POSITIVE DEFLECTIONS OF ELASTIC BEAMS*

By

W. A. DAY

Hertford College, Oxford

1. Introduction. This article concerns the fourth-order boundary value problem

$$\left[\frac{1}{k} u'' \right]'' = f^+ - f^-, \quad u(\pm 1) = u'(\pm 1) = 0, \quad (1.1)$$

which governs the deflection of a clamped elastic beam. The beam is identified with the interval $[-1, 1]$ of the x -axis, the primes denote derivatives with respect to x , and k is the (possibly variable) flexural compliance – the reciprocal of the flexural rigidity. For reasons to emerge presently, it is convenient to express the load (per unit length) as the difference of two nonnegative functions f^+ and f^- .

The deflection can be represented as the integral

$$u(x) = \int_{-1}^1 G(x, y)(f^+(y) - f^-(y)) dy \quad (1.2)$$

in which the kernel, or Green's function, is symmetric and is strictly positive in the open square $(-1, 1) \times (-1, 1)$ but vanishes everywhere on the boundary of the square.

The positive character of G has the implication that $u > 0$ in $(-1, 1)$ if $f^+ > f^-$ in $(-1, 1)$. However, the condition $f^+ > f^-$ is certainly not necessary in order to ensure that $u > 0$, and we propose to establish the sufficiency of an alternative condition which depends upon the lemma which follows.

LEMMA. Let g and h be defined on $[-1, 1]$ by

$$g(x) = \max\{G(x, y) : y \in [-1, 1]\}, \quad (1.3)$$

$$h(y) = \inf\left\{\frac{G(x, y)}{g(x)} : x \in (-1, 1)\right\}. \quad (1.4)$$

Then g and h are strictly positive in the open interval $(-1, 1)$ and each vanishes at the end points. Moreover $h \leq 1$ and

$$G(x, y) \geq g(x)h(y) \quad \text{in } [-1, 1] \times [-1, 1]. \quad (1.5)$$

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It is clear, from the fact that G is positive, that g itself is strictly positive in $(-1, 1)$ and it is clear that g and h both vanish at the end points. In addition, the inequality (1.5) is an immediate consequence of the definition of h , and so is the inequality $h \leq 1$. What is not clear, and it is here that we encounter a genuine difficulty, is that h is strictly positive in $(-1, 1)$; to prove this requires detailed examination of the behaviour of G .

When the beam is uniform, that is when the flexural compliance k is constant, g and h can be determined explicitly and turn out to be

$$g(x) = \frac{k(1-x^2)^2(1+|x|)}{6(2+|x|)^2}, \quad h(y) = \frac{27}{32}(1-y^2)(1-|y|). \quad (1.6)$$

If, momentarily, we take the lemma for granted and return to the representation (1.2), we see that (1.5) implies that

$$\int_{-1}^1 G(x, y) f^+(y) dy \geq g(x) \int_{-1}^1 h(y) f^+(y) dy,$$

while the definition (1.3) implies that

$$\int_{-1}^1 G(x, y) f^-(y) dy \leq g(x) \int_{-1}^1 f^-(y) dy.$$

Thus,

$$u(x) \geq g(x) \left[\int_{-1}^1 h(y) f^+(y) dy - \int_{-1}^1 f^-(y) dy \right] \quad (1.7)$$

and we conclude

THEOREM 1. The condition

$$\int_{-1}^1 h(y) f^+(y) dy > \int_{-1}^1 f^-(y) dy \quad (1.8)$$

is sufficient to ensure that $u > 0$ in $(-1, 1)$.

Theorem 1 has an interesting interpretation. A horizontal beam, whose ends are clamped, is subjected to a nonuniform load, of amount $f^- (\geq 0)$ per unit length, which is applied from above and acts in the downward vertical direction, causing the beam to sag. It is required to shore up the beam by applying from below a load, of amount $f^+ (\geq 0)$ per unit length, which acts vertically upwards. Thus the question arises as to which loads f^+ are sufficient to ensure that the upward deflection of the beam is positive.

As we have seen, one way of ensuring what is required is to arrange that $f^+ > f^-$ in $(-1, 1)$, but such an apparently straightforward choice of f^+ suffers from the defect that it demands prior knowledge of f^- at each point of the beam. By contrast, the criterion (1.8) demands only minimal knowledge of f^- , through the value of the net downward force

$$\int_{-1}^1 f^-(y) dy \quad (1.9)$$

causing the beam to sag.

The presence on the left-hand side of (1.8) of some such weighting factor as h , which vanishes at the ends of the beam, is only to be expected. It would be highly inefficient to attempt to shore up the beam by applying a load f^+ concentrated near to a clamped end,

and to do so would require the expenditure of a large net upward force

$$\int_{-1}^1 f^+(y) dy. \quad (1.10)$$

We can make these considerations more precise by considering what happens when f^+ has compact support in the sense that the closure of the set $\{x: f^+(x) > 0\}$ is contained within the open interval $(-1, 1)$. In these circumstances we do obtain a condition for positive deflection which involves f^+ and f^- through the net forces (1.9) and (1.10).

COROLLARY 1. If f^+ has compact support and γ is the least upper bound of the set

$$\left\{ \frac{1}{h(y)} : y \in (-1, 1) \text{ and } f^+(y) > 0 \right\},$$

the condition

$$\int_{-1}^1 f^+(y) dy > \gamma \int_{-1}^1 f^-(y) dy$$

is sufficient to ensure that $u > 0$ in $(-1, 1)$.

In accordance with the remarks we have just made, the constant γ increases without limit if there are points of the support of f^+ which are sufficiently close to an end point.

There are circumstances in which f^+ does not have compact support and yet (1.8) can still be replaced by a condition involving f^+ only through a net force; this is so if, for example, f^+ is a monotone function. A new feature enters, though, in that the net force is the force exerted on part of the beam rather than on the entire beam.

To illustrate what happens let us suppose that f^+ decreases. Steffensen's inequality [1, p. 107] assures us that

$$\int_{-1}^1 h(y) f^+(y) dy \geq \int_{1-\lambda}^1 f^+(y) dy,$$

where

$$\lambda = \int_{-1}^1 h(y) dy$$

and, thus, we have

COROLLARY 2. The condition

$$\int_{1-\lambda}^1 f^+(y) dy > \int_{-1}^1 f^-(y) dy$$

is sufficient to ensure that $u > 0$ in $(-1, 1)$ provided that f^+ decreases.

For the uniform beam $1 - \lambda = 19/64$, which is approximately 0.2969.

Further light on the significance of the function h is provided by a second theorem which tells us something about the possible shapes which can be adopted by a clamped beam under the action of loads of constant sign.

THEOREM 2. If $f^- = 0$ the deflection satisfies the inequality $u \geq \mu h$ in $[-1, 1]$, the constant μ being the maximum deflection $\max\{u(x) : x \in [-1, 1]\}$.

Theorem 2 may be regarded as the counterpart, in the context of the fourth-order problem (1.1), of an elementary assertion about concave functions, namely that if $u'' \leq 0$ and $u(\pm 1) = 0$ then

$$u(x) \geq \mu \cdot \frac{1}{2}(1 - |x|),$$

where, once again, μ is the maximum value of u .

The verification of Theorem 2 is straightforward. Indeed, if x, y, z are any points of $[-1, 1]$, the symmetry of G , the lower bound (1.5), and the definition (1.3) of g , imply the inequalities

$$G(x, y) = G(y, x) \geq g(y)h(x) \geq G(y, z)h(x) = G(z, y)h(x).$$

Thus, we have

$$\int_{-1}^1 G(x, y)f^+(y) dy \geq h(x) \int_{-1}^1 G(z, y)f^+(y) dy.$$

Since $f^- = 0$ this says that $u(x) \geq u(z)h(x)$, and on choosing z so that $u(z) = \mu$ we arrive at the inequality $u \geq \mu h$.

It would be of interest to extend our considerations from beams to plates, or even to shells. However, the difficulties in the way of such an extension appear to be formidable, not least because, as the reader will recall, the Green's function of a clamped plate need not be of constant sign [Talenti, 2].

2. Proof that h is positive. It might be thought possible to make the proof depend upon the fact that the Green's function is totally positive, in the sense of Karlin's treatise [3], but I have not succeeded in constructing a proof on these lines and have had to rely on an ad hoc argument. (Such a proof is supplied by S. Friedland in a paper which follows this one.) The ad hoc argument, though, establishes rather more than is required in that it shows $h(y)$ to be bounded from below by a positive multiple of $(1 - y^2)(1 - |y|)^2$.

We suppose throughout that the flexural compliance is continuous and strictly positive, and we choose constants m and M in such a way that

$$0 < m \leq k(x) \leq M \quad \text{in } [-1, 1]. \quad (2.1)$$

It will be convenient to write

$$L = \{(x, y): -1 \leq y \leq x \leq 1\}, \quad U = \{(x, y): -1 \leq x \leq y \leq 1\},$$

respectively, for the lower and upper triangles into which the square $[-1, 1] \times [-1, 1]$ is divided by the diagonal line $x = y$, and to introduce the moments

$$a_n = \int_{-1}^1 x^n k(x) dx \quad (n = 0, 1, 2)$$

and the associated numbers

$$b_n = a_n / (a_0 a_2 - a_1^2).$$

Schwarz's inequality tells us that the denominator $a_0 a_2 - a_1^2$ is positive. The numbers a_0, a_2, b_0, b_2 are positive, but a_1 and b_1 may have either sign or may vanish.

It is a routine, if tedious, matter to check that the Green's function $G(x, y)$ equals

$$\begin{aligned} & -b_0 \int_x^1 s(s-x)k(s) ds \int_{-1}^y s(y-s)k(s) ds \\ & + b_1 \int_x^1 s(s-x)k(s) ds \int_{-1}^y (y-s)k(s) ds + b_1 \int_x^1 (s-x)k(s) ds \int_{-1}^y s(y-s)k(s) ds \\ & - b_2 \int_x^1 (s-x)k(s) ds \int_{-1}^y (y-s)k(s) ds \end{aligned} \quad (2.2)$$

in L , and equals

$$\begin{aligned} & -b_0 \int_{-1}^x s(x-s)k(s) ds \int_y^1 s(s-y)k(s) ds \\ & + b_1 \int_{-1}^x s(x-s)k(s) ds \int_y^1 (s-y)k(s) ds + b_1 \int_{-1}^x (x-s)k(s) ds \int_y^1 s(s-y)k(s) ds \\ & - b_2 \int_{-1}^x (x-s)k(s) ds \int_y^1 (s-y)k(s) ds \end{aligned} \quad (2.3)$$

in U .

It is clear from (2.2) and (2.3) that G is continuous in the closed square $[-1, 1] \times [-1, 1]$, that it vanishes everywhere on the boundary of the square, and that it is symmetric. It is by no means clear, though, that G is strictly positive in the interior of the square. We proceed to derive estimates which are the key to our proof and, at the same time, make plain that G is positive: we claim that there are positive constants A and B such that

$$A(1-x^2)(1-|x|)^2(1-|y|)^2 \leq G(x, y) \leq B(1-|y|)^2. \quad (2.4)$$

Let us fix an arbitrary $x \in (-1, 1)$. We shall show, to begin with, that the ratio

$$G(x, y) / \int_{-1}^y (y-s)k(s) ds, \quad (2.5)$$

whose denominator is positive, is a decreasing function of $y \in (-1, x)$. For, (2.2) can be rearranged as

$$\begin{aligned} G(x, y) &= \int_x^1 (b_1 - b_0 s)(s-x)k(s) ds \int_{-1}^y s(y-s)k(s) ds \\ &+ \int_x^1 (b_1 s - b_2)(s-x)k(s) ds \int_{-1}^y (y-s)k(s) ds \end{aligned} \quad (2.6)$$

and, therefore, in order to prove that (2.5) decreases it will be enough to prove that the ratio

$$\int_{-1}^y s(y-s)k(s) ds / \int_{-1}^y (y-s)k(s) ds \quad (2.7)$$

increases for $y \in (-1, x)$ and that

$$\int_x^1 (b_1 - b_0 s)(s-x)k(s) ds < 0. \quad (2.8)$$

The derivative of the ratio (2.7) with respect to y equals

$$\left[\int_{-1}^y sk(s) ds \int_{-1}^y (y-s)k(s) ds - \int_{-1}^y k(s) ds \int_{-1}^y s(y-s)k(s) ds \right] / \left[\int_{-1}^y (y-s)k(s) ds \right]^2.$$

The denominator is certainly positive, and so is the numerator, as Čebyšev's inequality [1, p. 40] tells us, and so (2.7) must increase.

With a view to verifying (2.8), let us denote the left-hand side by $\theta(x)$ and let us consider its dependence upon x . The first two derivatives are

$$\theta'(x) = - \int_x^1 (b_1 - b_0 s) k(s) ds,$$

$$\theta''(x) = (b_1 - b_0 x) k(x).$$

Now let $x_0 = b_1/b_0$. Since $|b_1| < b_0$, the number $x_0 \in (-1, 1)$ and, on considering the sign of the second derivative, we see that θ is a convex function in $[-1, x_0]$ and is concave in $[x_0, 1]$. Since $\theta(1) = \theta'(1) = 0$, the concave nature of θ implies that $\theta < 0$ in $[x_0, 1)$ and, in particular, $\theta(x_0) < 0$. On the other hand, θ is convex in $[-1, x_0]$ and its value at the left-hand end point of that interval is

$$\begin{aligned} \theta(-1) &= \int_{-1}^1 (b_1 - b_0 s)(s+1)k(s) ds = \int_{-1}^1 (b_1 + (b_1 - b_0)s - b_0 s^2)k(s) ds \\ &= a_0 b_1 + a_1(b_1 - b_0) - a_2 b_0 = a_1 b_1 - a_2 b_0 = -1. \end{aligned}$$

It follows that $\theta < 0$ in $[-1, x_0]$ and, hence, that $\theta < 0$ in $[-1, 1)$, which proves (2.8).

Thus, it is indeed the case that the ratio (2.5) is a decreasing function of $y \in (-1, x)$. On appealing to (2.6), and noting that the ratio (2.7) tends to the limit -1 as $y \rightarrow -1$, we deduce that the ratio (2.5) tends to

$$\begin{aligned} & - \int_x^1 (b_1 - b_0 s)(s-x)k(s) ds + \int_x^1 (b_1 x - b_2)(s-x)k(s) ds \\ &= \int_x^1 (-b_1 - b_2 + (b_0 + b_1)s)(s-x)k(s) ds \leq (b_0 - b_2) \int_x^1 (s-x)k(s) ds, \end{aligned}$$

where we have used the fact that $-b_1 - b_2 + (b_0 + b_1)s$ is an increasing function of s and, therefore, does not exceed $b_0 - b_2$, the value it takes at $s = 1$.

In summary, we have established that the inequalities

$$\frac{G(x, x)}{\int_{-1}^x (x-s)k(s) ds} \leq \frac{G(x, y)}{\int_{-1}^y (y-s)k(s) ds} \leq (b_0 - b_2) \int_x^1 (s-x)k(s) ds$$

holds in the lower triangle L . Since the integral on the extreme right-hand side is a decreasing function of x we can replace x by y in that integral to deduce that

$$\frac{G(x, x)}{\int_{-1}^x (x-s)k(s) ds} \leq \frac{G(x, y)}{\int_{-1}^y (y-s)k(s) ds} \leq (b_0 - b_2) \int_y^1 (s-y)k(s) ds \quad (2.9)$$

in L .

On the other hand, if we start from (2.3) and argue in much the same way we can readily deduce that the inequalities

$$\frac{G(x, x)}{\int_x^1 (s - x)k(s) ds} \leq \frac{G(x, y)}{\int_y^1 (s - y)k(s) ds} \leq (b_0 - b_2) \int_{-1}^y (y - s)k(s) ds \quad (2.10)$$

hold in U .

It follows from (2.9) and (2.10) that, in either triangle,

$$\begin{aligned} G(x, y) &\leq (b_0 - b_2) \int_{-1}^y (y - s)k(s) ds \int_y^1 (s - y)k(s) ds \\ &\leq \frac{1}{4} (b_0 - b_2) M^2 (1 + y)^2 (1 - y)^2 \\ &= \frac{1}{4} (b_0 - b_2) M^2 (1 + |y|)^2 (1 - |y|)^2 \\ &\leq (b_0 - b_2) M^2 (1 - |y|)^2, \end{aligned}$$

which establishes the upper bound in (2.4).

In order to establish the lower bound we need to estimate $G(x, x)$ from below: we proceed to show that

$$G(x, x) \geq \frac{m^5 (1 - x^2)^3}{18M^2 (a_0 a_2 - a_1^2)}. \quad (2.11)$$

Upon setting $y = x$ in (2.3) we see that $G(x, x)$ equals

$$\begin{aligned} &-b_0 \int_{-1}^x s(x - s)k(s) ds \int_x^1 s(s - x)k(s) ds \\ &+ b_1 \int_{-1}^x s(x - s)k(s) ds \int_x^1 (s - x)k(s) ds + b_1 \int_{-1}^x (x - s)k(s) ds \int_x^1 s(s - x)k(s) ds \\ &-b_2 \int_{-1}^x (x - s)k(s) ds \int_x^1 (s - x)k(s) ds. \end{aligned} \quad (2.12)$$

Next, we introduce the function

$$\phi(x) = \frac{1}{2} \int_{-1}^x (x - s)^2 k(s) ds,$$

which is positive in $(-1, 1]$ and whose derivative is

$$\phi'(x) = \int_{-1}^x (x - s)k(s) ds.$$

In view of the equations

$$\begin{aligned} \int_{-1}^x s(x - s)k(s) ds &= x\phi'(x) - 2\phi(x), \\ \int_x^1 (s - x)k(s) ds &= \int_{-1}^1 (s - x)k(s) ds + \int_{-1}^x (x - s)k(s) ds = a_1 - a_0 x + \phi'(x), \\ \int_x^1 s(s - x)k(s) ds &= \int_{-1}^1 s(s - x)k(s) ds + \int_{-1}^x s(x - s)k(s) ds \\ &= a_2 - a_1 x + x\phi'(x) - 2\phi(x), \end{aligned}$$

we can rewrite (2.12) in terms of ϕ and ϕ' and, after some rearrangement, we discover the identity

$$G(x, x) = 2\phi(x) - 4b_0\phi(x)^2 + 4(b_0x - b_1)\phi(x)\phi'(x) - (b_0x^2 - 2b_1x + b_2)\phi'(x)^2,$$

which provides the means of estimating $G(x, x)$ from below.

For, let

$$\psi(x) = -\frac{d}{dx} \left(\frac{G(x, x)}{\phi(x)} \right).$$

On calculating this derivative with the help of the identity just discovered, and rewriting b_0, b_1, b_2 in terms of a_0, a_1, a_2 we find that $\psi(x)$ equals

$$\frac{(2\phi(x)\phi''(x) - \phi'(x)^2)((a_0x^2 - 2a_1x + a_2)\phi'(x) - 2(a_0x - a_1)\phi(x))}{(a_0a_2 - a_1^2)\phi(x)^2}$$

or, since

$$a_0x^2 - 2a_1x + a_2 = \int_{-1}^1 (x-s)^2 k(s) ds = 2\phi(x) + \int_x^1 (s-x)^2 k(s) ds,$$

$$a_0x - a_1 = \int_{-1}^1 (x-s)k(s) ds = \phi'(x) - \int_x^1 (s-x)k(s) ds,$$

that $\psi(x)$ equals

$$\frac{(\phi(x)\phi''(x) - \phi'(x)^2)(\phi'(x)\int_x^1 (s-x)^2 k(s) ds + 2\phi(x)\int_x^1 (s-x)k(s) ds)}{(a_0a_2 - a_1^2)\phi(x)^2}. \quad (2.13)$$

Each of the terms in this fraction can be estimated readily and, to take two examples,

$$\frac{1}{6}m(1+x)^3 \leq \phi(x) \leq \frac{1}{6}M(1+x)^3,$$

$$\frac{1}{2}m(1+x)^2 \leq \phi'(x) \leq \frac{1}{2}M(1+x)^2.$$

The only combination which causes even slight trouble is $2\phi(x)\phi''(x) - \phi'(x)^2$. This, however, equals

$$\begin{aligned} & \int_{-1}^x (x-s)^2 k(s) ds \int_{-1}^x k(s) ds - \left(\int_{-1}^x (x-s)k(s) ds \right)^2 \\ &= \frac{1}{2} \int_{-1}^x \int_{-1}^x ((x-s) - (x-t))^2 k(s)k(t) ds dt \\ &\geq \frac{1}{2} m^2 \int_{-1}^x \int_{-1}^x (s-t)^2 ds dt = \frac{1}{12} m^2 (1+x)^4, \end{aligned}$$

and, hence, we conclude from (2.13) that $\psi(x)$ is not less than

$$\frac{\frac{1}{12}m^2(1+x)^4 \left[\frac{1}{2}m(1+x)^2 \cdot \frac{1}{3}m(1-x)^3 + \frac{1}{3}m(1+x)^3 \cdot \frac{1}{2}m(1-x)^2 \right]}{(a_0a_2 - a_1^2)M^2(1+x)^6/36} = m^4(1-x)^2/(a_0a_2 - a_1^2)M^2.$$

The manner in which ψ has been defined, and the fact that $G(1, 1) = 0$, enable us to write

$$G(x, x) = \phi(x) \int_x^1 \psi(s) ds$$

and so we have the estimate

$$G(x, x) \geq \frac{1}{6}m(1+x)^3 \cdot \frac{m^4(1-x)^3}{3(a_0a_2 - a_1^2)M^2} = \frac{m^5(1-x^2)^3}{18M^2(a_0a_2 - a_1^2)},$$

which is (2.11).

When we combine (2.11) with (2.9) and (2.10) we deduce that $G(x, y)$ is bounded from below by

$$\frac{m^5(1-x^2)^3}{18M^2(a_0a_2 - a_1^2)} \cdot \frac{1}{M(1+x)^2} \cdot m(1+y)^2 = \frac{m^6(1-x^2)(1-x)^2(1+y)^2}{18M^3(a_0a_2 - a_1^2)}$$

in L , and by

$$\frac{m^5(1-x^2)^3}{18M^2(a_0a_2 - a_1^2)} \cdot \frac{1}{M(1-x)^2} \cdot m(1-y)^2 = \frac{m^6(1-x^2)(1+x)^2(1-y)^2}{18M^3(a_0a_2 - a_1^2)}$$

in U . In either case

$$G(x, y) \geq \frac{m^6(1-x^2)(1-|x|)^2(1-|y|)^2}{18M^3(a_0a_2 - a_1^2)}$$

and so we have established the inequalities (2.4).

We are now in position to show that h is strictly positive. If we interchange x and y in (2.4) and appeal to the symmetry of G we see that

$$A(1-y^2)(1-|y|)^2(1-|x|)^2 \leq G(x, y) \leq B(1-|x|)^2.$$

The upper bound, and the definition (1.3) of g , imply that

$$g(x) \leq B(1-|x|)^2$$

and, therefore, the lower bound implies

$$G(x, y) \geq \frac{A}{B}g(x)(1-y^2)(1-|y|)^2.$$

It now follows from (1.4) that

$$h(y) \geq \frac{A}{B}(1-y^2)(1-|y|)^2$$

and so $h > 0$ in $(-1, 1)$, as required.

3. The uniform beam. If the beam is uniform, the flexural compliance k is a positive constant and the Green's function equals

$$\frac{k}{24}(1-x)^2(1+y)^2[(1+x)(1-y)+x-y]$$

in L , and equals

$$\frac{k}{24}(1+x)^2(1-y)^2[(1-x)(1+y)+y-x]$$

in U .

Elementary considerations show that for any fixed $x \in [-1, 1]$ the maximum value of $G(x, y)$ occurs at $y = x/(2 + |x|)$. Thus

$$g(x) = G\left(x, \frac{x}{2 + |x|}\right) = \frac{k(1-x^2)^2(1+|x|)}{6(2+|x|)^2}.$$

Furthermore, the ratio

$$G(x, y)/g(x) \tag{3.1}$$

equals

$$\frac{(2+|x|)^2[(1+x)(1-y)+x-y](1+y)^2}{4(1+x)^2(1+|x|)} \tag{3.2}$$

in L , and equals

$$\frac{(2+|x|)^2[(1-x)(1+y)+y-x](1-y)^2}{4(1-x)^2(1+|x|)} \tag{3.3}$$

in U . Once again, elementary considerations suffice to show that for any fixed $y \in [-1, 0]$ the greatest lower bound of the ratio (3.1), as a function of x , is the limit to which the ratio tends when $x \rightarrow 1$; on substituting $x = 1$ in (3.2) we see that this is

$$\frac{27}{32}(1-y)(1+y)^2 = \frac{27}{32}(1-y^2)(1-|y|).$$

If $y \in [0, 1]$ the greatest lower bound is the limit to which the ratio tends when $x \rightarrow -1$, and we see from (3.3) that this is

$$\frac{27}{32}(1+y)(1-y)^2 = \frac{27}{32}(1-y^2)(1-|y|).$$

Thus

$$h(y) = \frac{27}{32}(1-y^2)(1-|y|)$$

for every $y \in [-1, 1]$.

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