THIRD-ORDER SOLUTIONS OF BURGERS' EQUATION*

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Abstract. Burgers' equation with small dissipation coefficient and general initial conditions is considered. The first three terms are calculated in both the inner (i.e., close to the shock) and outer (away from the shock) expansions. It is shown that these two expansions can be matched and that this third-order matching essentially completes the determination of the second-order inner solution but leaves an undetermined function in the third-order solution. It is also shown that the second-order inner solution can be determined completely, without use of the third-order inner solution, by use of an integral conservation property.

1. Introduction. Burgers' equation and its various generalizations, several of which arise in models of shock formation and propagation in nonlinear dissipative media of different kinds, can, when the dissipation coefficient is sufficiently small, be solved by the method of matched asymptotic expansions [1-5]. For Burgers' equation itself, this method can be more useful [4] than a direct use of the Cole-Hopf solution [6-8] while for the generalized forms of the equation it is usually the only method available.

In his original paper on a class of generalized Burgers' equations [1], Murray calculated the outer solution (i.e., the solution away from the shock) to both first and second orders, and the inner solution (i.e., the solution close to the shock) to first order, and performed the first-order matching of the two solutions. Since he believed that matching at second order would not be sufficient to determine the remaining unknown function in the inner solution—which in fact is the quantity which Lighthill [9] terms the “shock displacement due to diffusion”—Murray made use of an integral conservation property to complete this determination. Subsequently, Crighton and Scott [2] and Arya and Lardner [3] showed that for certain specific generalized Burgers' equations the second-order matching is indeed feasible and does in fact determine the shock displacement due to diffusion. More recently [5] the complete second-order calculation has been given for a class of equations almost as broad as Murray's original class.

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At each order of the calculation, the outer solution is determined completely (in terms of given initial values). For the inner solution, the unknowns are the shock position plus two unknown functions at each order. Matching at first order determines the shock position and one of the unknowns in the first-order inner solution (the shock amplitude). Matching at second order determines the remaining unknown in this first-order solution (Lighthill's displacement) plus one of the unknowns in the second-order inner solution. It seems plausible to speculate [5] that this process of partial determination of the n th-order inner solution through matching at n th-order followed by its complete determination through matching at $(n + 1)$ th-order will continue to higher orders. However such a speculation is disputed in certain quarters (see for example [4] p. 360) on the grounds that the remaining unknown in the second-order inner solution enters that solution in a term which is transcendentally small in the matching region.¹

The main point of the present investigation is to determine whether the matching process can in fact be carried to higher order than the second. We shall consider Burgers' equation itself, since this will suffice to settle the point of principle that is in dispute, and this is in any case the most important of the equations. We shall calculate the outer and inner solutions to third order and perform the third-order matching. It turns out that this matching does determine the remaining unknown in the second-order inner solution apart from a constant which (we speculate!) can be determined in principle by a matching at the shock formation point.

The integral conservation property introduced by Murray [1] and used extensively by Crighton and Scott [2] offers an alternative means of completing the calculation of the inner solution at each order. Its use does not require the inner solution at the next higher order. This approach is also used to complete the second-order inner solution, the results being consistent with those obtained from the third-order matching. In this case, no undetermined constant remains in the solution.

2. Calculation of the inner and outer expansions. We consider Burgers' equation

$$u_\tau - uu_x = ku_{xx} \quad (1)$$

where the dissipation coefficient $k \ll 1$ and $u(x, \tau)$ satisfies the given initial condition $u(x, 0) = \phi(x)$.

Outer solution. We seek the solution in the form of an outer expansion:

$$u \equiv u^{(0)} = u_0(x, \tau) + ku_1(x, \tau) + k^2u_2(x, \tau) + \dots \quad (2)$$

Substituting and comparing coefficients of each k^n we obtain from the differential equation

$$u_{0\tau} - u_0u_{0x} = 0, \quad (3)$$

$$u_{1\tau} - u_0u_{1x} - u_{0x}u_1 = u_{0xx}, \quad (4)$$

$$u_{2\tau} - u_0u_{2x} - u_{0x}u_2 = u_{1xx} + u_1u_{1x}, \text{ etc.}, \quad (5)$$

¹ This contention was brought to my attention by two of the (anonymous) reviewers of reference [5] and I should like to thank them for stimulating the present investigation.

and from the initial conditions,

$$u_0(x, 0) = \phi(x), \quad u_n(x, 0) = 0 \quad (n \geq 1). \quad (6)$$

The solution of (3) satisfying (6) is given by

$$u_0(x, \tau) = \phi(\beta) \quad (7)$$

where β is determined implicitly in terms of (x, τ) by the equation

$$\beta = x + \tau\phi(\beta). \quad (8)$$

It is well known [10, 11] that the solution given by (7) and (8) ceases to be uniquely defined for $\tau \geq \tau_f$ where $\tau_f = [\phi'_m]^{-1}$ and ϕ'_m is the maximum value of $\phi'(\beta)$. For such values of τ the solution must include a shock. The shock starts at the point (x_f, τ_f) where $x_f = \beta_m - \tau_f\phi(\beta_m)$ and β_m is the value of β at which $\phi'(\beta)$ is maximum.

To calculate u_1, u_2 , etc. it is simplest to use (β, τ) as variables in place of (x, τ) . Equation (4) then becomes, after substituting for u_0 ,

$$\frac{\partial}{\partial \tau}(Du_1) = \frac{\partial}{\partial \beta} \left(\frac{\phi'(\beta)}{D} \right) = \frac{\phi''(\beta)}{D^2}$$

where

$$D = 1 - \tau\phi'(\beta). \quad (9)$$

The solution satisfying (6) is then given by

$$u_1 = \tau\phi''(\beta)/D^2. \quad (10)$$

A similar calculation starting with equation (5) gives

$$u_2 = \frac{\tau^2\phi^{iv}(\beta)}{2D^3} + \frac{8\tau^3\phi''(\beta)\phi'''(\beta)}{3D^4} + \frac{5\tau^4\phi''(\beta)^3}{2D^5}. \quad (11)$$

Inner solution. For $\tau > \tau_f$, let the shock have position $x = x_s(\tau)$ and introduce the inner stretched variable η by

$$x - x_s(\tau) = k\eta. \quad (12)$$

In terms of (η, τ) , equation (1) takes the form

$$u_{\eta\eta} - (u + x'_s)u_\eta = ku_\tau. \quad (13)$$

We now seek the inner solution as an expansion of the form

$$u = u^{(i)} = u_0^*(\eta, \tau) + ku_1^*(\eta, \tau) + k^2u_2^*(\eta, \tau) + \dots \quad (14)$$

Substituting this into (13) and comparing the coefficients of each k^n we get

$$u_{0\eta\eta}^* + (u_0^* + x'_s)u_{0\eta}^* = 0, \quad (15)$$

$$u_{1\eta\eta}^* + (u_0^* + x'_s)u_{1\eta}^* + u_{0\eta}^*u_1^* = u_{0\tau}^*, \quad (16)$$

$$u_{2\eta\eta}^* + (u_0^* + x'_s)u_{2\eta}^* + u_{0\eta}^*u_2^* = u_{1\tau}^* - u_1^*u_{1\eta}^*. \quad (17)$$

The solution of (15) is given by

$$u_0^* = -x'_s(\tau) + a(\tau)\tanh \xi \quad (18)$$

where

$$\xi = \frac{1}{2}a(\tau)[\eta + b(\tau)] \quad (19)$$

and $a(\tau)$, $b(\tau)$ are the “constants” of integration.

We note that the left side of (16) can be written in the form

$$\frac{\partial}{\partial \eta} \left[u_{0\eta}^* \frac{\partial}{\partial \eta} \left(\frac{u_1^*}{u_{0\eta}^*} \right) \right]$$

and thus this equation can readily be integrated twice. The result is

$$\begin{aligned} u_1^* = & b'(\tau) \tanh^2 \xi + \frac{x_s''(\tau)}{a(\tau)^2} (\tanh^2 \xi - 2\xi \tanh \xi - \xi^2 \operatorname{sech}^2 \xi) \\ & + \frac{2a'(\tau)}{a^2(\tau)} \xi + c(\tau) (\tanh \xi + \xi \operatorname{sech}^2 \xi) + d(\tau) \operatorname{sech}^2 \xi \end{aligned} \quad (20)$$

where $c(\tau)$, $d(\tau)$ are constants of integration. A similar procedure for (17) gives the following rather lengthy expression for u_2^* :

$$\begin{aligned} u_2^* = & \frac{2a'(b' - d)}{a^3} + \frac{2c(b' - d)}{a} \xi \tanh \xi \operatorname{sech}^2 \xi - \frac{(b' - d)^2}{a} \tanh \xi \operatorname{sech}^2 \xi \\ & + \frac{2x_s''(b' - d)}{a^3} [\xi \operatorname{sech}^2 \xi - (1 + \xi^2) \tanh \xi \operatorname{sech}^2 \xi] + \frac{2(ac' - a'c)}{a^3} \xi \\ & + \frac{aa'' - 2a'^2}{a^5} \left(\frac{2}{3} \xi^3 \operatorname{sech}^2 \xi + 2\xi^2 \tanh \xi - 2\xi \right) + \frac{2d'}{a^2} \\ & + \frac{6a'x_s''}{a^5} (2 + \xi^2 - 2\xi \tanh \xi - \xi^2 \operatorname{sech}^2 \xi) + \frac{b''}{a^2} (\xi^2 \operatorname{sech}^2 \xi + 2\xi \tanh \xi - 3) \\ & + \frac{2x_s'''}{a^4} (\xi^2 \operatorname{sech}^2 \xi - \xi^2 + 2\xi \tanh \xi - 2) + \frac{c^2}{a} (\xi \operatorname{sech}^2 \xi - \xi^2 \tanh \xi \operatorname{sech}^2 \xi) \\ & + \frac{x_s''^2}{a^5} \left[4\xi + 2\xi \operatorname{sech}^2 \xi - (1 + \xi^2)^2 \tanh \xi \operatorname{sech}^2 \xi + \frac{4}{3} \xi^3 \operatorname{sech}^2 \xi - 2\xi^2 \tanh \xi \right] \\ & + \frac{2x_s''c}{a^3} [-1 + \xi \tanh \xi - \xi^2 \operatorname{sech}^2 \xi + \xi(1 + \xi^2) \tanh \xi \operatorname{sech}^2 \xi] \\ & + e(\tau) (\tanh \xi + \xi \operatorname{sech}^2 \xi) + f(\tau) \operatorname{sech}^2 \xi \end{aligned} \quad (21)$$

where $e(\tau)$, $f(\tau)$ are the constants of integration.

3. Matching. We examine first the outer limit of the inner solution given by (14), (18), (20), and (21). Allowing $\xi \rightarrow \pm \infty$ and replacing ξ from (19) we find that

$$\begin{aligned} u \sim & -x_s' \pm a + k [P_1 + Q_1 \eta \pm (R_1 + S_1 \eta)] \\ & + k^2 [P_2 + Q_2 \eta + T_2 \eta^2 \pm (R_2 + S_2 \eta + U_2 \eta^2)] \end{aligned} \quad (22)$$

plus transcendently small terms, where

$$Q_1 \pm S_1 = \frac{a'}{a} \mp \frac{x_s''}{a}, \quad (23)$$

$$P_1 \pm R_1 = b' + \frac{x_s''}{a^2} + \frac{a'b}{a} \pm \left(c - \frac{x_s''b}{a} \right), \quad (24)$$

$$T_2 \pm U_2 = \frac{3a'x_s'' - ax_s'''}{2a^3} \pm \frac{1}{2a^3} (aa'' - 2a'^2 - x_s''^2), \quad (25)$$

$$Q_2 \pm S_2 = \left(\frac{2a'^2}{a^4} - \frac{a''}{a^3} + \frac{c'}{a} + \frac{2x_s''^2}{a^4} - \frac{a'c}{a^2} \right) \\ \pm \left(\frac{b''}{a} + \frac{2x_s'''}{a^3} + \frac{x_s''c}{a^2} - \frac{6a'x_s''}{a^4} \right) + 2b(T_2 \pm U_2), \quad (26)$$

$$P_2 \pm R_2 = \frac{2a'(b' - d)}{a^3} + \frac{12a'x_s''}{a^5} - \frac{3b''}{a^2} + \frac{2d'}{a^2} - \frac{4x_s'''}{a^4} \\ - \frac{2x_s''c}{a^3} \pm e + b(Q_2 \pm S_1) - b^2(T_2 \pm U_2). \quad (27)$$

Observe that $b(\tau)$, which occurs in the first-order solution (18), enters the asymptotic form (22) in the term of order k , while $d(\tau)$, which occurs in the second-order solution (20), enters (22) in the term of order k^2 .

We next calculate the asymptotic behavior of the outer solution given by (2), (7), (10), and (11) at points close to the shock. At any point $x = x_s(\tau)$ on the shock path there are two values of β satisfying equation (8), and we denote them by $\beta_{\pm}$. The situation in the $x\tau$ -plane is illustrated in Figure 1 (see [9] for example). We use the notation $\phi_{\pm}$, $\phi'_{\pm}$, $\phi''_{\pm}$,

FIG. 1.

etc. for $\phi(\beta_{\pm})$, $\phi'(\beta_{\pm})$, $\phi''(\beta_{\pm})$, etc., so that equation (8) on the shock becomes

$$\beta_{\pm} = x_s + \tau\phi_{\pm}. \quad (28)$$

For points close to the shock we express the outer solution in terms of the inner variable. Thus writing $x - x_s = k\eta$ in (8) and expanding β in powers of k , we obtain

$$\beta = \beta_{\pm} + \frac{k\eta}{D_{\pm}} + \frac{k^2\eta^2\tau\phi''_{\pm}}{2D_{\pm}^3} + \dots \quad (29)$$

where $D_{\pm} = 1 - \tau\phi'_{\pm}$. Then expanding (2), (7), (10), and (11) in like manner we find that, to order k^2 ,

$$u \sim \phi_{\pm} + k \left[\frac{\phi'_{\pm}}{D_{\pm}}\eta + \frac{\tau\phi''_{\pm}}{D_{\pm}^2} \right] + k^2 \left[\frac{\phi''_{\pm}}{2D_{\pm}^3}\eta^2 + \left(\frac{\tau\phi'''_{\pm}}{D_{\pm}^3} + \frac{2\tau^2\phi''_{\pm}{}^2}{D_{\pm}^4} \right)\eta + \left(\frac{\tau^2\phi_{\pm}^{iv}}{2D_{\pm}^3} + \frac{8\tau^3\phi'_{\pm}\phi''_{\pm}}{3D_{\pm}^4} + \frac{5\tau^4\phi_{\pm}^{v3}}{2D_{\pm}^5} \right) \right]. \quad (30)$$

The two expressions (22) and (30) can now be matched at each order. At the lowest order we get the conditions

$$-x'_s \pm a = \phi_{\pm} \quad (31)$$

which, together with (28), allow the four quantities x_s , a , $\beta_{\pm}$ to be determined as functions of τ . They lead to the well-known equal areas construction for the shock path [9, 11]. The lowest-order inner solution given by (18) is then determined except for the quantity $b(\tau)$. Physically this quantity has the significance that the center of the zero-order shock layer given by (18) and (19) is shifted from the shock path $x_s(\tau)$ by a displacement $-kb(\tau)$.

Next, matching (22) and (30) to order k , we obtain

$$Q_1 \pm S_1 = \frac{\phi'_{\pm}}{D_{\pm}}, \quad P_1 \pm R_1 = \frac{\tau\phi''_{\pm}}{D_{\pm}^2}. \quad (32)$$

It can readily be seen that the first of these pairs of conditions can be derived by differentiating (31), so that only two new matching conditions arise. After substituting from (24) and adding and subtracting the two conditions, they can be written in the form

$$(ab)' = -\frac{x''_s}{a} + \frac{a\tau}{2} \left[\frac{\phi''_+}{D_+^2} + \frac{\phi''_-}{D_-^2} \right], \quad (33)$$

$$c = \frac{x''_s b}{a} + \frac{\tau}{2} \left[\frac{\phi''_+}{D_+^2} - \frac{\phi''_-}{D_-^2} \right]. \quad (34)$$

The first of these may be integrated with use of (28) and (31), and we obtain (using the fact that $a = 0$ and $\beta_+ = \beta_-$ at the shock formation point $\tau = \tau_f$) that

$$b = \frac{1}{2a} \ln \left| \frac{D_-}{D_+} \right|. \quad (35)$$

This agrees with Lighthill's expression [9] for the displacement due to diffusion. We note that at this stage the lowest-order inner solution (18) is now complete while in the term u_1^* of order k , $c(\tau)$ is determined by (34) but $d(\tau)$ remains unknown.

Finally, matching (22) and (30) to order k^2 , we obtain

$$T_2 \pm U_2 = \frac{\phi_{\pm}''}{2D_{\pm}^3}, \quad (36)$$

$$Q_2 \pm S_2 = \frac{\tau \phi_{\pm}'''}{D_{\pm}^3} + \frac{2\tau^2 \phi_{\pm}''^2}{D_{\pm}^4}, \quad (37)$$

$$P_2 \pm R_2 = \frac{\tau^2 \phi_{\pm}^{iv}}{2D_{\pm}^3} + \frac{8\tau^3 \phi_{\pm}'' \phi_{\pm}'''}{3D_{\pm}^4} + \frac{5\tau^4 \phi_{\pm}''^3}{2D_{\pm}^5}. \quad (38)$$

It is again not difficult to see that (36) and (37) can be obtained by differentiating the conditions (32), so that again only two new matching conditions arise. These are obtained in full by substituting from (25)-(27) into (38) and it is clear from (27) that they are sufficient to determine $e(\tau)$ completely as well as the derivative of $d(\tau)/a(\tau)$. After some manipulation the expression for $(d/a)'$ may be integrated and the result is that

$$\begin{aligned} \frac{d}{a} - \frac{x_s''}{a^3} + \frac{bc}{2} - \frac{b^2 x_s''}{4a} - \frac{b'}{a} \\ - \frac{\tau^2}{8} \left(\frac{\phi_+'''}{D_+^2} - \frac{\phi_-'''}{D_-^2} \right) - \frac{5\tau^3}{24} \left(\frac{\phi_+''^2}{D_+^3} - \frac{\phi_-''^2}{D_-^3} \right) = \text{constant}. \end{aligned} \quad (39)$$

The remaining unknown constant can presumably be determined in principle by matching with a solution at the formation point (x_f, τ_f) . It can however be determined more readily by using the integral conservation property [2], and we shall show in the next section that this constant is in fact zero.

It is clear however that the matching process can be carried through successfully to order k^2 , and that matching at order k^2 determines the inner solution of order k apart from a single constant.

4. The integral conservation rule. Integrating equation (1) from $x = -\infty$ to $+\infty$, assuming that $u, u_x \rightarrow 0$ as $x \rightarrow \pm\infty$, we obtain the conservation property

$$\frac{d}{d\tau} \int_{-\infty}^{\infty} u \, dx = 0.$$

Thus,

$$\int_{-\infty}^{\infty} u(x, \tau) \, dx = \int_{-\infty}^{\infty} \phi(x) \, dx. \quad (40)$$

We use the notation $u^{(0)}(x, \tau)$ to denote the outer expansion given by (2), $u^{(i)}(\eta, \tau)$ to denote the inner expansion given by (14) and $u^{(0)(i)}(\eta, \tau)$ to denote the inner expansion of

the outer expansion given by (30). Then if

$$f(\eta, \tau) = u^{(i)}(\eta, \tau) - u^{(0)(i)}(\eta, \tau), \quad (41)$$

the expression $u = u^{(0)} + f$ gives a uniform approximation valid in both regions [2]. Equation (40) can then be written

$$\int_{-\infty}^{\infty} u^{(0)}(x, \tau) dx + k \int_{-\infty}^{\infty} f(\eta, \tau) d\eta = \int_{-\infty}^{\infty} \phi(x) dx. \quad (42)$$

Clearly, to calculate the left side to order k^2 we must retain the u_2 -term in $u^{(0)}$ but we need only retain the terms of order k in $f(\eta, \tau)$. From (14), (18), (20), (30), and the first-order matching conditions (31) and (32), these are given by

$$f(\eta, \tau) = f_0(\eta, \tau) + kf_1(\eta, \tau) + O(k^2),$$

where

$$f_0(\eta, \tau) = a[\tanh \xi - \sigma(\eta)], \quad (43)$$

$$\begin{aligned} f_1(\eta, \tau) = & \left(d - b' - \frac{x_s''}{a^2} \right) \text{sech}^2 \xi + c\xi \text{sech}^2 \xi - \frac{x_s''}{a^2} \xi^2 \text{sech}^2 \xi \\ & + \left(c - \frac{2x_s''}{a^2} \xi \right) [\tanh \xi - \sigma(\eta)], \end{aligned} \quad (44)$$

and $\sigma(\eta) = +1$ if $\eta > 0$, $\sigma(\eta) = -1$ if $\eta < 0$.

The terms in (42) of order 1, after substituting from (7) for u_0 and changing to β as integration variable, become

$$\int_{\beta_-}^{\beta_+} \phi(\beta) d\beta = \frac{1}{2} \tau [\phi_+^2 - \phi_-^2],$$

which is precisely the equal areas rule. The terms of order k are

$$\int_{-\infty}^{\infty} u_1(x, \tau) dx + \int_{-\infty}^{\infty} f_0(\eta, \tau) d\eta = 0,$$

and after substituting from (10) and (43) and evaluating the integrals, we again obtain the result (35) for Lighthill's displacement. Finally the terms of order k^2 are

$$\int_{-\infty}^{\infty} u_2(x, \tau) dx + \int_{-\infty}^{\infty} f_1(\eta, \tau) d\eta = 0.$$

After substituting from (11) and (44) and evaluating the resulting integrals (changing from x to β in the first) we reproduce the result (39) for $d(\tau)$ with the constant on the right side equal to zero.

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