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ON THE PROPERTIES OF ENTROPY RATE IN THE ENTROPY RATE ADMISSIBILITY CRITERION*

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1. Introduction. In this paper we discuss the properties of entropy rate, along shock curves and phase boundary curves, associated with the system of conservation laws

$$\begin{aligned} u_t + p_x &= 0, \\ v_t - u_x &= 0, \\ E_t + (pu)_x &= 0. \end{aligned} \tag{1.1}$$

The above system expresses the one dimensional flow of a compressible fluid in Lagrangian coordinates. In this system, u, v, E, and p denote the velocity, the specific volume, the total energy, and the pressure of the fluid, respectively. Here, the total energy is given by $E = e + \frac{1}{2}u^2$, where e is the internal energy, and the pressure is a function of two state variables, namely,

$$p = p(v, e) = p(v, s) = p(v, \theta), \tag{1.2}$$

where s and θ are the entropy and the temperature, respectively.

The entropy rate arises in the entropy rate admissibility criterion proposed by Dafermos [1], [2]. It is well known that weak solutions to the Cauchy problem for a system of conservation laws

$$u_t + f(u)_x = 0 \tag{1.3}$$

are not unique (by weak solutions we mean bounded measurable functions which satisfy (1.3) in the sense of distributions). In order to choose a physically relevant solution, Dafermos has proposed the above admissibility criterion. This criterion roughly says that the entropy decays with the highest possible rate for the admissible solution (the entropy increases if it is the physical entropy).

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Originally, this criterion was proposed for hyperbolic systems. Recently, the applicability of this criterion has been extended in [3] to a nonhyperbolic system given by

$$\begin{aligned} u_t + p(v)_x &= 0, \\ v_t - u_x &= 0. \end{aligned} \tag{1.4}$$

Unlike in an ideal gas, we assume that $p'(v)$ is positive on an interval (α, β), where (1.4) becomes elliptic. A typical example of this type of fluid is a van der Waals fluid.

This paper consists of 4 sections. In Sec. 2 we explain the assumption on system (1.1) and the terminology necessary to subsequent sections. Specifically, we introduce the van der Waals fluid and describe the shock and phase boundary curves and the entropy rate admissibility criterion. In Sec. 3 we consider the entropy rate for the system (1.4) for completeness. In Sec. 4 we consider the entropy rate for the system (1.1). We shall show that the entropy rate for system (1.1) has similar properties to that for system (1.4).

2. Preliminaries.

A. *Assumptions for system* (1.1). In system (1.1), the main assumptions in the hyperbolic case are

$$p_v(v, e) < 0, \quad p_e(v, e) > 0, \quad p_s(v, s) > 0. \tag{2.1}$$

If we use v and s as the state variables for the pressure, the characteristic speeds are

$$\lambda_1 = -\sqrt{-p_v(v, s)}, \quad \lambda_2 = 0, \quad \lambda_3 = \sqrt{-p_v(v, s)}. \tag{2.2}$$

From the thermodynamic relation

$$de = -pdv + \theta\, ds, \tag{2.3}$$

we infer

$$p_v(v, s) = -pp_e(v, e) + p_v(v, e). \tag{2.4}$$

Assumptions (2.1) and the above relation assure the hyperbolicity of (1.1).

REMARK 2.1. In the above argument we use u, v and E as the state variables of the system. In what follows, we continue to do so, unless stated explicitly otherwise. See Liu [4] for the detailed theory of system (1.1).

If we use u, v, and θ as the state variables for system (1.1), the main assumptions for hyperbolicity are

$$p_v(v, \theta) < 0, \qquad e_\theta(v, \theta) > 0, \tag{2.5}$$

and the characteristic speeds become

$$\lambda_1^2, \quad \lambda_3^2 = -p_v(v, \theta) + \frac{\theta p_\theta^2}{e_\theta}, \quad \lambda_2 = 0. \tag{2.6}$$

We assume, in the nonhyperbolic case, that there is a region of v and s in which

$$p_v(v, s) > 0 \tag{2.7}$$

is satisfied. We also assume that

$$p_e(v, e) > 0, \quad p_s(v, s) > 0, \quad e_\theta(v, \theta) > 0. \tag{2.8}$$

are satisfied even in the nonhyperbolic region.

REMARK 2.2. It is interesting to observe that because of (2.8c) the condition $p_v(v, \theta) > 0$ does not necessarily imply that the system (1.1) is nonhyperbolic. The region where $p_v(v, \theta) > 0$ is called the spinoidal region and is assume to be unstable [5]. As we will see in Section 2.B, the condition (2.8) is satisfied for a van der Waals fluid.

B. *van der Waals Fluid*. If we use v and θ as the state variables, the equation of state for a van der Waals fluid is given by

$$p(v, \theta) = \frac{R\theta}{v - b} - \frac{a}{v^2}, \tag{2.9}$$

where R, a, and b are positive constants. In Fig. 1 a few isotherms are drawn. If the temperature is below the critical temperature given by $\theta_c = 8a/27bR$, the isotherm has the following features:

(i) $p_v(v, \theta_0) < 0$ on $(b, \alpha) \cup (\beta, \infty)$,
(ii) $p_v(v, \theta_0) > 0$ on (α, β),
(iii) $p_v(\alpha, \theta_0) = p_v(\beta, \theta_0) = 0$.

The domain (b, α) is called the α-phase (the liquid phase), and the domain (β, ∞) is called the β-phase (the vapor phase). As stated before, the domain (α, β) is assumed to be unstable and is referred to as the spinoidal region.

In the isothermal case the horizontal line for which the areas A and B are equal is called the Maxwell line. We denote the pressure at the Maxwell line by p_m. The values of v in the α-phase and the β-phase at which the pressure is equal to p_m are denoted by α_m and β_m, respectively. For the equilibrium case in the isothermal flow, the liquid and the vapor coexist if the pressure is equal to p_m. This is based on the Gibbs function and discussed in [6].

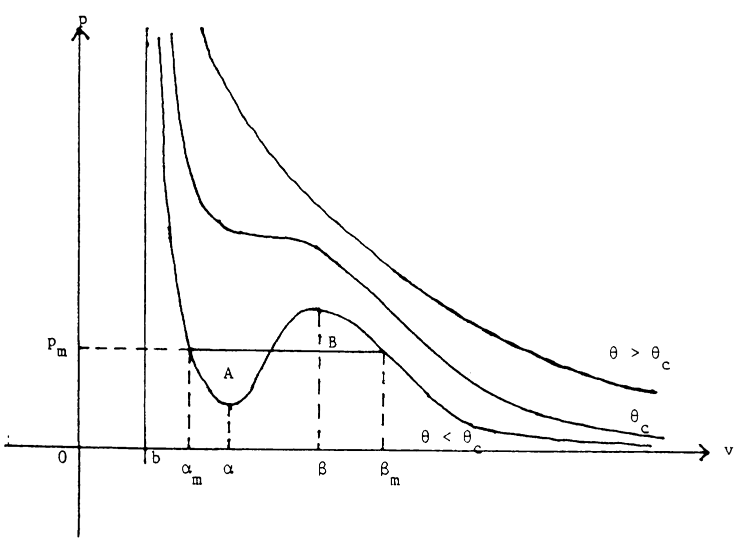

FIG. 1. Isotherms for different values of θ.

Since we need the thermodynamic relations, we summarize them in the context of the van der Waals fluid. Suppose ψ is the specific Helmholtz free energy, then we have

$$s = -\frac{\partial \psi}{\partial \theta}, \qquad p = -\frac{\partial \psi}{\partial v}. \tag{2.11}$$

For the van der Waals equation of state given by (2.9), using (2.11) we get

$$\psi = -R\theta \ln(v - b) - \frac{a}{v} + F(\theta),$$

where $F(\theta)$ is an arbitrary function of θ. As $e = \psi + \theta s$, we have

$$e(v, \theta) = -a/v + F(\theta) - F'(\theta).$$

Following [6], [7], if we set

$$F(\theta) = -c_v \theta \ln \theta + \text{constant},$$

where c_v is a positive constant, then we obtain

$$\begin{aligned} e(v, \theta) &= -a/v + c_v \theta + \text{constant}, \\ s(v, \theta) &= R \ln(v - b) + c_v \ln \theta + c_v. \end{aligned} \tag{2.12}$$

Using (2.12), we can express the van der Waals equation of state (2.4) in terms of v and e or v and s.

C. *Shock, contact discontinuity, and phase boundary curves.* A jump discontinuity is a singular line across which the Rankine-Hugoniot condition is satisfied. For the system (1.1), the conditions are given by

$$\begin{aligned} \sigma[u - u_L] &= [p - p_L], \\ \sigma[v - v_L] &= -[u - u_L], \\ \sigma[E - E_L] &= [pu - p_L u_L], \end{aligned} \tag{2.13}$$

where σ is the speed of propagation of the jump discontinuity, and (u, v, E) and (u_L, v_L, E_L) are the states on the left and the right of the jump discontinuity, respectively. There are three types of jump discontinuities, namely, the shock, the contact discontinuity, and the phase boundary. The states on both sides of jump discontinuity belong to the same phase in case of a shock or a contact discontinuity, and belong to different phases in case of a phase boundary. A shock with a positive (negative) speed is called a forward (backward) shock. The same terminology applies to a phase boundary.

For a given (u_L, v_L, E_L) the set of (u, v, E) forms a one parameter family of states which can be connected to (u_L, v_L, E_L) on the right by a jump discontinuity. We call this set a forward (backward) shock curve, a contact discontinuity curve, or a forward (backward) phase boundary curve, depending on the type of jump discontinuity. We employ the idea of Liu [4], and denote the above parameter by ξ. Then, differentiating (2.13) with respect to ξ, we obtain the different equations

$$du/d\xi = h_i, \tag{2.14}$$

where $u = (u, v, E)^T$ and $h_i (i = 1, 2, 3)$ are given by

$$h_1^T = \left(1, \frac{2\sigma - p_e(u - u_L)}{p_v - \sigma^2 + p_e(p - p_L)},\right.$$
$$\left. u + \frac{(u - u_L)(p_v + \sigma^2) + (u - u_L)pp_e - 2\sigma p}{p_v - \sigma^2 + p_e(p - p_L)}\right),$$
$$h_2^T = (0, p_e, -\bar{p}_v), \tag{2.15}$$
$$h_3^T = \left(1, \frac{2\sigma - p_e(u - u_L)}{p_v - \sigma^2 + p_e(p - p_L)},\right.$$
$$\left. u + \frac{(u - u_L)(p_v + \sigma^2) + (u - u_L)pp_e - 2\sigma p}{p_v - \sigma^2 + p_e(p - p_L)}\right).$$

Here, $p_v = p_v(v, s)$, $\bar{p}_v = p_v(v, e)$, and $p_e = p_e(v, e)$. The equation (2.14; $i = 1$) is for the backward shock and the backward phase boundary and (2.14; $i = 3$) is for the forward shock and the forward phase boundary. Therefore, σ is negative in (2.14; $i = 1$) and positive in (2.14; $i = 3$). The equation (2.14; $i = 2$) is for the contact discontinuity, and σ is zero in this case.

In the isothermal case (1.4), the Rankine-Hugoniot conditions are given by

$$\begin{aligned} \sigma[u - u_L] &= [p - p_L], \\ \sigma[v - v_L] &= -[u - u_L], \end{aligned} \tag{2.16}$$

where (u, v) and (u_L, v_L) are the states on the right and the left of the jump discontinuity. Solving (2.16) for σ, we obtain

$$\sigma = \pm \sqrt{-\frac{p - p_L}{v - v_L}}\,. \tag{2.17}$$

On substituting (2.17) in (2.16b), we obtain

$$u - u_L = \mp \sqrt{-\frac{p - p_L}{v - v_L}}\,(v - v_L). \tag{2.18}$$

The set of (u, v) which is connected by a shock or a phase boundary on the right to a given (u_L, v_L) satisfies the above relation, and forms a shock curve or a phase boundary curve.

D. *Entropy rate admissibility criterion.* A convex function $\eta(u)$ is called an entropy for (1.3), with entroy flux $q(u)$, if

$$\eta(u)_t + q(u)_x = 0$$

holds identically for any smooth vector field $u(x, t)$ which satisfies (1.3), i.e., if

$$\sum_{j=1}^{n} \frac{\partial \eta}{\partial u_j} \cdot \frac{\partial f_j}{\partial u_k} = \frac{\partial q}{\partial u_k}, \qquad k = 1, \ldots, n.$$

Dafermos [1], [2] has proposed the entropy rate admissibility criterion to choose an admissible weak solution. A solution $u(x, t)$ will be called admissible if there is no solution $\bar{u}(x, t)$ with the property that for some $\eta \in [0, T]$, $u(x, t) = \bar{u}(x, t)$ on $(-\infty, \infty) \times [0, \tau]$ and $D_+ H_{\bar{u}}(\tau) < D_+ H_u(\tau)$, where $\eta(u)$ is an entropy and

$$H_u(t) = \int_{-\infty}^{\infty} \eta(u(x, t))\, dx.$$

For the system (1.1), we use the minus physical entropy as the entropy. The corresponding entropy flux is zero (see Hsiao [8]). Then, the entropy rate is given by

$$D_+ H_u(\tau) = \sum_{\substack{\text{jump} \\ \text{discontinuities}}} \sigma(s_R - s_L). \tag{2.17}$$

where s is the physical entropy, and s_R and s_L are the values of entropy on the right and the left of the jump discontinuity. Notice that along a shock or a phase boundary curve $D_+ H$ does not depend on τ.

For the isothermal case (1.4) we use the mechanical energy $\frac{1}{2}u^2 + \int^v(-p(w))\, dw$ as the entropy. The corresponding entropy flux is given by $up(v)$. For the above choice, the rate of entropy decay is

$$D_+ H_u(\tau) = \sum_{\substack{\text{jump} \\ \text{discontinuites}}} \sigma\left\{ -\frac{1}{2}(p(v_R) + p(v_L))(v_R - v_L) + \int_{v_L}^{v_R} p(w)\, dw \right\}, \tag{2.18}$$

where v_R and v_L are the values of v on the right and the left of the jump discontinuity. Since the mechanical energy is employed as the entropy, it may be more appropriate here to refer the criterion as the energy rate admissibility criterion.

It will be interesting to study the properties of entropy rate along the shock and the phase boundary curves. As mentioned above, in this case $D_+ H$ does not depend on time, so that we will use Φ for the entropy rate.

3. Isothermal case. The entropy rate along a shock curve or a phase boundary curve in the isothermal case is given by

$$\Phi(v; v_L) = \sigma\left\{ -\frac{1}{2}[p(v) + p(v_L)](v - v_L) + \int_{v_L}^{v} p(w)\, dw \right\}, \tag{3.1}$$

where σ is given by (2.17) (we have omitted the subscript R).Differentiation of Φ implies

$$\frac{d\Phi}{dv} = -\frac{1}{4\sigma}(-\lambda^2 + \sigma^2)\left(3p - p_L - \frac{2\int_{v_L}^{v} p(w)\, dw}{v - v_L}\right), \tag{3.2}$$

where $\lambda = \sqrt{-p'(v)}$. Although most of the results are discussed in [3], we summarize the properties of Φ in this section for the completeness.

THEOREM 3.1. For shocks the contact of Φ at $v = v_L$ to the line $\Phi = 0$ is at least of the third order. Namely, $d\Phi/dv$ and $d^2\Phi/dv^2$ are zero at $v = v_L$.

The above theorem is an easy consequence of (3.2) and Taylor's expansion. If $p'' \neq 0$, we have the monotonicity property for the entropy rate along the shock curves.

THEOREM 3.2. If $p'' > 0$ and $p' < 0$, then the entropy rate is a decreasing (increasing) function of v along the forward (backward) shock curves.

Proof. If $p'' > 0$, we see that

$$\lambda^2 \gtrless \sigma^2, \quad p \gtrless p_L, \quad p \gtrless \frac{\int_{v_L}^{v} p(w)\,dw}{v - v_L}, \quad \text{for } v \gtrless v_L.$$

Since $\lambda^2 - \sigma^2$ is negative (positive) along the forward (backward) shock curves, $d\Phi/dv$ is negative (positive) along the forward (backward) shock curves. Q.E.D.

REMARK 3.1. If $p'' < 0$ and $p' < 0$, "decreasing" and "increasing" should be interchanged in the above theorem.

In the following theorem we show that the entropy rate along the phase boundary curves is not necessarily monotone. We assume that v_L is close to α_m, so that there exists v_0 in the β-phase such that $P(v_0) = p(v_L)$. We also assume that if v_L is less than α_m, then there exists v_1 in the β-phase such that

$$\frac{1}{2}(p(v_1) + p(v_L))(v_1 - v_L) - \int_{v_L}^{v_1} p(w)\,dw = 0.$$

If this value v_1 exists, v_1 should be less than the value v_2 at which $\sigma^2 - \lambda^2 = 0$ (see Fig. 2).

THEOREM 3.3. Suppose $p(v)$ has the form of the graph in Fig. 2 and $0(v^{-\gamma})(\gamma > 0)$ as v approaches infinity. Then, if $v_L < \alpha_m$ and there exist v_0, v_1, and $v_2 (v_0 < v_1 < v_2)$, then Φ has a relative minimum (maximum) between v_0 and v_1 and has a relative maximum (minimum) at v_2 along the forward (backward) phase boundary curves. If $v_L \geqslant \alpha_m$, and v_0 and v_2 exist, then Φ has a relative maximum (minimum) at v_2 along the forward (backward) phase boundary curves. In either case, Φ approaches minus (plus) infinity as v approaches infinity.

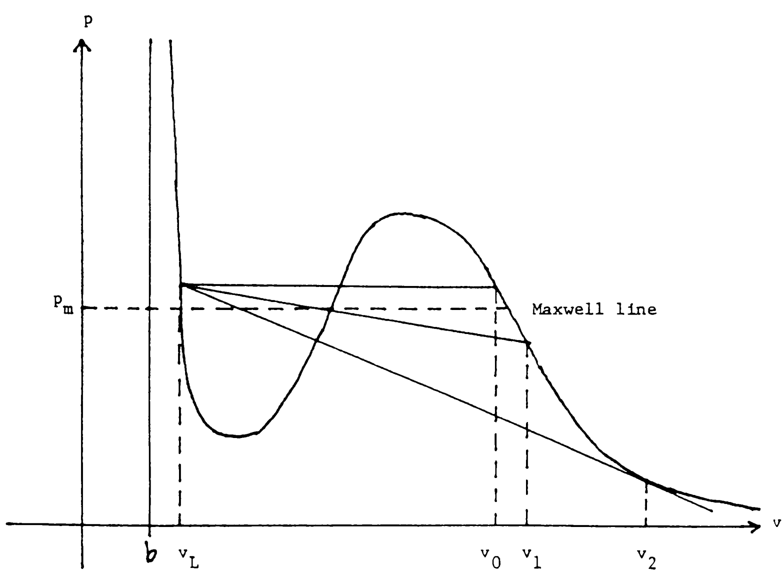

FIG. 2. Relation for v_0, v_1, and v_2.

Proof. If we set

$$Q = (v - v_L)(3p - p_L) - 2\int_{v_L}^{v} p(w)\,dw,$$

we see that dQ/dv is negative. Therefore, if $v_L < \alpha_m$, Q is positive at v_0 and negative at v_1, and if $v_L \geqslant \alpha_m$, Q is nonpositive. It is easy to see that

$$\frac{1}{2}(p + p_L)(v - v_L) - \int_{v_L}^{v} p(w)\,dw$$

approaches infinity as v approaches infinity, if p is $0(v^{-\gamma})$ as v approaches infinity. Q.E.D.

4. Nonisothermal case. In this section we study the properties of entropy rate of the system (1.1) along the shock curve and the phase boundary curves given by (2.5; $i = 1, 3$). The entropy rate along a shock curve or a phase boundary curve is given by

$$\Phi(\xi) = \sigma(\xi)(s(\xi) - s_L), \tag{4.1}$$

where ξ is the parameter introduced in Sec. 2B., s is the physical entropy, and σ is the speed of the jump discontinuity given by

$$\sigma = \pm\sqrt{-\frac{p - p_L}{v - v_L}}\,.$$

We shall show that Φ in (4.1) has similar properties to Φ in (3.1). We shall, also, see that $d\Phi/d\xi$ is related to the adiabatic transformation [6].

First, we discuss the properties of the entropy rate along the shock curves. In this case we set the parameter ξ so that $(u(\xi), v(\xi), E(\xi)) = (u_L, v_L, E_L)$ at $\xi = 0$.

THEOREM 4.1. The contact of $\Phi(\xi)$ at $\xi = 0$ to the line $\Phi = 0$ is at least of the third order. Namely, $d\Phi/d\xi$ and $d^2\Phi/d\xi^2$ are zero at $\xi = 0$.

Proof. If we differentiate Φ, we have

$$\frac{d\Phi}{d\xi} = \frac{d\sigma}{d\xi}(s - s_L) + \sigma\frac{ds}{d\xi}. \tag{4.2}$$

To see how each term on the right hand side behaves as ξ approaches zero, we differentiate the Rankine-Hugoniot conditions (2.13a,b) to obtain

$$\begin{aligned} &\frac{d\sigma}{d\xi}(u - u_L) + \sigma = p_s\frac{ds}{d\xi} + p_v\frac{dv}{d\xi}, \\ &\frac{d\sigma}{d\xi}(v - v_L) + \sigma\frac{dv}{d\xi} = -1. \end{aligned} \tag{4.3}$$

Using

$$\frac{dv}{d\xi} = \frac{2\sigma - p_e(u - u_L)}{p_v - \sigma^2 + p_e(p - p_L)}, \tag{4.4}$$

we find

$$\frac{ds}{d\xi} = \frac{p_e(\sigma^2 + p_v)(u - u_L)}{p_s\{p_v - \sigma^2 + p_e(p - p_L)\}},$$
$$\frac{d\sigma}{d\xi} = -\frac{\sigma^2 + p_v}{(v - v_L)\{p_v - \sigma^2 + p_e(p - p_L)\}}. \tag{4.5}$$

Substituting (4.5) in (4.2a), we see

$$\frac{d\Phi}{d\xi} = -\frac{\sigma^2 + p_v}{p_v - \sigma^2 + p_e(p - p_L)}\left\{\frac{s - s_L}{v - v_L} - \frac{p_e(p - p_L)}{p_s}\right\}. \tag{4.6}$$

If we differentiate (4.3) and (4.4), we find that $d^2s/d\xi^2$ is zero at $\xi = 0$. It follows that $(\sigma^2 + p_v)$ and $(s - s_L)/(v - v_L)$ approach zero as ξ approaches zero. This implies that $d\Phi/d\xi$ and $d^2\Phi/d\xi^2$ approaches zero as ξ approaches zero. Q.E.D.

Concerning the monotonicity of the entropy rate along the shock curves, we have the following

THEOREM 4.2. Along the forward shock curves the following relations hold:

(i) $d\Phi/d\xi > 0$ if $\sigma^2 + p_v < 0$,

(ii) $d\Phi/d\xi < 0$ if $\sigma^2 + p_v > 0$.

Along the backward shock curves we have the following relations:

(iii) $d\Phi/d\xi < 0$ if $\sigma^2 + p_v < 0$,

(iv) $d\Phi/d\xi > 0$ if $\sigma^2 + p_v > 0$.

Proof. Consider case (i). In (4.6) using relation (2.4), we obtain

$$p_v - \sigma^2 + p_e(p - p_L) = \bar{p}_v - \sigma^2 - p_e p_L < 0.$$

Next, we examine the sign of the quantity inside the braces in (4.6). If ξ is positive, from (2.13) we see that $p > p_L$ and $v < v_L$, and from (4.5a) we find $ds/d\xi > 0$, therefore $s > s_L$. If ξ is negative, $p < p_L$, $v > v_L$, and $s < s_L$. Hence, we conclude that $d\Phi/d\xi > 0$ if $\sigma < \lambda$ along the forward shock curves. The other cases are proved in a similar manner. Q.E.D.

In the following, we consider the entropy rate along the phase boundary curves. As stated in Section 2, we assume that there is a region in which p_v is positive (this guarantees the existence of the region in which $p_v(v, \theta)$ is positive). We also assume that it is possible to choose the values of v_L in the α-phase, so there exist values of v in the β-phase at which σ is zero. We adjust the parameter ξ so that σ is zero at $\xi = 0$. The question is whether or not the entropy rate is monotone along the phase boundary curves. Checking the sign of (4.6), we obtain the following

THEOREM 4.3. Suppose $\sigma^2 + p_v$ does not change sign (namely, remain less than zero) along the phase boundary curves as v varies in the β-phase. If $s < s_L$ and s and s_L are close, then Φ has a relative minimum (maximum) for the forward (backward) phase boundary curves as v increases. If $s \geqslant s_L$, then Φ is monotone along the phase boundary curves.

Proof. Since v_L is in the α-phase and v is in the β-phase, v is always greater than v_L. Then from (2.13) and from the fact that σ is real, $u < u_L (u > u_L)$ for forward (backward) phase boundaries. Therefore, ξ has to decrease (increase) along forward (backward) phase boundaries. Let us now examine the sign of $d\Phi/d\xi$. In this case the second term in the braces in (4.6) is nonpositive. From (4.5), we see that $ds/d\xi < 0$ and $\xi < 0$ along forward phase boundaries and that $ds/d\xi > 0$ and $\xi > 0$ along backward phase boundaries. This implies that if s at $\xi = 0$ is less than s_L and close to s_L, $s - s_L$ will change sign as v increases. Therefore, $d\Phi/d\xi$ changes sign from positive to negative along the forward (backward) phase boundary curves as ξ decreases (increases). Hence, Φ has a relative minimum (maximum) along forward (backward) phase boundary curves. It is easy to see that Φ is monotonically increasing (decreasing) along the forward (backward) phase boundaries, if $s \geqslant s_L$. Q.E.D.

Several remarks are in order concerning the entropy rate.

REMARK 4.1. If v is in the β-phase $\sigma^2 + p_v$ might change sign as v increases along the phase boundary curves, as in the isothermal case. Whether we can set s arbitrarily close to s_L at $\xi = 0$ is an open question. If the pressure and the entropy are given by (2.9) and (2.12), respectively, then

$$p(v, s) = \frac{R\exp(s/c_v - 1)}{(v - b)^{1 + R/c_v}} - \frac{\sigma}{v^2}.$$

Since c_v is greater than R, it is not difficult to show that if the entropy is a small constant s_0, then $p(v, s_0)$ has essentially the same features as (2.10). Then, from the continuity we can easily choose s arbitrarily close to s_L satisfying $p = p_L$.

REMARK 4.2. At $\xi = 0$ ($\sigma = 0$), $d\Phi/d\xi$ for the phase boundary is given by

$$\left.\frac{d\Phi}{d\xi}\right|_{\xi=0} = -\frac{s - s_L}{v - v_L}. \tag{4.7}$$

If there is no shock, and the stationary phase boundary which joins two constant states is observed, then $d\Phi/d\xi$ should be zero at $\xi = 0$. If the fluid is a van der Waals fluid in which the entropy is given by (2.12), then substituting (2.12) in (4.7), we find

$$\left(\frac{\theta}{\theta_L}\right)^{c_v}\left(\frac{v - b}{v_L - b}\right)^R = 1. \tag{4.8}$$

Since the specific volume in the vapor phase v is greater than that in the liquid phase v_L, this implies that if there is no shock and the stationary phase boundary separating two constant states is observed, the temperature in the vapor phase is lower than the temperature in the liquid phase. This is in accordance with the result of Slemrod [7]. We should note that (4.8) is a necessary condition. It is interesting to see if the above argument is reasonable, at least, in terms of the Riemann problem. It is also interesting to note that (4.8) is the same as the adiabatic transformation [6].

REMARK 4.3. In Sec. 2 we used v as the parameter. As a matter of fact we can use u as the parameter as in this section. Then, from (2.16) we see

$$\frac{d\sigma}{du}(u - u_L) + \sigma = p'\frac{dv}{du},$$

$$\frac{d\sigma}{du}(v - v_L) + \sigma\frac{dv}{du} = -1.$$

Then,

$$\frac{d\Phi}{du} = \frac{\sigma^2 - \lambda^2}{2(\sigma^2 + \lambda^2)}\left(3p - p_L - \frac{2\int_{v_L}^{v} p(w)\,dw}{v - v_L}\right).$$

Since $d\Phi/du$ does not have a singularity at $\sigma = 0$, this parametrization may be better.

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