BACKWARD ENERGY-FLUX FOR SPHERICALLY SYMMETRIC SCALAR WAVES*

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Abstract. I consider spherically symmetric scalar waves of two types: explosive waves in an infinite medium generated by a spherically symmetric source, and implosive waves in which the medium is contained inside a spherically symmetric source. The explosive waves travel outward, the implosive waves inward. Although there are exceptional cases, in general there are, at any time, shells in which the energy-flux is opposed to the sense of propagation of the waves.

1. Introduction. The transmission of energy by waves is so familiar that it is rather surprising to find that there may be locations and times at which energy is flowing the wrong way, i.e., against the direction in which the waves are travelling. Although the argument is purely mathematical, it is useful to think of the medium as a gas and the disturbance ϕ as the (small) excess of pressure over its equilibrium value.

The scalar wave equation is

$$\phi_{xx} + \phi_{yy} + \phi_{zz} - \phi_{tt} = 0, \quad (1.1)$$

the units being chosen to make the speed of propagation unity. The energy-density is

$$E = \frac{1}{2} [(\text{grad } \phi)^2 + \phi_t^2], \quad (1.2)$$

and the energy-flux is the vector

$$\mathbf{F} = -\phi_t \text{grad } \phi, \quad (1.3)$$

a common dimensional factor being reduced to unity by a further choice of units.

Henceforth we shall consider only cases of spherical symmetry about an origin O . If r is the radius vector from O , the above equations become

$$(r\phi)_{rr} - (r\phi)_{tt} = 0, \quad (1.4)$$

$$E = \frac{1}{2} (\phi_r^2 + \phi_t^2), \quad (1.5)$$

$$F = -\phi_t \phi_r; \quad (1.6)$$

F is now a scalar, positive if energy flows in the sense of r increasing.

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2. Explosive and implosive waves. Let the medium extend to infinity, being bounded inside by a sphere $r = a$. This sphere contains machinery which generates on its surface a disturbance $\phi(a, t)$. Suppose the medium is initially at rest. By virtue of (1.4) we have, throughout the medium,

$$\phi(r, t) = (a/r)f(s), \quad s = t + a - r. \quad (2.1)$$

This rather artificial form for the d'Alembert solution for outgoing waves is adopted because, on putting $r = a$, it gives the function f in terms of the surface datum,

$$\phi(a, t) = f(t). \quad (2.2)$$

These are *explosive waves*.

Now suppose the medium is contained in a sphere $r = a$, outside of which there is machinery which generates a disturbance $\phi(a, t)$ on $r = a$. Supposing the medium initially at rest, waves travel inward according to the formula

$$\phi(r, t) = (a/r)f(s), \quad s = t - a + r. \quad (2.3)$$

Again (2.2) holds. These are *implosive waves*. There is a singularity at $r = 0$, so our argument must break off when the wave reaches the origin.

Apart from this singularity, the two types, explosive and implosive, differ rather trivially in a matter of sign, and we can deal with both by a common argument, introducing a parameter

$$\begin{aligned} k &= 1 \text{ for explosive waves,} \\ k &= -1 \text{ for implosive waves.} \end{aligned} \quad (2.4)$$

Now (2.1) and (2.3) are both contained in the single formula:

$$\phi(r, t) = (a/r)f(s), \quad s = t + k(a - r). \quad (2.5)$$

3. Critical characteristics and extremals. We have

$$\phi_t = (a/r)f', \quad \phi_r = -k(a/r)f' - (a/r^2)f, \quad (3.1)$$

and so, by (1.6), the energy-flux is

$$F = (a^2/r^3)f'^2(kr + f/f'). \quad (3.2)$$

Here the argument of f is of course s as in (2.5). Now although the *value* of F may depend chiefly on the leading factors, its *sign* is that of

$$g(r, s) = kr + f/f'. \quad (3.3)$$

The characteristics of our problem are $s = \text{const}$, i.e.,

$$t + k(a - r) = \text{const}. \quad (3.4)$$

I define the *critical characteristics* by the conditions

$$f'(s) = 0, \quad f(s) \neq 0. \quad (3.5)$$

By virtue of (2.2) they have a simple physical meaning since $\phi_t(a, t) = 0$ means that the activity of the source on the boundary $r = a$ has a stationary value but is not zero. On such a characteristic g is not defined, but near it f/f' is very large and changes sign as we

pass through the critical characteristic. Thus, no matter whether $k = 1$ or -1 or how large r may be, F changes sign: *if the flux is forward on one side, it is negative on the other side.*

There are cases where there are no critical characteristics, but they are exceptional. If the source provides a single one-sided pulse, there is at least one critical characteristic. If it provides a quasi-sinusoidal disturbance, there may be a large number of critical characteristics. We are entitled to make the statement that *in general, for both explosive and impulsive disturbances, there are shells in which the energy-flux is opposed to the sense of wave propagation.* They certainly exist if there exists at least one value of t for which the disturbance $\phi(a, t)$ supplied by the source satisfies

$$\phi_t(a, t) = 0, \quad \phi(a, t) \neq 0. \quad (3.6)$$

We can make the regions of backward energy-flux more precise by using the *extremals* for which $\phi_r = 0$. These may be described as crests and troughs of waves. It is clear from (3.1) and (3.3) that g is zero on an extremal. In general then g will change sign on crossing an extremal, and so, if F is positive on one side, it will be negative on the other.

It is hard to make statements valid in all circumstances. For example, g may not change sign on passing through a zero. What follows is a *general* description of the domains of backward energy-flux in the (r, t) plane.

Consider two adjacent critical characteristics. In the domain between them, g ranges from $-\infty$ on one to $+\infty$ on the other. Between the two characteristics there lies an extremal, because in passing from $-\infty$ to $+\infty$, g must pass through the value zero. We then have two bands, in one of which the energy-flux is in the sense of propagation while in the other it is backward.

4. An explosive model. Let the source provide the disturbance

$$\phi(a, t) = c \exp(-bt^2), \quad (4.1)$$

where b is a positive constant. This appears to be a good model for an explosive pulse; it would not do for the impulsive case because, although in the remote past the disturbance was very small, it would get in to $r = 0$ and create a singularity with which the wave equation is not competent to deal.

As in (2.2) we have

$$\begin{aligned} f(s) &= c \exp(-bs^2), \\ f'(s) &= -2cbs \exp(-bs^2), \\ f/f' &= -1/(2bs). \end{aligned} \quad (4.2)$$

There is then only one critical characteristic, $s = t + a - r = 0$. Below this line (reduce t) s is negative and f/f' is positive. Since the problem is explosive, $k = 1$ in (3.3) and so

$$g(r, s) = r + f/f', \quad (4.3)$$

which is positive. Thus the flux is outward throughout that part of the (r, t) plane below the characteristic $s = 0$, which is a line with slope 45° through the point $r = a$, $t = 0$.

We now look *above* the characteristic $s = 0$. By (4.2) and (4.3)

$$g(r, s) = r - 1/(2bs). \quad (4.4)$$

Thus $g = 0$ on the hyperbola

$$rs = \frac{1}{2}b^{-1}, \quad (4.5)$$

or, more explicitly,

$$r(t + a - r) = \frac{1}{2}b^{-1}. \quad (4.6)$$

There is inward flux in the domain bounded by the line $r = a$, the characteristic $s = 0$, and this hyperbola (Fig. 1).

5. Conclusion. There is no doubt that for radiation obeying the usual linear wave equation, backward energy-flux occurs in the case of spherical symmetry. It is an interesting phenomenon, but not of serious importance in the transmission of energy. In order to discuss the *sign* of energy-flux, the formula for it was put in the artificial form (3.2), in which f' appeared of great importance because $f' = 0$ made the last term

Fig. 1. Space-time diagram for the exceptional explosive pulse as in (4.1). The coordinates are (r, t) where r is the distance from the centre of the source. The history of the source is the block marked S on the left, standing on $0 < r < a$. The diagonal PQ is the only critical characteristic, and the hyperbola running from R to T is the only extremal. The critical characteristic and the extremal divide the region outside the source into three parts. In the lowest part, energy flows to the right, which is the direction of propagation. In the middle region it flows to the left, against the direction of propagation. In the top part it flows to the right.

If observations are made at some time t , the relevant events lie on a horizontal line. One such line is $ABCDE$. When we pass from the diagram to three-dimensional space, AB represents a shell adjacent to the source in which energy is flowing backward: for BC it is forward: for CD backward: and from D on it flows forward.

If we seek the history at some fixed station $r = \text{const.}$, this is represented by the line GHJ . From the past up to H , energy flows out: from H to I it flows in: and from J on into the future it flows out.

infinite. To get a better perspective regarding flux of energy, we write (3.2) as

$$F = a^2(f'^2/r^2 + ff'/r^3), \quad (5.1)$$

putting $k = 1$ for the explosive case. Thus the total flux of energy in the period (t_1, t_2) across the sphere $r = \text{const}$, in the sense of r increasing, is

$$\begin{aligned} 4\pi r^2 \int_{t_1}^{t_2} F dt &= 4\pi a^2 \int_{t_1}^{t_2} [f'(t + a - r)]^2 dt + 2\pi a^2 [f(t_2 + a - r)]^2/r \\ &\quad - 2\pi a^2 [f(t_1 + a - r)]^2/r. \end{aligned} \quad (5.2)$$

The first integral on the right represents the accumulation of positive contributions, making a fairly steady flux, in comparison with which the last terms are trivial in the long run; it is the last term in (5.2) which indicates the effect of backward flux.

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