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ON THE POTENTIAL IN A STRIP WITH A PAIR OF *U* NOTCHES

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1. Synopsis and introduction. The potential in a strip with a pair of V or U notches was solved by the author in a recent paper [1]. The solution is expressed as a linear combination of a set of harmonic functions in addition to some linear terms. Each harmonic function satisfies the requirements of the problem, except the boundary conditions on the curves of notch. Subsequently, one curve of notch is transformed conformally into a portion of the circumference of a unit circle, on which the boundary condition is adjusted. The corresponding condition on the curve of the other notch is automatically satisfied by antisymmetry.

The present paper is endeavored to give an alternate solution of the problem. Since the strip with V notches is perhaps the less crucial case, the solution is formulated with reference to the U notches only. The present solution differs from the previous one essentially in the following two aspects:

(i) The set of harmonic functions is a subset of the harmonic functions derived systematically from a single function.

(ii) The boundary conditions on the curves of notch are adjusted without the use of conformal transformation. Each curve is merely regarded as a piecewise continuous curve.

Finally, several numerical examples are given for illustration.

2. The problem. The geometry of the given strip in the xy plane is shown in Fig. 1. The lines $y = 0$ and $y = 2a$ are the lower and upper edges, respectively. The curve of the lower U notch is denoted by $ADCGA'$. AD and GA' are two parallel line segments, each of length h, intersecting the lower edge normally. They are connected smoothly by a semicircle DCG of radius λ. Thus the opening AA' of the notch is 2λ and the depth OC is $h + \lambda$. b is the distance from C to the line $y = a$ so that

$$a = h + \lambda + b. \tag{1}$$

The upper U notch is of the same size as the lower one and located symmetrically with respect to the line $y = a$. Let the potential on the lower boundary of the strip be unity and that on the upper boundary be negative unity.

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3. Set of harmonic functions. Define a function Q_0 of a complex variable $z = x + iy$ by

$$Q_0(z) = -\ln \sinh(\pi z/2a), \tag{2}$$

where a is a positive real constant. Again, define a set of functions Q_s by

$$Q_s(z) = \frac{(-1)^s}{(s-1)!} \frac{d^s}{dz^s} Q_0(z) \qquad (s \geqslant 1). \tag{3}$$

With the aid of the infinite product

$$\sinh \frac{\pi z}{2a} = \frac{\pi z}{2a} \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4n^2a^2}\right), \tag{4}$$

we find for $s \geqslant 1$,

$$Q_s(z) = \frac{1}{z^s} + \sum_{n=1}^{\infty} \left\{ \frac{1}{(z - 2nai)^s} + \frac{1}{(z + 2nai)^s} \right\}. \tag{5}$$

The sinh in (2) has zeros on the y axis at the points $z = 2nai$, where n takes all positive and negative integers and zero. These points are the poles of Q_s. Split Q_s into real and imaginary parts in the form:

$$Q_s(z) = S_s(x, y) - iT_s(x, y). \tag{6}$$

In particular, the subset of functions T_{2s-1} is useful in constructing the required solution. Some properties of T_{2s-1} are given below:

(i) It is harmonic in the entire xy plane, save at the poles.

(ii) It has poles on the y axis at the points $y = 2na$ of order $2s - 1$; n being an integer or zero.

(iii) It is periodic in y of periodicity $2a$.

(iv) It has a line of symmetry at $x = 0$ so that it is even in x.

(v) It has lines of antisymmetry at $y = na$ such that, save at the poles,

$$T_{2s-1}(x, na) = 0; \tag{7}$$

n being an integer or zero. Thus, it is odd in y.

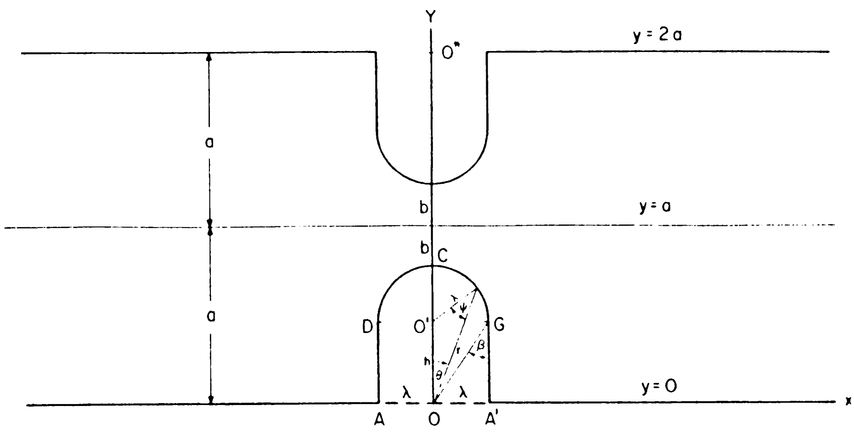

FIGURE 1. Strip with a pair of U notches.

The corresponding set of functions used in the previous paper [1] is H_{2s}. It is not difficult to see that

$$H_{2s}(x, y) = T_{2s+1}(x, y) \qquad (s \geqslant 0). \tag{8}$$

Unlike the present paper, the set H_{2s} was derived from an integral.

4. The solution. The required solution is likewise constructed in the form:

$$V(x, y) = 1 - \frac{y}{a} + \sum_{s=1}^{\infty} A_{2s}T_{2s-1}(x, y), \tag{9}$$

where A_{2s} are parametric coefficients. The function is harmonic, even in x, and antisymmetric with respect to the line $y = a$ as desired. It has two singularities with alternate signs on the edges of the strip, one at the origin and the other at the point $(x, y) = (0, 2a)$ or 0^*. Both singularities are excluded from the strip due to presence of the notches. The function gives potentials on the lower and upper edges of values 1 and -1, respectively, save at the singularities.

5. Boundary conditions on notches. The remaining boundary conditions to be satisfied are those on the notches. Each notch is a piecewise continuous curve. We define a pair of polar coordinates (r, θ) as follows:

$$z = ire^{-\theta}, \tag{10}$$

so that

$$x = r\sin\theta, \qquad y = r\cos\theta. \tag{11}$$

The right half CGA' of the lower notch is composed of two parts. One part is a circular arc CG with θ varying from 0 to β and the other part a line segment GA' with θ varying from β to $\pi/2$, where

$$\beta = \tan^{-1}(\lambda/h). \tag{12}$$

Let $z_0 = x_0 + iy_0$ be a point on the curve of lower notch. On the circular arc CG,

$$x_0 = \lambda\sin(\theta + \psi), \qquad y_0 = h + \lambda\cos(\theta + \psi), \tag{13}$$

and on the line segment GA',

$$x_0 = \lambda, \qquad y_0 = \lambda\cot\theta. \tag{14}$$

Here, when $\theta \leqslant \beta$,

$$\psi = \sin^{-1}(h\sin\theta/\lambda). \tag{15}$$

Thus, $T_{2s-1}(x_0, y_0)$ is a function of θ and so also is $V(x_0, y_0)$. By symmetry, they are even in θ. Hence, $V(x_0, y_0)$ can be expanded into a Fourier cosine series in θ over the range $-\pi/2$ to $\pi/2$ in the following form:

$$V(x_0, y_0) = \frac{1}{2}a_0 + \sum_{m=1}^{\infty} a_m\cos 2m\theta, \tag{16}$$

where, for $m = 0, 1, 2, \ldots,$

$$a_m = \frac{4}{\pi}\int_0^{\pi/2} V(x_0, y_0)\cos 2m\theta\, d\theta. \tag{17}$$

The boundary condition of unit potential on the curve of lower notch is therefore satisfied if for $m = 0, 1, 2, \ldots,$

$$a_m = 2\delta_{0,m}, \tag{18}$$

where $\delta_{n,m}$ is a Kronecker delta. This leads to a set of equations in A_{2s} as follows: For $m = 0, 1, 2, \ldots,$

$$\sum_{s=1}^{\infty} {}^{m}\alpha_{2s} A_{2s} = 2\delta_{0,m} - g_m, \tag{19}$$

where

$$\begin{aligned} g_m &= \frac{4}{\pi}\int_0^{\pi/2}\left(1 - \frac{y_0}{a}\right)\cos 2m\theta\, d\theta \\ &= 2\delta_{0,m} - \frac{4}{\pi a}\int_0^{\pi/2} y_0 \cos 2m\theta\, d\theta, \\ {}^{m}\alpha_{2s} &= \frac{4}{\pi}\int_0^{\pi/2} T_{2s-1}(x_0, y_0)\cos 2m\theta\, d\theta. \end{aligned} \tag{20}$$

By antisymmetry, when the boundary condition on the lower notch is satisfied, the relevant boundary condition on the upper notch is automatically satisfied.

6. Evaluation of coefficients and solution of equations. Simpson rule is used in evaluating the integrals in (20). In particular, the integral g_0 is readily integrable. The value is

$$g_0 = 2 - \frac{h}{\pi a}\{4\beta + \pi \tan^2\beta - (2\beta - \sin 2\beta)\sec^2\beta - 4\tan\beta \ln \sin\beta\}, \tag{21}$$

which may serve as a check. The function T_{2s-1} involved in the second integral is given by

$$T_{2s-1}(x_0, y_0) = \mathrm{Re}[iQ_{2s-1}(z_0)]. \tag{22}$$

Here, Q_s may be evaluated from the following:

$$\begin{aligned} Q_1(z) &= (\pi/2a)\coth(\pi z/2a), \\ Q_2(z) &= (\pi/2a)^2 \operatorname{csch}^2(\pi z/2a), \\ Q_3(z) &= Q_1(z)Q_2(z), \end{aligned} \tag{23}$$

and recurrently for $s \geqslant 1$,

$$Q_{s+3}(z) = \frac{2}{(s+1)(s+2)}\sum_{n=0}^{s}(n+1)Q_{n+2}(z)Q_{s-n+1}(z). \tag{24}$$

Or, alternately, T_{2s-1} may be evaluated directly for the points on the curve from the series:

$$T_{2s-1}(x_0, y_0) = \mathrm{Re}\left[\frac{i}{z_0^{2s-1}}\right] + \sum_{n=1}^{\infty}\mathrm{Re}\left[\frac{i}{(z_0 - 2nai)^{2s-1}} + \frac{i}{(z_0 + 2nai)^{2s-1}}\right], \tag{25}$$

which converges rapidly when s is large.

The set of equations in (19) can be solved by matrix inversion or otherwise after truncated into a finite set containing the first N equations and the first N coefficients of A_{2s}. The integer N may be called an index of truncation. When the equations are solved, it is then straightforward to compute the potential at any point in the strip. In particular, the potential along the line $x = 0$ across the narrowest section of the strip is

$$V(0, y) = 1 - \frac{y}{a} + \sum_{s=1}^{\infty} A_{2s} T_{2s-1}(0, y) \qquad (a - b \leqslant y \leqslant a). \tag{26}$$

7. Numerical examples. The solution is illustrated by the following numerical examples:

b	1	1	1
λ	0.5	0.4	0.3
h	0.5	0.6	0.7
a	2	2	2

In applying the Simpson rule, the integrand of each integral is divided into S double subintervals. To ascertain the accuracy, the computation is repeated by increasing S by 10% until the resulting value becomes stable at the desired accuracy. The values of the potential $V(0, y)$ are shown in Table 1 to $4D$ together with the values of S and N. The computation indicates that the coefficient A_{2s} forms an alternating sequence and the series in (26) is a positive series with good convergence. The effect of truncation is generally small whenever a suitable N is chosen.

TABLE 1. Values of $V(0, y)$ across narrowest section $x = 0$ for $a = 2$ and $b = 1$.

y	$\lambda = 0.5$ $h = 0.5$	$\lambda = 0.4$ $h = 0.6$	$\lambda = 0.3$ $h = 0.7$
1	1.0000	1.0000	0.9984
1.1	0.8604	0.8541	0.8451
1.2	0.7396	0.7312	0.7202
1.3	0.6307	0.6220	0.6112
1.4	0.5299	0.5219	0.5121
1.5	0.4349	0.4279	0.4195
1.6	0.3439	0.3381	0.3313
1.7	0.2557	0.2513	0.2462
1.8	0.1694	0.1665	0.1631
1.9	0.0844	0.0830	0.0812
2	0.0000	0.0000	0.0000
Remarks	$S = 100$ $N = 12$	$S = 100$ $N = 14$	$S = 100$ $N = 16$

In the examples as shown, the value of $V(0, y)$ at $y = 1$ or the point C of the notch is unity. The computed value at C shows, however, a slight descrepancy in the case $\lambda = 0.3$ of amount 0.16%. Further computation reveals that such a discrepancy increases rapidly with decrease of λ.

REFERENCES

[1] C. B. Ling, *Potential related to a strip with a pair of V or U notches*, Quarterly of Applied Mathematics, **41**, 369–377 (1983)