UNSTEADY TRANSONIC FLOWS IN AN AXIAL FLOW COMPRESSOR*

By

A. R. MANWELL

Cambridge, England

Introduction. This paper deals with some transonic flow problems for the supersonic axial flow compressor: these arise in connection with flutter analysis for fan blading. Very detailed solutions for unsteady plane transonic flows through a cascade of flat plates have been given by Goldstein Braun and Adamczyk (1977) and Nagashima and Whitehead (1977), see also Goldstein (1975), Kurosaka (1974), Mani and Horvay (1970), Miles (1959), Morfey and Fisher (1970) and Verdon and McCune (1975). The present paper treats the subject from the point of view of boundary value problems for the reduced wave equations of supersonic and subsonic flow, with particular reference to (a) the rotor in a finite duct and (b) a non-uniform incident stream.

In Secs. (1) and (2) we solve the supersonic entry problem for $\sigma = 0, \pi$ by Fredholm theory. If the frequencies are small we can show that the solutions depend on the duct length, in some cases very significantly. The uniqueness proof of Sec. (3) requires strong conditions on the incident stream and, see App. (B), we give a counter-example for a differential equation of the same general class. If we could strengthen this result to do the same for a specific case of entry solutions in the two-dimensional model of the ducted rotor, (i.e. for Eq. (3.1) with given ψ) then we would have a basis for a strict analytical discussion of the 'buzz-saw' phenomenon, cf. Morfey and Fisher *loc cit.* Sec. (4) shows that the exit problems for the subsonic stream ought likewise to be set in a bounded region.

The above analysis of the differential equations has some interesting points of contact with the physical arguments of Ferri (1940), (1964) and Kantrowitz (1950). Ferri envisaged a supersonic biplane such that the external shock wave system might be eliminated. Kantrowitz, having developed a working supersonic compressor, explained its operation in terms of a two-dimensional model which is now standard, c.f. Levine (1957), Stratford and Sansome (1959), Hutton (1974) and Lichtfuss and Starken (1974), from an extensive literature.

* Received July 25, 1984.

Kantrowitz reasoned that in order to attain a quasi-smooth steady entry flow, avoiding instability due to strong shocks extending far ahead of the rotor face, we need a precise balance between the weak shock compressions associated with the forward blade edges and the (simple wave) expansions over the rear blade segments. If this balance is not achieved then only the propagation of unsteady waves upstream can restore quasi-smooth flow. Ferri (1964), p. 387, goes further, proposing that the interference between the unsteady waves coming off successive blades produces changes in the axial component of velocity only: this is, in effect, condition (A) of Secs. (1), (2) below.

Because the physical flow problem involves such a fine balance the formulation of the boundary problem for the two-dimensional model needs careful study. In Secs. (1) (2) we propose three possibilities: either of these might prove most appropriate under various prescribed working conditions and each involves a finite entry region. The investigation of (G), on the other hand, is based on the cascade in an unbounded stream. Here the uniqueness proofs, Manwell (1981) for the entry flows and Sec. (1) below for the flow behind the in-passage shock, certainly support the analysis. However, see App. (A) below, the necessary delicate balance has been achieved only by the introduction of a small dissipation and then, only in the strictly analytical sense. Difficulties arise as soon as we look into the effect of duct length or, again, non-uniformities in the entry flow. From the physical point of view it would therefore seem preferable to keep to the usual treatment of acoustic waves in a duct where end conditions are of basic importance.

1. The Kantrowitz model for supersonic entry flow; uniqueness proofs for supersonic and subsonic cascade flows. In the idealisation of flow into a ducted rotor which provides the two-dimensional cascade model, see Figs. (1) (2), we first suppose the rotor blades to be very stubby and consider a quasi-two-dimensional flow in a thin annular region bounded by circular cylinders each coaxial with the axis of the rotor. (This primitive model can be elaborated by 'strip theory'.) Thus we agree to ignore all radial motion, centripetal forces are not considered and the velocity along OX , see Fig. (2), corresponds to the axial component measured relative to the moving blade under consideration. Similarly, the velocity along CC' in the model corresponds to the 'swirling' component of the (unsteady) flow measured at the plane of entry to the duct. Ferri's hypothesis suggests that this latter component will be determined by the incident stream, or better, by the mean steady flow. We shall therefore set up a boundary value problem for slightly perturbed, usually unsteady, flows, prescribing (a) φ , the velocity potential along CC' , (b) the velocity component normal to the blade segment and (c) the usual blade-to-blade periodicity (1.5) below. These conditions uniquely define the corresponding solutions of the reduced wave equation for a uniform stream and this result can be extended to slightly non-uniform incident flows. We call this Problem (A).

There are, on a linear theory, other possible choices of the entry conditions at CC' . For example we might fix the perturbation pressure and, compare (G), Eq. (2.5) take as the independent variable

(B)

$$p = (i\omega_1 - \partial/\partial x)\varphi \quad (1.1)$$

where ω_1 is a frequency parameter. Again, there is a (formal) generalisation of the outgoing-wave condition of Sommerfeld, originally stated for the Helmholtz equation but often applied to the analysis of waves in subsonic flow. The analogue for entry conditions in the case of a finite duct would appear to be best expressed by setting $\tilde{\varphi} = 0$ at CC' where, compare Copson (1975) p. 227 *et seq* also Courant and Hilbert (1962), Ch. IV, §5, we set

(C)

$$\tilde{\varphi} = (iM_1\omega_1 - \partial/\partial x)\varphi. \quad (1.2)$$

Here $M_1 > 1$ denotes the Mach number of the entry flow and the right hand member is the usual Helmholtz expressions after the introduction of non-dimensional parameters. Here and in what follows the notation is that of (G).

Let φ_1, φ_2 be solutions of the reduced wave equation as set in the region $\Omega: ABA'C'C$ and for a uniform supersonic entry flow. Then we have, with $\varphi_1 - \varphi_2 = \varphi$,

$$\begin{aligned} L[\varphi] &\equiv \varphi_{xx} - \varphi_{yy}/\beta_1^2 - 2iM_1k_1\varphi_x - k_1^2\beta_1^2\varphi = 0, \\ k_1\beta_1^2 &= M_1\omega_1, \end{aligned} \quad (1.3)$$

compare (G), Eq. (2.2). If we write $\psi = e^{-iM_1k_1x}\varphi$ we find

$$\psi_{xx} - \psi_{yy}/\beta_1^2 + k_1^2\psi = 0. \quad (1.4)$$

Moreover not only φ but both p and $\tilde{\varphi}$ satisfy (1.3) and also the blade-to-blade periodicity condition

$$\varphi' \equiv \varphi(x', y') = e^{i\theta}\varphi(x, y); \quad x' = x + 1, y' = y + m; m\beta_1 < 1. \quad (1.5)$$

which applies, in particular, between congruent points of the characteristics $AC, A'C'$. Again, the boundary conditions in the uniqueness proof, $\varphi = 0$ at CC' and $\partial\varphi/\partial y = 0$ at AB , respectively, go over to the same conditions on ψ of Eq. (1.4); this extends to the analogous discussion based on (1.3) (1.4) for problems (B) and (C).

If we multiply (1.4) by $2\bar{\psi}_x$ and apply Green's theorem in the region Ω this being bounded by $\Gamma: ABA'C'CA$ we get

$$I = \oint_{\Gamma} \left\{ (|\psi_x|^2 + |\psi_y|^2/\beta_1^2 + k_1^2|\psi|^2) dy + 2\Re \psi_x \bar{\psi}_y/\beta_1^2 dx \right\}. \quad (1.6)$$

The contribution from AB vanishes with φ_y and for BA' we find

$$I_2 = \int_B^{A'} \left(\left| \frac{d\psi}{dx} \right|^2 + k^2|\psi|^2 \right) dy \geq 0. \quad (1.7)$$

Also $I_3 = \int_{A'}^C$ has a similar form and this cancels with $L_5 = -\int_A^C$ because between corresponding points $\psi'/\psi = \exp(i(\sigma + M_1k_1))$. Finally, since $\varphi = 0 = \psi$ along CC' we get

$$I_4 = \int_C^{C'} D|\psi_y|^2\beta_1^{-2} dy \geq 0; \quad D = 1 - m^2\beta_1^2 > 0. \quad (1.8)$$

It now follows that the non-negative contributions L_2 and I_4 each vanish and so both ψ and $\partial\psi/\partial n$ vanish along CC' . This ensures the vanishing of the solutions in the vicinity of

the non-characteristic arc CC' and by using standard properties of hyperbolic equations also the periodic property (1.5) we see that it vanishes throughout the region Ω . Finally the vanishing of the solution along BA' and the boundary condition at the blade segment BD' ensures that it vanishes in $\Delta BD'A'$.

To extend the proof to problem (B) we apply the previous arguments to show that p vanishes in Ω and then observe that the potential must be of the form

$$\varphi_0 = e^{i\omega_1 x} (ae^{\omega_1 y} + be^{-\omega_1 y}), \quad (1.9)$$

which cannot satisfy the periodicity condition and so must vanish. A similar argument starting with the condition $\tilde{\varphi} = 0$ at CC' leads to the uniqueness proof for problem (C) and subject to $1 < M_1 < 2$.

It is shown in (G) that the unsteady flow behind the strong in-passage shock, that is to say the solution in the blade passages and behind the cascade, may be expressed in terms of Φ_2, Ψ_2 where

$$U_2 = (\Phi_2)_x + (\Psi_2)_y; \quad V_2 = (\Phi_2)_y - (\Psi_2)_x \quad (1.10)$$

and the problem can be reduced to determining Φ_2 satisfying the reduced wave equation for subsonic flow and the following boundary conditions at the (normal) shock:

$$U_2 + h_2 P_2 = h_1 P_1; \quad (P_2)_y + h_4 \left(\frac{\partial^2}{\partial y^2} - \omega_2^2 \right) \Psi_2 = h_3 \frac{\partial P_1}{\partial y}. \quad (1.11)$$

Here we have

$$P_2 = (i\omega_2 - \partial/\partial x)\Phi_2, \quad h_2 = \frac{1}{2}(1 + M_1^{-2}),$$

$$h_4 = (\gamma + 1)(M_1 M_2 / \beta_1 \beta_2)^2, \quad M_2^2 = 1 + \frac{\gamma - 1}{2} M_1^2 \left(\gamma M_1^2 - \frac{\gamma - 1}{2} \right), \quad (1.12)$$

the form of h_1, h_3 being immaterial in the present discussion. The potential function $\varphi = \Phi_2$ satisfies

$$\varphi_{xx} + \varphi_{yy}/\beta_2^2 + 2iM_2 k_2 \varphi_x + k_2^2 \beta_2^2 \varphi = 0,$$

$$k_2 \beta_2^2 = M_2 \omega_2, \quad \beta_2^2 = 1 - M_2^2 \quad (1.13)$$

and the boundary condition at the shock line may be written

$$(A(\Phi_2)_x + iB\omega_2 \Phi_2) \Big|_0^y + \omega_2^2 \int_0^y (y-t)(\Psi_2)_y \Big|_{y=t} dt = \dots, \quad (1.14)$$

$$A = 1 + h_4(1 - k_2), \quad B = h_2 h_4 - 1,$$

where

$$(\Psi_2)_y + (1 - h_2)(\Phi_2)_x + ih_2 \omega_2 \Phi_2 = \dots \quad (1.15)$$

and the right hand members are known from the entry solutions.

We now consider solutions of Eqn. (1.13) in the region $AD'LL'$ say,

$$\tilde{\Omega}: x > 0, \quad y > m(x - l), \quad 0 < y < m. \quad (1.16)$$

Setting $\phi = F + iG$ for the difference of two possible solutions we find

$$\oint (FG_x - GF_x + M_2 k_2 (F^2 + G^2)) dy + (GF_y - FG_y) dx / \beta_2^2 = 0. \quad (1.17)$$

If ϕ_y vanishes at the blade segments and we also require $\phi = 0$ at the exit line LL' then, taking account of the periodicity between points of $y = 0$, $y = m$ (or $y = 2m$ in case $\sigma = \pi$) the only contribution to (1.17) comes from the shock line(s). In the simplest case, if squares of ω_2 may be neglected we get

$$\omega_2 (AM_2^2 - B\beta_2^2) \int_{D'}^{A'} (F^2 + G^2) dy = 0, \quad (1.18)$$

and some straightforward reduction shows that the factor before the integral sign does not vanish provided $1 < M_1^2 < (3 + \gamma)/2$. The uniqueness proof follows from (1.14) showing that we have zero Cauchy data at the shock line. In general we treat (1.14) as a Volterra equation linking ϕ_x with ϕ and, after solving this, we get the same conclusion for sufficiently small values of ω_2 .

The numerical process of shock fitting as employed in (G) is such as to suggest that some restriction on the magnitude of ω_2 is probably necessary for the solution to be valid.

It will be observed that, in addition to the requirements of (G) our uniqueness proof requires the prescription of one constant at the point D . We can avoid this by working with $\chi = \partial\phi/\partial y$ rather than ϕ but then we would have to prescribe χ instead of ϕ along the exit line LL' .

2. Construction of the solutions Φ_1 for $\sigma = 0, \pi$. To exhibit the dependence of the solutions on the choice of the entry line CC' we introduce a representation in terms of trigonometrical series. The first step is to define new coordinates X, Y according to (2.1). Here, dropping the suffix on β_1 , we set

$$DX = 2(x - m\beta^2 y), \quad DY = 2\beta(y - mx), \\ D = 1 - m^2\beta^2 > 0 \quad (2.1)$$

where $D > 0$ as before, and in view of the subsonic axial property. At $y = 0$ we have $X = X(x) = 2x/D$, $Y = Y(X) = -2m\beta x/D$ and with point $C(-\beta c, c)$ we find $Y(C) = 2\beta c/(1 - m\beta)$. We seek solutions for $\sigma = 0$ or π in the form

$$\phi = \sum_n \frac{c_n e^{cn\pi X}}{\mu_n - m\beta n\pi} (\exp(i\mu_n Y) - \exp(i(\overline{\mu_n - \lambda_n} Y(c) + \lambda_n Y))), \quad (2.2)$$

where $\text{Sgn}(n) = \text{sgn}(n)$ for $n \neq 0$ and $\text{Sgn}(0) = -1$. Here we have

$$\lambda_n, \mu_n = \frac{1}{2} Mkm\beta \pm \text{Sgn}(n), \Delta(n), \\ = \frac{1}{2} Mkm\beta \pm \left(n\pi - M\frac{k}{2} \right) + O(k^2/n), \\ \Delta(n)^2 = (n\pi - Mk/2)^2 - 2D/H, \quad (2.3)$$

these estimates holding with respect to k as k tends to 0 with the single exception of $n = 0$ which arises only for $\sigma = 0$. (In the case $\sigma = \pi$ it is convenient to replace n by $n + \frac{1}{2}$.) To avoid the case of equal roots in the indices we require $k < 2\pi/(M + 1)$ for $\sigma = 0$ and k less than half this value for $\sigma = \pi$. The solution under (2.2) satisfies the periodicity condition and vanishes along CC' . To satisfy the boundary condition at the blade segment we must have

$$\begin{aligned} \left. \frac{\partial \varphi}{\partial y} \right|_{y=0} &= \frac{2i\beta}{D} \left\{ \sum c_n \{ \exp(i(n\pi - m\beta\mu_n)) 2x/D \right. \\ &\quad \left. + \Theta_n \exp(i(n\pi - m\beta\lambda_n)) 2x/D \right\}, \\ &= W'(x) - i\omega W(x), \end{aligned} \quad (2.4)$$

where

$$\begin{aligned} |\Theta_n| &= \left| \frac{\lambda n - m\beta n\pi}{m\beta n\pi - \mu_n} \cdot \exp(i(\mu_n - \lambda_n)Y(c)) \right| \\ &\leq \Theta = \frac{1 - m\beta}{1 + m\beta}. \end{aligned} \quad (2.5)$$

We write $f(\tilde{x}) = \sum c_n \exp(in2\tilde{x}\pi)$, $\tilde{x} = x/(1 - m\beta)$ and, using the asymptotic bounds of (2.3) to arrange (2.4) as an integral equation,

$$\begin{aligned} f(\tilde{x}) + \Theta f(\Theta x^2 - Y(C)) \exp\{iMk(Y(C) + m\beta(1 + \Theta)\tilde{x})\} \\ = k \int_0^1 K(\tilde{x}, t; k \cdots) f(t) dt + g(x) \end{aligned} \quad (2.6)$$

where K and g are known; for $\sigma = \pi$ the factor k before K may be improved to k^2 .

Regarding the right hand member as known (2.6) is of the form

$$f(x) = k(x) - \Theta \exp[i(\lambda + \mu x)] f(x_1), \quad x_1 = \Theta x - Y(C) \quad (2.7)$$

and this, compare Picard (1927), is solved by the series

$$\begin{aligned} f(x) &= h(x) - \Theta \exp i(\lambda + \mu x) h(x_1) \\ &\quad + \Theta^2 \exp i(2\lambda + \mu(x + x_1)) h(x_2), \dots, \\ x_{n+1} &= \Theta x_n - Y(C) \cdots \end{aligned} \quad (2.8)$$

leading to a regular Fredholm equation equivalent to (2.6). Here f is periodic and convergence is assured for $\Theta < 1$. It should also be noted that the known function g is given on a finite interval and the solution will, in general, have discontinuities in its first order derivatives across the characteristics $X = Y$, $X + Y = 2Y(C) \pmod{2}$. However the uniqueness proof still applies and the solution of Eq. (2.6) and the corresponding boundary problem can be completed by using the Fredholm alternative.

In the case of rigid body motion of the blades we set, in our notation, the unit of length being AB ,

$$W = H + A(x - d) \quad (2.9)$$

and define Y^* as the positive non-integral part of $-Y(C)$. In case $0 < \Theta + Y^*(C) < 1$ the weak shock discontinuities reflected from CC' do not intersect the segment AB . Then, in case $\sigma = \pi$ we can solve (2.6) to order ω by simply equating coefficients and after setting $g(x) = f(x)\exp(-iMkx)$ with $g = g_0 + g_1x$. We then find

$$\begin{aligned} g_0 &= -\frac{D}{2\beta(1+\Theta)} [iA/(1-m\beta) + \omega(H-Ad)] + \frac{\Theta Y^* DA \omega (1 + M^2/\beta^2)}{2\beta(1+\Theta)(1+\Theta^2)}, \\ g_1 &= -\frac{DA\omega [1 + M^2/\beta^2]}{2\beta(1+\Theta^2)}. \end{aligned} \quad (2.10)$$

It can then be shown that the velocity component along the blade is

$$\begin{aligned} \frac{\partial \varphi}{\partial x} &= \frac{1}{\beta} \frac{\partial \varphi}{\partial y} - \frac{2i}{D} \{2g(\tilde{x})\exp i(Mk\tilde{x}(1-m\beta)) \\ &\quad + \frac{M^2\omega^2}{\beta^2} [(1-m\beta)\tilde{g}(\tilde{x}) - \Theta(1+m\beta)\tilde{g}(\Theta\tilde{x} + Y^*)] + O(\omega^2), \\ \tilde{g}(\tilde{x}) &= \frac{DA/(1-m\beta)}{2\beta(1+\Theta^2)} \left(-\frac{1}{2} + \tilde{x}\right), \end{aligned} \quad (2.11)$$

and hence, neglecting ω^2 ,

$$\begin{aligned} \text{Im}\left(\frac{\partial \varphi}{\partial x}\right) &= \frac{\omega}{\beta} \left\{ \frac{1-\Theta}{1+\Theta} (H-Ad) + \frac{AY^*\Theta}{1+\Theta} \left[\frac{M^2(1+m\beta)}{\beta^2(1-m\beta)} - \frac{2(1+M^2/\beta^2)}{1+\Theta^2} \right] \right\} \\ &\quad + O(x). \end{aligned} \quad (2.12)$$

As shown in App. (A), the solutions with φ set equal to zero are equivalent to the solutions of (G) for the unbounded region. The introduction of the finite duct length modifies the solutions significantly. This is particularly striking if $W(x)$ vanishes near $x = 0$ and the item under (2.12) and involving Y^* does not vanish. Moreover, for fixed Mach number and blade geometry but variable CC' , the correcting term has an oscillatory character, appearing and disappearing for changes of the duct length which are comparable with the blade chord. This feature, however, disappears from the first order solution if $M^2 = 1/3 m^2 + 1/3$.

3. Non-uniform entry flows. The equation for acoustic waves in a non-uniform streaming supersonic flow may be written as

$$\begin{aligned} (\lambda \chi_\xi)_\eta + (\lambda \chi_\eta)_\xi + \left(\frac{\partial(\chi/a^2, \psi)}{\partial(\xi_0 \eta)} + \frac{1}{a^2} \frac{\partial(\chi, \psi)}{\partial(\xi, \eta)} \right)_t + \rho \frac{J}{a^2} \chi_{tt} &= 0, \\ \lambda = \lambda(a), \quad J = \frac{\partial(x, y)}{\partial(\xi, \eta)}, \quad q^2 = u^2 + v^2, \quad a = a(q) \end{aligned} \quad (3.1)$$

see Manwell (1982). Here χ is the perturbation velocity potential, ξ, η are characteristic coordinates for the steady flow and ψ is the stream function for the latter. To find a

non-trivial example of such a steady flow we seek simple wave solutions of the characteristic system

$$\begin{aligned} u_\xi &= -T_1 v_\xi, & y_\xi &= T_2 x_\xi, \\ u_\eta &= -T_2 v_\eta, & y_\eta &= T_1 x_\eta, \end{aligned} \quad (3.2)$$

$$T_{1,2} = \tan(\theta + \varepsilon), \quad \varepsilon = \varepsilon(q). \quad (3.3)$$

compare Courant and Freidrichs (1948), Ch. IV B and in particular Fig. 20. For the present purposes it is sufficient to consider such solutions as

$$x = A(\xi) + B(\eta)C(\xi), \quad (3.4)$$

$$y = \int T_2(\xi) dA + T_1(\xi)B(\eta)C(\xi), \quad (3.5)$$

$$C'(\xi)/C(\xi) = T_1'(\xi)/(T_2 - T_1). \quad (3.6)$$

We first construct the solution between two parallel lines along which ξ is constant as are also the quantities u , v and p . The periodic solution may therefore be generated by replacing (x, y) by $(x + 1, y + m)$ and so on. Evidently the profile must be a concave/convex one, compare Lichtfuss and Starken (1974) pp. 56-61. A salient feature of such a simple wave solution is that it can be continued forward from the blade surface by only a finite distance before we encounter a shock line. From the standpoint of supersonic aerodynamic (the case $\omega = 0$) we must expect some complications in the theory of boundary value problems and this seems to apply also in the case of acoustic waves.

We consider the case where Θ , the inclination of the velocity vector to the main stream direction is small and we suppose that the axes are taken so that we have, with $T_0 = \tan E$,

$$x = A_0 \xi + B_0 \eta + \dots, \quad y = T_0(B_0 \eta - A_0 \xi) + \dots. \quad (3.7)$$

Then equation (3.1) may be written as

$$\begin{aligned} \chi_{\xi\eta} + r_1 \chi_\xi + r_2 \chi_\eta - c \frac{Mk}{2} ((b_0 + r_3) \chi_\xi + (q_0 + r_4) \chi_\eta + r_5 \chi) \\ p_0 = 1 + m\beta, \quad q_0 = 1 - m\beta, \quad -\beta^2 \frac{k^2}{4} p_0 q_0 (1 + r_0) \chi = 0, \end{aligned} \quad (3.8)$$

the r_j being small and having the periodic property with, say, $\sigma = 0$. We set

$$\chi = e^{ik(M/2(q_0 \xi + p_0 \eta) + w(\xi, \eta))} Z \quad (3.9)$$

where w satisfies

$$w_{\xi\eta} + r_1 w_\xi + r_2 w_\eta + = \frac{M}{2} (r_5 - q_0 r_1 - p_0 r_2) \quad (3.10)$$

and is defined by zero Cauchy data along CC' : $\xi = \eta$. This leads to

$$\tilde{\mathcal{L}}(Z) \equiv Z_{\xi\eta} + \tau_1 Z_{\xi} + \tau_2 Z_{\eta} + k_1^2(1 + t_0)Z = 0, \quad (3.11)$$

with

$$\begin{aligned} \tau_1 &= r_1 + ik\left(\omega_{\eta} - \frac{1}{2}Mr_3\right), & k_1^2 &= k^2 p_0 q_0 / 4, \\ \tau_2 &= r_2 + ik\left(\omega_{\xi} - \frac{1}{2}Mr_4\right), \end{aligned} \quad (3.12)$$

and t_0 real and small of the same order of magnitude as the r_j . From its construction $w(\xi, \eta)$ and hence t_0 also $|Z|$ retains the periodic property between AC and $A'C'$.

We now extend the uniqueness theorem of Section 1 under the condition $\chi = 0$ along CC' , the periodic property between the characteristics and prescribed smooth values of $\partial\chi/\partial n$ along AB , the image of a rear blade segment in the (ξ, η) plane. We first remark that by an elementary analysis based on (3.2), (3.3) the condition $\partial\chi/\partial n = 0$ is equivalent to

$$\chi_{\xi} - d\eta\chi_{\eta}/d\xi = 0, \quad (3.13)$$

in the characteristic coordinates. We note also that, by a classical result, see Picard (1928), we may suppose that the characteristic coordinates are re-mapped (in the present case only slightly changed) so that the blade segment goes into $\eta = \Theta\xi + \text{const}$ and CC' remains on the line $\xi = \eta$. We now choose real multipliers B, C in

$$\mathcal{M} = 2B\bar{Z}_{\xi} + 2C\bar{Z}_{\eta} \quad (3.14)$$

according to

$$B = B_0 + b_1\xi - b_2\eta; \quad C = C_0 - c_1\xi - c_2\eta. \quad (3.15)$$

Here we first require

$$B_0 > C_0; \quad c_1 > 0; \quad c_2 > b_1 > b_2 > 0. \quad (3.16)$$

Also since $\xi > \eta$, $0 < \eta < 1$ holds for points of the closure of Ω we can, by taking the b_j, c_j sufficiently small in comparison with B_0, C_0 , arrange that

$$B_0 \leq B < B_*, \quad \text{say,} \quad C_0 \geq C > C_* > 0. \quad (3.17)$$

If we now multiply $\tilde{\mathcal{L}}(Z)$ by $\bar{\mathcal{M}}$, apply Green's theorem and take the real parts we get

$$J_0 + J_1 + \sum_{j=1}^5 I_j = 0. \quad (3.18)$$

Here the area integrals are

$$\begin{aligned} J_0 &= \iint_{\Omega} \left(b_2 |Z_{\xi}|^2 + c_1 |Z_{\eta}|^2 \right. \\ &\quad \left. + \left\{ k_1^2(1 + t_0)(c_2 - b_1) - \left(B \frac{\partial}{\partial \xi} + C \frac{\partial}{\partial \eta} \right) t_0 \right\} |Z|^2 \right) d\xi d\eta, \end{aligned} \quad (3.19)$$

$$J_1 = 2 \iint_{\Omega} \mathbf{R}((\tau_1 Z_{\xi} + \tau_2 Z_{\eta})(B\bar{Z}_{\xi} + C\bar{Z}_{\eta})) d\xi d\eta \quad (3.20)$$

and the I_j are the line integrals over $AB, BA' \dots$ taken in anti-clockwise order. It will be seen that having fixed B, C as above the sum $J_0 + J_1$ will be positive definite for sufficiently small $|\tau_j|$ and $|t_0| < 1$. Then we have

$$I_2 = \int_B^{A'} (C|Z_\eta|^2 + k_1^2(1 + t_0)B|Z|^2) d\eta \geq 0, \quad (3.21)$$

$$I_4 = \int_0^1 \{B_0 - C_0 + (b_1 + c_1 + c_2 - b_2)\xi\} |Z_\xi|^2 d\xi \geq 0. \quad (3.22)$$

Also

$$I_1 = \int_A^B \{C|Z_\eta|^2 d\eta - B|Z_\xi|^2 d\xi - k_1^2(1 + t_0)|Z|^2(Cd\xi - B d\eta)\}. \quad (3.23)$$

which, after use of (3.13), yields

$$\begin{aligned} I_1 &= \int_A^B \{H_1 - k^2(1 + t_0)|Z|^2(C - \Theta B)\} d\xi, \\ H_1 &= \frac{1}{4}(C/\Theta - B) \left| \frac{d\chi}{d\xi} \right|^2 + k(Bt_2 - Ct_1) \operatorname{Im} \left(\bar{\chi} \frac{d\chi}{d\xi} \right) \\ &\quad + k^2(C\Theta t_1^2 - Bt_2^2) |\chi|^2. \end{aligned} \quad (3.24)$$

Again, in view of the periodicity of w and χ we find

$$\begin{aligned} I_3 + I_5 &= \int_C^A \{ (b_1 - b_2) |Z'(\xi, 0)|^2 - k_1^2(c_1 + c_2) \\ &\quad \times (1 + t_0) |Z(\xi, 0)|^2 \} d\xi. \end{aligned} \quad (3.25)$$

It is now an elementary, if slightly tedious matter, to show that the sum (3.24) with (3.25) can be cast in the form

$$\mathcal{K} = \int \{ \mathcal{H}_2(\xi) \Phi'(\xi)^2 - \mathcal{H}_3 \Phi(\xi)^2 \} d\xi, \quad (3.26)$$

with $H_2(\xi) > 0$, $\Phi(0) = 0$ and H_3 small for nearly uniform flows. Then, by using the Schwarz inequality, we can show that $\mathcal{K}$ is non-negative for some small k_1 . Hence, from (3.18) *et seq* the solutions X and Z vanish in Ω .

The proof demands strong conditions. For example if B, C are to be linear then (3.16) is necessary to ensure that J_0, I_4 also $I_3 + I_5$ are non-negative and, at the same time, $C - \Theta B > 0$ in (3.24) holds only if we make c_1 and b_1 small $O(1/CA)$, as measured in the characteristic plane. Likewise, to ensure the positiveness of $J_0 + J_1$, the same restriction must be imposed on $|\tau_j|, k$ and the $|r_j|$.

These observations, the great simplification of the preceding discussion for the special case of flat plates and the example of Appendix (B) suggests that uniqueness may not hold in general. However, it remains to be proved that this is so for any one member of (3.1).

Failure of uniqueness in the two-dimensional model, particularly for a very long entry region, might be linked with the general behaviour of disturbances in plane supersonic flow. Again, the 'buzzsaw' property of rotors would arise as a natural consequence of the indetermination of the strictly periodic property between blade and blade

4. A property of the exit flows for $\sigma = 0$. We consider periodic solutions of (1.13) in the series form

$$\varphi = \sum_n \varphi_n = \sum \{ a_n \exp(in\pi\tilde{X} + \tilde{\lambda}_n Y) - b_n \exp(in\pi\tilde{X} + \tilde{\mu}_n Y) \}. \quad (4.1)$$

Here we write

$$\begin{aligned} \tilde{D}\tilde{X} &= 2(x + m\beta_2^2 y), & \tilde{D}\tilde{Y} &= 2\beta_2(y - mx), \\ \tilde{D} &= 1 + m^2\beta_2^2 \end{aligned} \quad (4.2)$$

with

$$\begin{aligned} \tilde{\lambda}_n, \tilde{\mu}_n &= iM_2 k_2 R_2 m\beta_2 \pm |\tilde{\Delta}(n)|, & n &\neq 0, \\ \tilde{\Delta}(n)^2 &= (n\pi + M_2 k_2/2)^2 - k_2^2 \tilde{D}/4, \\ k_2(M_2 + \tilde{D}^{1/2}) &< 2\pi, \end{aligned} \quad (4.3)$$

and

$$\bar{\chi}_0, \tilde{\mu}_0 = i \frac{k_2 \beta_2}{2} (mM_2 \pm (1 + m^2)^{1/2}). \quad (4.4)$$

Each term of (4.1) satisfies the periodicity condition for the case $\sigma = 0$ and the series provides a complete set of solutions in the vicinity of EE' , say $\tilde{Y} = 0$.

We can arrange that φ vanishes on LL' and then we find for $n \neq 0$,

$$b_n = a_n \exp(-2|\tilde{\Delta}(n)\tilde{Y}(l, 0)|), \quad (4.5)$$

showing that the b_n are uniformly small for large $|l|$. We note also that

$$\mathcal{D}_n \varphi_n(\tilde{X}, 0) = \frac{\partial \varphi_n}{\partial \tilde{Y}}(\tilde{X}, 0), \quad (4.6)$$

with

$$\mathcal{D}_n = |\tilde{\Delta}(n)| \coth |\tilde{\Delta}(n)\tilde{Y}(l, 0)| + i \frac{M_2 k_2 m}{2} \beta_2, \quad (4.7)$$

and

$$\mathcal{D}_0 = k_2 \beta_0 \left\{ (1 + m^2)^{1/2} \cot \left| \frac{k_2 \beta_2}{2} (1 + m^2)^{1/2} \tilde{Y}(l, 0) \right| + i M_2 m \right\}. \quad (4.8)$$

An elementary observation is that for a whole sequence of values of $\tilde{Y}(l, 0)$ the solution $\varphi_{(0)}$ vanishes along both EE' and LL' .

We now show how, in principle, the boundary value problem for the solution Φ_2 set in $\tilde{\Omega}$ may be reduced to an auxiliary problem set in the blade passage $\tilde{\Omega}_1: A'D'EE'$. The first step is to use the Fourier series expansions for φ , $\varphi_{\tilde{Y}}$ noting (4.7), to write down the relation, based on the solution in $\tilde{\Omega}_2$,

$$\varphi(\tilde{X}, 0) = \int L_1(\tilde{X}, t; l) \varphi_y(t, 0) dt \quad (4.9)$$

which holds for general Cauchy data along EE' . Then a simple modification of the uniqueness proof of (1.13) *et seq* shows that the corresponding problem set in $\tilde{\Omega}_1$ is

uniquely defined by the original conditions also values of φ along the boundary line EE' . The actual construction of the solutions requires a fairly lengthy use of integral equations. However, it is reasonable to assume that, as in the case of Laplace's equation, (to which equation (1.13) tends for small values of the frequency parameter) we can find a second relation connecting φ , φ_y with the data, and that this is of the form

$$\varphi(\tilde{X}, 0) = \int L_2(\tilde{X}, t; l) \varphi_y(t, 0) dt + g(\bar{X}) \quad (4.10)$$

Combining (4.9) and (4.10) we get another linear relation to determine φ_y and so the values of φ along EE' . The significant observation is that because of the possible item in $\varphi_{(0)}$ and equation (4.8) the kernel of (4.9) contains a periodic item. Hence even for the exit flows it would be very surprising if the solutions tend smoothly into an asymptotic value for $l \rightarrow \infty$.

Acknowledgement. I would like to thank Dr D. S. Whitehead for drawing my attention to the combination tone which characterises supersonic rotors.

Appendix A. We show that if we admit a very small dissipative term in the reduced wave equation then the solutions which are bounded at infinity in the upstream direction are precisely those which result from dropping the Θ -terms in (2.4) *et seq.* The modified equation is

$$L[\varphi] + \varepsilon \left(i\omega_1 - \frac{\partial}{\partial x} \right) \nabla^2 \varphi = 0 \quad (A.1)$$

compare Lamb (1952) p. 328, equations 4, also p. 359. If ε is very small we may replace φ in the correction terms by φ_{λ_n} or φ_{μ_n} as the case may be and an easy calculation gives the small perturbations of the indices according to

$$2|\Delta(n)| \text{Sgn}(n) \delta(i\lambda_n) = \frac{\varepsilon(\lambda_n, n)}{D} (2n\pi - 2m\beta_1\lambda_n - \omega_1 D). \quad (A.2)$$

It then follows from a check of the signs that the λ_n -functions of equation (2.2) increase exponentially as Y tends to infinity and so must be excluded from the series solution. To show that the other functions decrease it is sufficient to show that the product of the last factor on the right of (A.2) with the corresponding item involving μ_n remains positive with

$$D((2n\pi - \omega_1)^2 + k_1^2 m^2 \beta_1^4 / M_1^2). \quad (A.3)$$

In a slight re-arrangement of this discussion we note also that for fixed (x, y) the terms of the second series on the right of (2.2) tend to zero as c , $Y(C)$ increase without limit.

Formally the same conclusions may be derived by the damping device of (G) which leads to (A.2) as before, although, strictly speaking, the ε are different for each n .

Appendix B. Equations for which Problem A has eigenfunctions. Let φ be any solution of (1.3) subject to the periodicity condition (1.5) with $\sigma = 0$ and suppose φ vanishes on CC' . Let

$$\varphi = e^{kw} \psi \quad (B.1)$$

and suppose w satisfies

$$w(x+1, y+m) = w(x, y) \quad (B.2)$$

for points of $A'C'$, AC , respectively. Then ψ satisfies

$$L(\psi) + 2k(w_x\psi_x - w_y\psi_y/\beta^2) + \psi(kL_0(w) + k^2(w_x^2 - w_y^2/\beta^2)) = 0, \quad (B.3)$$

where

$$L_0(w) = w_{xx} - w_{yy}/\beta^2 - 2iMkw_x. \quad (B.4)$$

If, in addition, w satisfies the following

$$\varphi_y/\varphi = kw_y, 0 < x < 1 - m\beta; y = 0 \quad (B.5)$$

for points of AB then ψ_y vanishes on this segment. Hence such a ψ has zero data on both CC' and AB and satisfies the periodic condition between congruent points of $A'C'$ and AC : it is, therefore, an eigenfunction for Eq. (B.3) with the boundary conditions of Problem A.

The simplest choice is to take $n = 0$ in (2.2) and then (B.5) gives

$$w_y(x, 0) = \frac{\beta}{D} \{ Mm\beta + i\beta(1 + m^2)^{1/2} \cot \mathcal{A} \}; \quad (B.6)$$

$$\mathcal{A} = k\beta(1 + m^2)^{1/2} (\beta c/(1 - m\beta) + m\beta x/D).$$

Here we choose k, c according to

$$kc\beta^2(1 + m^2)^{1/2} = \frac{\pi}{2}(1 - m\beta) \quad (B.7)$$

and suppose that k is small making c large $O(1/k)$. If also $L_0(w) = M^2k^2w$ we have solutions

$$w = \exp(iMkx) \{ f(x + \beta y) + ig(x + \beta y) \} \quad (B.8)$$

and we will take $f(0) = g(0) = 0$ the derivatives of f and g being determined by (B.6) for $0 < x < 1 - m\beta$ and remaining disposable in $1 - m\beta < x < 1 + m\beta$. Hence we can satisfy the periodicity condition between congruent points of AC : $x + \beta y = 0$ and $A'C'$: $x + \beta y = 1 + m\beta$.

Some routine calculations give

$$w_x = iMm\beta \exp(-iMk\beta y) + O(mk), \quad (B.9)$$

$$w_y = \beta w_x + O(mk),$$

$$w_x^2 - w_y^2/\beta^2 = O(k),$$

from which it follows that we may write B3 in the form

$$\psi_{xx} - \psi_{yy}/\beta^2 - 2cMk(1 + O(m))\psi_x + O(mk\psi_y) - \beta^2k^2\psi(1 + O(k)) = 0, \quad (B.10)$$

as required.

It should be added that this example, taken with Sec. (3), certainly suggests that the entry problem for a long duct is badly set but we must remember that we are working with complex solutions and have not shown that any specific member of the (very wide) class of equations (3.1) has eigenfunctions for the problems under consideration.

FIGURE 1: Supersonic compressor (after Kantrowitz)

FIGURE 2: Flow plane for supersonic entry

FIGURE 3: Characteristic plane

FIGURE 4: Scheme for transonic flutter analysis

REFERENCES

- Copson, E. T., *Partial differential equations*, Cambridge, 1975
- Courant, R. and Friedrichs, K. O., *Supersonic flow and shock waves*, 1948 Interscience
- Courant, R. and Hilbert, D., *Methods of mathematical physics*, Vol. II, Interscience, 1962
- Ferri, A., *Esperienze su di un biplano iperacustica tipo Busemann*, Atti di Guidonia No. 37-38, 1940
- Ferri, A., *Aerodynamic properties of supersonic compressors*, High Speed Aerodynamics and Jet Propulsion, Vol. X (Edit. W. R. Hawthorne) Princeton, 1964
- Goldstein, M. E., *Cascade with subsonic leading-edge locus*, AIAA Journ. **13** 1117 (1975)
- Goldstein, M. E., Braun, W. and Adamczyk, J. J., *Unsteady flow in a supersonic cascade with strong in-passage shocks*, J. Fluid Mech. **83**, 569 (1977)
- Hutton, M. F., *Supersonic inflow onto turbine blading: the unique incidence theorem*, Gen. Elect. Gen Board Res. Rept. R/M/M 159, 1974
- Kantrowitz, A., *The supersonic axial-flow compressor*, N.A.C.A. Tech. Rep. no. 974, 1950
- Kurosaka, M., *On the unsteady supersonic cascade with a subsonic leading edge- an exact first order theory-Part I*, Trans. ASME, **A.96** 13(1974)
- Lamb, R., *Hydrodynamics* (6th ed.)Cambridge, 1952
- Levine, P., *Two-dimensional inflow conditions for a supersonic compressor with curved blades*, J. Appl. Mech. **24**, 165 (1957)
- Lichtfuss, H-J and Starken, H., *Supersonic cascade flow*, Progress in Aerospace Sciences 15, Edit. D. Kuchemann, Pergamon, 1974
- Mani, R. and Horvay, G., *Sound transmission through blade rows*, J. Sound Vib. **12**, 59 (1970)
- Manwell, A. R., *A uniqueness theorem for the moving medium reduced wave equation*, Appl. Anal. **12**, 215 (1981)
- Manwell, A. R., *Supersonic cascade flows with subsonic axial Mach number*, Univ. Camb. Eng. Dept. Rep. CUED/A-Turbo/TR110, 1982
- Miles, J. W., *The potential theory of unsteady supersonic flow*, Cambridge
- Morfey, C. L. and Fisher, M. J., *Shock wave radiation from a supersonic ducted rotor*, Aer. Journ. **74**, 579 (1970)
- Nagashima, T and Whitehead, D. S., *Linearized supersonic unsteady flow in cascades*, Aer. Res. Coun. R & M No. 3811, 1977
- Picard, É., *Leçons sur quelque équations aux dérivée partielles*, Paris, Gauthier-Villars, 1927
- Picard, É., *Leçons sur quelque équations fonctionnelles*, Paris, Gauthier-Villars, 1928
- Stratford, B. S. and Sansome, G. E., *The performance of supersonic turbine muzzles*, Aer. Res. Coun. R & M No. 3273, 1959
- Verdon, J. M. and McCune, J. E., *Unsteady supersonic cascade in subsonic axial flow*, AIAA Journ. **13**, 193 (1975)