ASYMPTOTIC SOLUTION NEAR THE APEX OF AN ELASTIC WEDGE WITH CURVED BOUNDARIES*

By

T. C. T. TING

University of Illinois at Chicago

Abstract. When the boundaries of an elastic wedge are straight lines, the asymptotic solution near the apex $r = 0$ of the wedge is simply a series of eigenfunctions of the form $r^{\lambda}f(\theta, \lambda)$ in which (r, θ) is the polar coordinate with origin at the wedge apex and λ is the eigenvalue. When the wedge boundaries are curved, the eigenvalues remain the same but the curvatures of the boundaries change the form of the eigenfunctions. The eigenfunction associated with a λ contains not only the term r^{λ}, but also $r^{\lambda+1}, r^{\lambda+2}, \ldots$ In some cases it also contains the term $r^{\lambda+1}(\ln r)$. Therefore, the second and higher order terms of asymptotic solution are not simply the second and next eigenfunctions. Examples are given for the first few terms of asymptotic solution for wedges with wedge angle π and 2π. The latter corresponds to a crack with curved free boundaries and we show that there exists a term $r^{1/2}(\ln r)$ besides the familiar terms $r^{-1/2}$.

1. Introduction. It is well known that the stress distribution near the apex of an elastic wedge with straight boundaries can be expressed in terms of a series of eigenfunctions of the form $r^{\lambda}f(\theta, \lambda)$ where r is the radial distance from the apex of the wedge, λ is a constant and f is a function of λ and the polar angle θ [1]. For given wedge angle 2α and homogeneous boundary conditions at the wedge boudaries, there are in general infinitely many eigenvalues λ and associated eigenfunctions $r^{\lambda}f(\theta, \lambda)$. Particularly important in applications is when one of the λ's is negative and hence the stress is singular at the apex. For instance, for the specimen shown in Fig. 1 under a tensile loading, $\lambda = -1/2$ at the crack tip Q where the wedge angle 2α is 2π. At point M where $2\alpha = 3\pi/2$, it can be shown that there are two negative λ's [2]. In solving the stress distribution in the entire specimen numerically by using a finite element scheme, one may use regular finite elements everywhere except at the singular points Q and M. At the singular points Q and M, a special element is used in which the singular nature of the stress is given by the analytical expression $r^{\lambda}f(\theta, \lambda)$. It may be sufficient to consider only the first term (or

*Received September 7, 1983. This work is supported by the U.S. Army Materials and Mechanics Research Center, Watertown, Massachusetts, under contract DAAG 46-83-K-0159.

terms) for which λ is negative in the special elements. In many cases, however, more terms are required [3, 4].

When the wedge boundaries are straight lines, inclusion of the second and higher order terms in the asymptotic solution simply means the addition of eigenfunctions $r^\lambda f(\theta, \lambda)$ associated with the second and the subsequent smallest eigenvalues λ. If the wedge boundaries are not straight lines, the curvatures of the boundaries change the form of the eigenfunction. The eigenfunction associated with a given λ now contains not only the term r^λ, but also $r^{\lambda+1}$, $r^{\lambda+2}$,... In some instances it also contains the $r^{\lambda+1}(\ln r)$ terms. Therefore, in choosing the higher order terms of asymptotic solution for wedges with curved boundaries, one cannot simply add another eigenfunction. After presenting the basic theories of analyses for wedges with curved boundaries we give two examples; one with wedge angle π and the other with wedge angle 2π, and show how one can obtain the first few terms of the asymptotic solution.

2. Formulation of the problem. Consider an elastic wedge bounded by two boundaries Γ and Γ^* as shown in Fig. 2. Using a polar coordinate system (r, θ), the stress components σ_{rr}, $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ given by [1]

$$\left.\begin{aligned} \sigma_{rr} &= r^\lambda\{A\cos(2+\lambda)\theta - B\sin(2+\lambda)\theta + C(2-\lambda)\cos\lambda\theta \\ &\qquad - D(2-\lambda)(\sin\lambda\theta)/\lambda\}, \\ \sigma_{\theta\theta} &= r^\lambda\{-A\cos(2+\lambda)\theta + B\sin(2+\lambda)\theta + C(2+\lambda)\cos\lambda\theta \\ &\qquad - D(2+\lambda)(\sin\lambda\theta)/\lambda\}, \\ \sigma_{r\theta} &= r^\lambda\{-A\sin(2+\lambda)\theta - B\cos(2+\lambda)\theta + C\lambda\sin\lambda\theta + D\cos\lambda\theta\}, d \end{aligned}\right\} \tag{1}$$

where A, B, C, D and λ are arbitrary constants, satisfy the equations of equilibrium and the stress compatability conditions. Notice that placement of λ in the denominator of the terms containing D ensures that these terms do not vanish simultaneously when $\lambda = 0$ [2]. Using matrix notations, we write Eq. (1) as

$$\boldsymbol{\sigma} = r^\lambda \mathbf{S}(\theta, \lambda)\mathbf{q}, \tag{2}$$

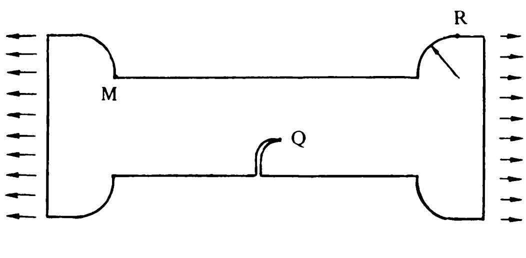

Fig. 1.

where

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{r\theta} \end{bmatrix}, \qquad \mathbf{q} = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}, \tag{3}$$

and $\mathbf{S}(\theta, \lambda)$ is a 3×4 matrix which can be defined accordingly.

I. *When* Γ *and* Γ^* *are straight lines.* We will assume that the boundaries Γ and Γ^* are stress-free. If Γ and Γ^* are straight lines with polar angles θ_0 and θ_0^*, respectively, we have

$$\sigma_{r\theta} = \sigma_{\theta\theta} = 0 \quad \text{at } \theta = \theta_0 \text{ and } \theta_0^*. \tag{4}$$

We may write Eqs. (4) as, using Eq. (2)

$$\mathbf{K}_0(\lambda)\mathbf{q} = \mathbf{0}, \tag{5}$$

where

$$\mathbf{K}_0(\lambda) = \begin{bmatrix} \mathbf{N}_0\mathbf{S}(\theta_0, \lambda) \\ \mathbf{N}_0\mathbf{S}(\theta_0^*, \lambda) \end{bmatrix}, \tag{6}$$

$$\mathbf{N}_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{7}$$

For a nontrivial solution of $\mathbf{q}$, we must have

$$\|\mathbf{K}_0(\lambda)\| = 0, \tag{8a}$$

which is, when the determinant is expanded,

$$\left\{[\sin 2(1+\lambda)\alpha]^2 - [(1+\lambda)\sin 2\alpha]^2\right\}/\lambda = 0, \tag{8b}$$

where $2\alpha = \theta_0 - \theta_0^*$ is the wedge angle. There are infinitely many roots (real and/or complex) for λ. For the strain energy to be bounded in a region containing $r = 0$, we must have $\lambda > -1$. For each eigenvalue λ, Eq. (5) gives the associated eigenvector $\mathbf{q}$ and Eq. (2) provides the eigenfunction for the stress $\boldsymbol{\sigma}$. Notice that $\lambda = 0$ is *not* a root for all values of α [2] as incorrectly claimed in some literatures.

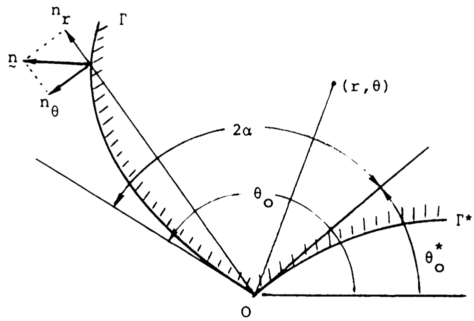

Fig. 2.

Let $\lambda = \lambda_1, \lambda_2, \cdots$ be the roots of Eq. (8). If the roots are real, we let $-1 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \ldots$ If they are complex, we arrange according to the real part of the complex roots. Then the asymptotic solution for $\boldsymbol{\sigma}$ as $r \to 0$ can be written as

$$\boldsymbol{\sigma} = k_1 r^{\lambda_1}\mathbf{S}(\theta, \lambda_1)\mathbf{q}_1 + k_2 r^{\lambda_2}\mathbf{S}(\theta, \lambda_2)\mathbf{q}_2 + k_3 r^{\lambda_3}\mathbf{S}(\theta, \lambda_3)\mathbf{q}_3 + \cdots \quad (9)$$

where $k_1, k_2, \cdots$ are arbitrary constants.

II. Γ *and* Γ^* *are curved.* When the boundaries Γ and Γ^* are not straight lines, the solution obtained in Eq. (9) does not satisfy the boundary conditions for all r because Eqs. (4) do not apply to curved boundaries. At any point on Γ, let (n_r, n_θ) be the polar components of the unit normal $\mathbf{n}$ to the boundary, Fig. 2. Then the components (t_r, t_θ) of surface traction vector $\mathbf{t}$ are given by

$$\left.\begin{aligned} t_r &= \sigma_{rr} n_r + \sigma_{r\theta} n_\theta, \\ t_\theta &= \sigma_{r\theta} n_r + \sigma_{\theta\theta} n_\theta \end{aligned}\right\} \quad (10)$$

or, using matrix notations,

$$\mathbf{t} = \mathbf{N}\boldsymbol{\sigma}, \quad (11)$$

where

$$\mathbf{t} = \begin{bmatrix} t_r \\ t_\theta \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} n_r & 0 & n_\theta \\ 0 & n_\theta & n_r \end{bmatrix}. \quad (12)$$

Along Γ, r is a function of θ and

$$\begin{aligned} n_r &= r(r^2 + r'^2)^{-1/2}, \\ n_\theta &= -r'(r^2 + r'^2)^{-1/2}, \end{aligned} \quad (13)$$

where a prime stands for differentiation with respect to θ. We will assume that, as $r \to 0$, the curve Γ can be represented by the asymptotic expression (see remarks in Sec. 5),

$$\theta = \theta_0 + \theta_1 r + \theta_2 r^2 + \cdots \quad (14)$$

We then have

$$-n_r/n_\theta = r\,d\theta/dr = \theta_1 r + 2\theta_2 r^2 + \cdots, \quad (15)$$

and $\mathbf{N}$ of Eq. (12) can be written as

$$\mathbf{N} = n_\theta\{\mathbf{N}_0 - \mathbf{N}_1(\theta_1 r + 2\theta_2 r^2 + \cdots)\}, \quad (16)$$

where $\mathbf{N}_0$ is defined in Eq. (7) and

$$\mathbf{N}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (17)$$

With θ given by Eq. (14), we can expand $\mathbf{S}(\theta, \lambda)$ of Eq. (2) in power seris of r as

$$\mathbf{S}(\theta, \lambda) = \mathbf{S}(\theta_0, \lambda) + \theta_1 \mathbf{S}'(\theta_0, \lambda) r + \left[2\theta_2 \mathbf{S}'(\theta_0, \lambda) + \theta_1^2 \mathbf{S}''(\theta_0, \lambda)\right] r^2 + \cdots, \quad (18)$$

where a prime stands for differentiation with respect to θ.

To satisfy the traction-free condition on Γ, we generalize the eigenfunction, Eq. (2), and write

$$\boldsymbol{\sigma} = r^{\lambda}\mathbf{S}(\theta, \lambda)\mathbf{q} + r^{\lambda+1}\mathbf{S}(\theta, \lambda + 1)\mathbf{q}^{(1)} + r^{\lambda+2}\mathbf{S}(\theta, \lambda + 2)\mathbf{q}^{(2)} + \cdots, \tag{19}$$

where $\mathbf{q}, \mathbf{q}^{(1)}, \mathbf{q}^{(2)}, \ldots$ are arbitrary constant matrices. Substituting Eq. (19) into (11) and making use of (18) and (16), we obtain

$$\begin{aligned} \mathbf{t}/n_{\theta} = {} & r^{\lambda}\mathbf{N}_0\mathbf{S}(\theta_0, \lambda)\mathbf{q} \\ & + r^{\lambda+1}\{\mathbf{N}_0\mathbf{S}(\theta_0, \lambda + 1)\mathbf{q}^{(1)} - \theta_1[\mathbf{N}_1\mathbf{S}(\theta_0, \lambda) - \mathbf{N}_0\mathbf{S}'(\theta_0, \lambda)]\mathbf{q}\} \\ & + r^{\lambda+2}\{\cdots\} + \cdots \end{aligned} \tag{20}$$

The same equation applies to $\mathbf{t}^*/n_\theta^*$ for the boundary Γ^* if a superscript * is added to $\theta_0, \theta_1, \ldots$ in Eq. (20). Imposition of the traction-free conditions $\mathbf{t} = \mathbf{t}^* = \mathbf{0}$ implies that the coefficients of r^{λ}, $r^{\lambda+1}$,.. must vanish. We obtain

$$K_0(\lambda)\mathbf{q} = \mathbf{0}, \tag{21}$$

$$\mathbf{K}_0(\lambda + 1)\mathbf{q}^{(1)} = \mathbf{K}_1(\lambda)\mathbf{q}, \tag{22}$$

................,

where $K_0(\lambda)$ is defined in Eq. (6) and

$$\mathbf{K}_1(\lambda) = \begin{bmatrix} \theta_1\{\mathbf{N}_1\mathbf{S}(\theta_0, \lambda) - \mathbf{N}_0\mathbf{S}'(\theta_0, \lambda)\} \\ \theta_1^*\{\mathbf{N}_1\mathbf{S}(\theta_0^*, \lambda) - \mathbf{N}_0\mathbf{S}'(\theta_0^*, \lambda)\} \end{bmatrix}. \tag{23}$$

Eq. (21) is identical to Eq. (5) and hence the eigenvalues λ and the eigenvectors $\mathbf{q}$ are identical to the case of a wedge with straight boundaries. However, the eigenfunction is now given by Eq. (19) for each λ in which $\mathbf{q}^{(1)}$ is determined in terms of $\mathbf{q}$ from Eq. (22). Likewise one can derive an equation for $\mathbf{q}^{(2)}$ which will depend on $\mathbf{q}$ also. By adding the eigenfunctions given by Eq. (19) for $\lambda = \lambda_1, \lambda_2, \ldots$, we write the asymptotic solution for $\boldsymbol{\sigma}$ when $r \to 0$ as

$$\begin{aligned} \boldsymbol{\sigma} = {} & k_1\{r^{\lambda_1}\mathbf{S}(\theta, \lambda_1)\mathbf{q}_1 + r^{\lambda_1+1}\mathbf{S}(\theta, \lambda_1 + 1)\mathbf{q}_1^{(1)} + \cdots\} \\ & + k_2\{r^{\lambda_2}\mathbf{S}(\theta, \lambda_2)\mathbf{q}_2 + r^{\lambda_2+1}\mathbf{S}(\theta, \lambda_2 + 1)\mathbf{q}_2^{(1)} + \cdots\} \\ & + k_3\{\ldots\} + \ldots, \end{aligned} \tag{24}$$

where $k_1, k_2, \ldots$ are arbitrary constants. Since $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$.., the r^{λ_1} term is the first order term. However, the second order term is r^{λ_2} (which is the case in Eq. (9)) if $\lambda_2 < \lambda_1 + 1$ and r^{λ_1+1} if $\lambda_2 > \lambda_1 + 1$. Thus the second and higher order terms are not necessarily the same as in wedges with straight boundaries.

Equation (22) gives a unique $\mathbf{q}^{(1)}$ provided $\lambda + 1$ is not a root of Eq. (8a). If it is, a solution $\mathbf{q}^{(1)}$ may still exist although the solution is not unique. In the next section, we consider the case in which $\mathbf{q}^{(1)}$ does not exist.

3. Modified eigenfunction. When $\lambda + 1$ is a root of Eq. (8a) and hence $\mathbf{K}_0(\lambda + 1)$ of Eq. (22) is singular, a solution for $\mathbf{q}^{(1)}$ exists if and only if [5]

$$\mathbf{l}^T\mathbf{K}_1(\lambda)\mathbf{q} = 0, \tag{25a}$$

where T stands for the transpose and $\mathbf{l}$ is the left eigenvector of $\mathbf{K}_0(\lambda + 1)$,

$$\mathbf{l}^T\mathbf{K}_0(\lambda + 1) = \mathbf{0}. \tag{25b}$$

When Eq. (25a) is satisfied, there exists a $\mathbf{q}^{(1)}$. Although $\mathbf{q}^{(1)}$ so obtained is not unique, the solution obtained in the previous section nevertheless remains valid.

If Eq. (25a) is not satisfied, there is no solution for $\mathbf{q}^{(1)}$ and the eigenfunction associated with the eigenvalue λ cannot be given by the expression, Eq. (19). In this case, consider the following modified eigenfunction [6, 7]

$$\begin{aligned}\boldsymbol{\sigma} = r^{\lambda}\mathbf{S}(\theta, \lambda)\mathbf{q} &+ \frac{\partial}{\partial\lambda}\left[r^{\lambda+1}\mathbf{S}(\theta, \lambda + 1)\mathbf{q}^{(1)}(\lambda)\right] \\ &+ \frac{\partial}{\partial\lambda}\left[r^{\lambda+2}\mathbf{S}(\theta, \lambda + 2)\mathbf{q}^{(2)}(\lambda)\right] + \dots\end{aligned} \tag{26a}$$

where $\mathbf{q}^{(1)}, \mathbf{q}^{(2)}, \dots$ are now assumed to depend on λ. Expanding the differentiation, we have

$$\begin{aligned}\boldsymbol{\sigma} = r^{\lambda}\mathbf{S}(\theta, \lambda)\mathbf{q} + r^{\lambda+1}&\Big[(\ln r)\mathbf{S}(\theta, \lambda + 1)\mathbf{q}^{(1)}(\lambda) \\ &+ \frac{\partial}{\partial\lambda}(\mathbf{S}(\theta, \lambda + 1))\mathbf{q}^{(1)}(\lambda) + \mathbf{S}(\theta, \lambda + 1)\frac{d}{d\lambda}\mathbf{q}^{(1)}(\lambda)\Big] \\ &+ r^{\lambda+2}[\cdots] + \cdots\end{aligned} \tag{26b}$$

If we substitute this into Eq. (11), make use of Eqs. (18) and (16), and set the traction-free conditions $\mathbf{t} = \mathbf{t}^* = \mathbf{0}$, we obtain

$$\mathbf{K}_0(\lambda)\mathbf{q} = \mathbf{0}, \tag{27}$$

$$\left.\begin{aligned}&\mathbf{K}_0(\lambda + 1)\mathbf{q}^{(1)}(\lambda) = \mathbf{0}, \\ &\mathbf{K}_0(\lambda + 1)\frac{d}{d\lambda}\mathbf{q}^{(1)}(\lambda) + \left[\frac{\partial}{\partial\lambda}\mathbf{K}_0(\lambda + 1)\right]\mathbf{q}^{(1)}(\lambda) = \mathbf{K}_1(\lambda)\mathbf{q},\end{aligned}\right\} \tag{28}$$

where $\mathbf{K}_0$ and $\mathbf{K}_1$ are defined in Eqs. (6) and (23). Similar equations may be derived for $\mathbf{q}^{(2)}$, $d\mathbf{q}^{(2)}(\lambda)/d\lambda, \dots$

Equation (27) is identical to (5) and hence λ and $\mathbf{q}$ are the same as in wedges with straight boundaries. Equation (28) provides $\mathbf{q}^{(1)}$ and $d\mathbf{q}^{(1)}(\lambda)/d\lambda$. A discussion on the existence of a solution for a system of equations similar to Eq. (28) can be found in [6]. When $\mathbf{q}^{(1)}(\lambda)$ exists and is non-zero, Eq. (26b) shows that there is a new term $r^{\lambda+1}(\ln r)$ in the eigenfunction. We see that while r^{λ} is the first order term, the second order term within the eigenfunction is $r^{\lambda+1}(\ln r)$, not $r^{\lambda+1}$.

If $\lambda + p$, where p is a positive integer, is a root of Eq. (8a) and a solution for $\mathbf{q}^{(p)}$ does not exist, a modification similar to Eq. (26a) can be made for the eigenfunction. We then would have the term $r^{\lambda+p}(\ln r)$.

4. Examples. In this section we will consider two wedges; one with wedge angle π and the other with wedge angle 2π. We will present the first few terms of asymptotic expansion for $r << 1$.

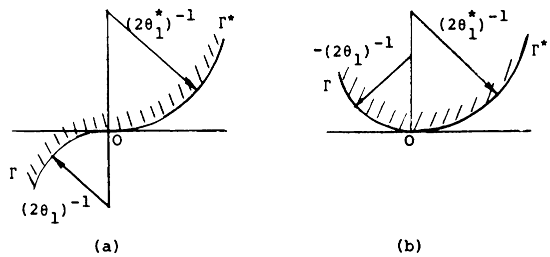

Fig. 3.

I. *Elastic wedge with wedge angle* π. Let $\theta_0 = \pi$ and $\theta_0^* = 0$ as shown in Fig. 3a. The boundaries Γ and Γ^* of the wedge are circles with radii $(2\theta_1)^{-1}$ and $(2\theta_1^*)^{-1}$, respectively. If $\theta_1 < 0$, we have the wedge shown in Fig. 3b. The first three roots of Eq. (8b) are [2]

$$\lambda_1 = 0, \qquad \lambda_2 = \lambda_3 = 1, \tag{29}$$

and Eq. (5) gives the associated eigenvectors $\mathbf{q}$:

$$\mathbf{q}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{q}_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{q}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}. \tag{30}$$

Using Eq. (24) and omitting the terms of order higher than r^2, the asymptotic solution is

$$\boldsymbol{\sigma} = k_1\{\mathbf{S}(\theta,0)\mathbf{q}_1 + r\mathbf{S}(\theta,1)\mathbf{q}_1^{(1)}\} + k_2 r\mathbf{S}(\theta,1)\mathbf{q}_2 + k_3 r\mathbf{S}(\theta,1)\mathbf{q}_3, \tag{31}$$

in which $\mathbf{q}_1^{(1)}$ is to be determined from (22). However, for $\lambda = \lambda_1 = 0$, $\lambda + 1$ is a root of Eq. (8b) and $\mathbf{K}_0(\lambda + 1)$ is singular. A solution for $\mathbf{q}_1^{(1)}$ exists if Eq. (25a) is satisfied. It can be shown that Eq. (25a) is satisfied if $\theta_1 = -\theta_1^*$, which implies that if Γ and Γ^* form the same circle. In this case, Eq. (22) has solutions for $\mathbf{q}_1^{(1)}$ and it can be shown that

$$\mathbf{q}_1^{(1)} = \begin{bmatrix} 0 \\ 8\theta_1 \\ 0 \\ 0 \end{bmatrix} + c_1\mathbf{q}_2 + c_2\mathbf{q}_3, \tag{32}$$

where c_1 and c_2 are arbitrary constants. Hence $q_1^{(1)}$ is non-unique. However, as we see from Eq. (31) the non-unique solutions associated with c_1 and c_2 can be ignored because they are already represented by the terms associated with k_2, and k_3.

If $\theta_1 \neq -\theta_1^*$, we use Eq. (26b) for the eigenfunction associated with λ_1. We then have, again excluding terms of order higher than r^2,

$$\begin{aligned} \boldsymbol{\sigma} = k_1\Big\{\mathbf{S}(\theta,0)\mathbf{q}_1 &+ r\Big[(\ln r)\mathbf{S}(\theta,1)\mathbf{q}^{(1)}{}_1 \\ &+ \frac{\partial}{\partial\lambda}\mathbf{S}(\theta,\lambda+1)|_{\lambda=0}\mathbf{q}_1^{(1)} + \mathbf{S}(\theta,1)\frac{d}{d\lambda}\mathbf{q}_1^{(1)}(\lambda)|_{\lambda=0}\Big]\Big\} \\ &+ k_2 r\mathbf{S}(\theta,1)\mathbf{q}_2 + k_3 r\mathbf{S}(\theta,1)\mathbf{q}_3. \end{aligned} \tag{33}$$

It can be shown that

$$\left.\begin{aligned} &\mathbf{q}_1^{(1)} = \gamma \mathbf{q}_2, \qquad \gamma = 4(\theta_1 + \theta_1^*)/\pi, \\ &\frac{d}{d\lambda}\mathbf{q}_1^{(1)}(\lambda)\bigg|_{\lambda=0} = \begin{bmatrix} \gamma \\ -8\theta_1^* \\ 0 \\ 0 \end{bmatrix} + c_1\mathbf{q}_2 + c_2\mathbf{q}_3, \end{aligned}\right\} \tag{34}$$

where c_1 and c_2 are arbitrary constants. Again, the terms associated with c_1 and c_2 can be ignored because they are represented by the terms associated with k_2 and k_3. We see that when $\theta_1 = -\theta_1^*$, $\gamma = 0$ and Eq. (33) reduces to (31). However, as long as $\theta_1 \neq -\theta_1^*$, the asymptotic solution given by Eq. (33) contains the new term $r(\ln r)$. The example presented here applies to the point R in Fig. 1 where $\theta_1 \neq 0$ while $\theta_1^* = 0$.

II. *Elastic wedge with wedge angle* 2π. Let $\theta_0 = \pi$ and $\theta_0^* = -\pi$ as shown in Fig. 4a. The radii of curvature for Γ and Γ^* are, respectively $(2\theta_1)^{-1}$ and $(2\theta_1^*)^{-1}$. It is clear that we must have $\theta_1^* - \theta_1 \geqslant 0$. If $\theta_1 < 0$, we have a cusped crack as shown in Fig. 4b.

The first five roots of Eq. (8b) are [2]

$$\lambda_1 = \lambda_2 = -1/2, \quad \lambda_3 = 0, \quad \lambda_4 = \lambda_5 = 1/2, \tag{35}$$

and the associated $\mathbf{q}$ are

$$\left.\begin{aligned} &\mathbf{q}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \qquad \mathbf{q}_2 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \\ &\mathbf{q}_3 = q_4 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{q}_5 = \begin{bmatrix} 0 \\ 5 \\ 0 \\ 1 \end{bmatrix}. \end{aligned}\right\} \tag{36}$$

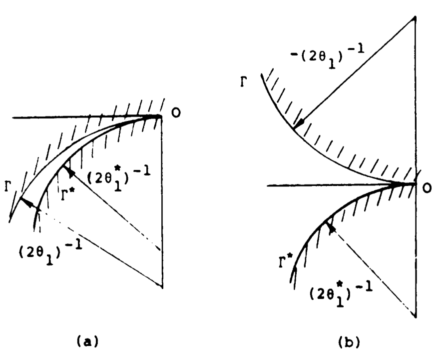

Fig. 4.

When Γ and Γ^* are straight lines, i.e., when $\theta_1 = \theta_1^* = 0$, we have from Eq. (9)

$$\begin{aligned}\boldsymbol{\sigma} = {} & k_1 r^{-1/2}\mathbf{S}(\theta,-1/2)\mathbf{q}_1 + k_2 r^{-1/2}\mathbf{S}(\theta,-1/2)\mathbf{q}_2 \\ & + k_3\mathbf{S}(\theta,0)\mathbf{q}_3 + k_4 r^{1/2}\mathbf{S}(\theta,1/2)\mathbf{q}_4 + k_5 r^{1/2}\mathbf{S}(\theta,1/2)\mathbf{q}_5\end{aligned} \tag{37}$$

where $k_1, k_2, \ldots$ are arbitrary constants.

If Γ and Γ^* are not straight lines, it can be shown that Eq. (24) applies to all λ's except that Eq. (26b) must be used in place of the eigenfunction associated with λ_2. The result can be written as, omitting the terms of order higher than $r^{1/2}$,

$$\begin{aligned}\boldsymbol{\sigma} = {} & k_1\left\{ r^{-1/2}\mathbf{S}(\theta,-1/2)\mathbf{q}_1 + r^{1/2}\mathbf{S}(\theta,1/2)\mathbf{q}_1^{(1)}\right\} \\ & + k_2\left\{ r^{-1/2}\mathbf{S}(\theta,-1/2)\mathbf{q}_2 + r^{1/2}\left[(\ln r)\mathbf{S}(\theta,1/2)\mathbf{q}_2^{(1)} \right.\right. \\ & \left.\left. + \frac{\partial}{\partial\lambda}\mathbf{S}(\theta,\lambda+1)\bigg|_{\lambda=-1/2}\mathbf{q}_2^{(1)} + \mathbf{S}(\theta,1/2)\frac{d}{d\lambda}\mathbf{q}_2^{(1)}(\lambda)\bigg|_{\lambda=-1/2}\right]\right\} \\ & + k_3\mathbf{S}(\theta,0)\mathbf{q}_3 + k_4 r^{1/2}\mathbf{S}(\theta,1/2)\mathbf{q}_4 + k_5 r^{1/2}\mathbf{S}(\theta,1/2)\mathbf{q}_5,\end{aligned} \tag{38}$$

where

$$\begin{aligned} & \mathbf{q}_1^{(1)} = c_1\mathbf{q}_4 + c_2\mathbf{q}_5, \qquad \mathbf{q}_2^{(1)} = -\frac{3(\theta_1^* - \theta_1)}{2\pi}\mathbf{q}_5, \\ & \frac{d}{d\lambda}\mathbf{q}_2^{(1)}(\lambda)\bigg|_{\lambda=-1/2} = 6\begin{bmatrix} -(\theta_1^* + \theta_1) \\ 2(\theta_1^* - \theta_1)/\pi \\ 0 \\ 0 \end{bmatrix} + c_3\mathbf{q}_4 + c_4\mathbf{q}_5,\end{aligned} \tag{39}$$

and c_1, c_2, c_3, c_4 are arbitrary constants. We see that the terms associated with c_1, c_2, c_3 and c_4 can be ignored because they are identical to the k_4 and k_5 terms. We see also that unless $\theta_1 = \theta_1^*$, the $r^{1/2}(\ln r)$ term is present. This term is not in Eq. (37) where the wedge boundaries are assumed to be straight.

The problem of stress distribution in an infinite plane containing a cusped crack of finite length has been studied in [8, 9] using a conformal mapping technique. One could recover the solution obtained in (38) by performing an asymptotic analysis of the solution obtained in [8, 9] for $r << 1$.

5. Concluding remarks. By introducing a new form of eigenfunction for the stresses in an elastic wedge with curved boundaries, the correct second and higher order asymptotic solutions are obtained which satisfy the traction-free boundary conditions on the curved boundaries. One could modify the analyses for other types of homogeneous boundary conditions such as vanishing of the displacements or mixed boundary conditions in which one component of stresses and one component of displacements vanish at a boundary. The analyses can also be extended to a composite wedge in which two or more wedges are glued together along curved interfaces.

The asymptotic nature of the curved boundary Γ as assumed in Eq. (14) is, of course, the simplest and is most likely to be encountered in practice. In general, however, one could have any other form of asymptotic expression for the wedge boundary Γ as long as

$\theta \to \theta_0$ when $r \to 0$. For instance, instead of Eq. (14) one could have

$$\theta = \theta_0 + \theta_1 r^{\beta} + \ldots \tag{40}$$

where $\beta > 0$ is not necessarily an integer. Then Eq. (19) for the eigenfunction is replaced by

$$\boldsymbol{\sigma} = r^{\lambda}\mathbf{S}(\theta, \lambda)\mathbf{q} + r^{\lambda+\beta}\mathbf{S}(\theta, \lambda + \beta)\mathbf{q}_1^{(1)} + \ldots \tag{41}$$

In the case of a crack with curved boundaries, the lowest λ is $-1/2$ and if $\beta = 1/6$ one would have, in addition to the $r^{-1/2}$ terms, the term $r^{-1/3}$ and possibly $r^{-1/3}(\ln r)$ depending on the asymptotic nature of the other curved boundary Γ^*.

REFERENCES

[1] M. L. Williams, *Stress singularities resulting from various boundary conditions in angular corners of plates in extension*, J. Appl. Mech., **19**, 526–528 (1952)

[2] T. C. T. Ting, *The wedge subjected to tractions: A paradox re-examined*, J. Elasticity, **14**, Sept. 1984

[3] P. Tong, T. H. H. Pian, and S. J. Lasry, *A hybrid-element approach to crack problems in plane elasticity*, Int. J. Numerical Meth. in Eng., **7**, 297–308 (1973)

[4] K. Y. Lin and J. W. Mar, *Finite element anaylysis of stress intensity factors for cracks at a bi-material interface*, Int. J. Fracture, **12** 521–531 (1976)

[5] F. B. Hilderbrand, *Methods of applied mathematics*, Prentice-Hall, Englewood Cliffs, NJ, 1954

[6] J. P. Dempsey and G. B. Sinclair, *On the stress singularities in the plane elasticity of composite wedge*, J. Elasticity, **9**, 373–391 (1979)

[7] R. L. Zwiers, T. C. T. Ting and R. L. Spilker, *On the logarithmic singularity of free-edge stress in laminated composites under uniform extension*, J. Appl. Mech., **49**, 561–569 (1982)

[8] C. H. Wu, *Unconventional internal cracks, Part I: Symmetric variations of a straight crack*, J. Appl. Mech., **49**, 62–68 (1982)

[9] C. H. Wu, *Unconventional internal cracks, Part II: Methods of generating simple cracks*, J. Appl. Mech. **49**, 383–388 (1982)