_1 and u_2 are the ellipses (i) and (ii) respectively in Fig. 2. Therefore we may decompose the problem into the cases

(I) $x_1 > x_2$ and

(II) $x_1 < x_2$.

For case (I), at time

$$t = \sqrt{\frac{\rho}{C_{11}}x_1^2 + \frac{\rho}{C_{66}}x_2^2} + \Delta t$$

the stress field is due to the displacement u_1 (with corresponding wave surface $1/C_{11}(x_1/t)^2 + 1/C_{66}(x_2/t)^2 = 1/\rho$)and has components $\sigma_{11} = C_1 u_{1,1}$, $\sigma_{22} = C_{11} u_{1,1}$, $\sigma_{12} = C_{66} u_{1,2}$, which by use of (16) and (21) give:

$$\begin{aligned}
\sigma_{11} &= \frac{C_{11}\Delta u}{2\pi} \frac{b\cos\theta\sin\theta}{\sqrt{\Delta t}\,\sqrt{2br}\left(-\cos\theta + \dfrac{\alpha}{b}\right)}, \\
\sigma_{22} &= \frac{C_{12}\Delta u}{2\pi} \frac{b\cos\theta\sin\theta}{\sqrt{\Delta t}\,\sqrt{2br}\left(-\cos\theta + \dfrac{\alpha}{b}\right)}, \\
\sigma_{12} &= \frac{\sqrt{C_{66}C_{11}}}{2\pi}\Delta u \frac{b\sin^2\theta}{\sqrt{\Delta t}\,\sqrt{2br}\left(-\cos\theta + \dfrac{\alpha}{b}\right)},
\end{aligned} \tag{22}$$

where $b = \sqrt{\rho/C_{11}}$, $r = \sqrt{x_1^2 + C_{11}x_2^2/C_{66}}$ and $\theta = \tan^{-1}(\sqrt{C_{11}/C_{66}}(x_2/x_1))$. For case (II), at time

$$t = \sqrt{\frac{\rho}{C_{66}}x_1^2 + \frac{\rho}{C_{11}}x_2^2} + \Delta t \tag{23}$$

the only non-zero displacement is u_2 and the corresponding stress components

$$\sigma_{11} = C_{12}u_{2,2}, \quad \sigma_{22} = C_{11}u_{2,2}, \quad \sigma_{12} = C_{66}u_{2,1}$$

by use of (17) and (23), yield near the wave front

$$\begin{aligned}
\sigma_{11} &= \frac{C_{12}\Delta u}{2\pi}\sqrt{\frac{C_{66}}{C_{11}}} \frac{d\cos\phi\sin\phi}{\sqrt{\Delta t}\,\sqrt{2d\hat{r}}\left(-\cos\phi + \dfrac{\alpha}{d}\right)}, \\
\sigma_{22} &= \frac{C_{11}\Delta u}{2\pi}\sqrt{\frac{C_{66}}{C_{11}}} \frac{d\cos\phi\sin\phi}{\sqrt{\Delta t}\,\sqrt{2d\hat{r}}\left(-\cos\phi + \dfrac{\alpha}{d}\right)}, \\
\sigma_{12} &= \frac{C_{66}\Delta u}{2\pi}\sqrt{\frac{C_{66}}{C_{11}}} \frac{d\cos^2\phi}{\sqrt{\Delta t}\,\sqrt{2d\hat{r}}\left(-\cos\phi + \dfrac{\alpha}{d}\right)},
\end{aligned}$$

where $d = \sqrt{\rho/C_{66}}$, $\hat{r} = \sqrt{x_2^2 + C_{66}x_2^2/C_{11}}$, and $\phi = \tan^{-1}(\sqrt{C_{66}/C_{11}}(x_2/x))$. Finally for $x_1 = x_2$ at time

$$t = \sqrt{\frac{\rho}{C_{11}}x_1^2 + \frac{\rho}{C_{66}}x_2^2} + \Delta t = \sqrt{\frac{\rho}{C_{66}}x_1^2 + \frac{\rho}{C_{11}}x_2^2} + \Delta t$$

we have contributions due to both wave surfaces as given above.

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