ON A RATIO THEOREM FOR A CLASS OF NONLINEAR BOUNDARY VALUE PROBLEMS*

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Abstract. A generalization of Ussing's flux ratio theorem is established as a consequence of a symmetry property of a Green's function. Bäcklund transformations are then used to construct ratio theorems for certain nonlinear boundary value problems.

1. Introduction. In many linear boundary value problems, reciprocity relations find their expression in terms of symmetry properties of appropriate Green's functions [1]. Such a symmetry is shown to lead directly to a flux ratio theorem of practical importance in membrane physiology [2]. In this context, a membrane is modelled as a slab occupying the interval $0 \leq x \leq h$ in which a tracer has diffusion coefficient $D(x) > 0$ and drift velocity $v(x)$. The concentration $c(x, t)$ of the tracer at time t satisfies the conservation equation

$$\frac{\partial c}{\partial t} + \frac{\partial j}{\partial x} = 0, \quad (1.1)$$

where the flux $j(x, t)$ is given by

$$j = -D(x) \frac{\partial c}{\partial x} + v(x)c, \quad (1.2)$$

whence

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x} \left[-D(x) \frac{\partial c}{\partial x} + v(x)c \right] = 0. \quad (1.3)$$

Two situations are then considered, both in theory and in practice. In the first, indicated by a subscript 1, no tracer is present in the slab at $t = 0$, and the boundary conditions for $t > 0$ are

$$c_1(0, t) = g_1(t) \geq 0, \quad c_1(h, t) = 0. \quad (1.4)$$

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The outflux $j_1(h, t)(\geq 0)$ is monitored for $t > 0$. The second situation, indicated by a subscript 2, is similar except that the boundary conditions are

$$c_2(0, t) = 0, \quad c_2(h, t) = g_2(t) \geq 0, \quad (1.5)$$

and the flux $j_2(0, t)(\leq 0)$ is monitored.

Ussing [3] showed that if g_1 and g_2 are constant, then in the limit as $t \rightarrow \infty$

$$\frac{-j_1(h, t)}{j_2(0, t)} = \frac{g_1}{g_2} \exp \left[\int_0^h \frac{v(x)}{D(x)} dx \right]. \quad (1.6)$$

Later, Sten-Knudsen and Ussing [4] obtained the much more striking result that, in fact, (1.6) holds exactly at all times $t > 0$ if g_1 and g_2 are constant. A further generalization was obtained by Bass and Bracken [5], who showed that the relation (1.6) continues to hold even if g_1 and g_2 vary with time, as long as their ratio is constant. Since (1.3) holds only for a membrane with a single transport pathway parallel to the x -axis, and no transport in other spatial directions, the practical significance of the result is that when a nonconstancy of the flux ratio is found in experiment, the presence of a heterogeneity transverse to the x -axis can be inferred [2].

In the next section, the result that the relation (1.6) holds for $t > 0$ if g_1/g_2 is a constant is derived in a novel manner as a simple consequence of a symmetry of a Green's function. The method of derivation not only throws new light on the nature of the result but also suggests further generalizations.

The main aim of the present work, however, is to show how new ratio theorems may be obtained via Bäcklund transformations for pairs of nonlinear boundary value problems associated with a generalized Burgers' equation and an important reciprocally linked nonlinear evolution equation.

2. The basic result. Consider the boundary value problem

$$\begin{aligned} \frac{\partial c}{\partial t} + \frac{\partial}{\partial x} \left[-D(x) \frac{\partial c}{\partial x} + v(x)c \right] &= 0, \quad 0 < x < h, \quad t > 0, \\ c(0, t) &= g_1(t), \quad t > 0, \\ c(h, t) &= g_2(t), \quad t > 0, \\ c(x, 0) &= 0, \quad 0 < x < h. \end{aligned} \quad (2.1)$$

If we set

$$r(x, t) = c(x, t)/w(x), \quad (2.2)$$

$$w(x) = \exp \left[\int_0^x \frac{v(y)}{D(y)} dy \right] \quad (2.3)$$

then (2.1) becomes

$$\begin{aligned} \frac{\partial r}{\partial t} &= \frac{1}{w(x)} \frac{\partial}{\partial x} \left[\hat{D}(x) \frac{\partial r}{\partial x} \right], \quad 0 < x < h, \quad t > 0, \\ r(0, t) &= r_1(t), \\ r(h, t) &= r_2(t), \\ r(x, 0) &= 0, \end{aligned} \quad (2.4)$$

where

$$\begin{aligned}\hat{D}(x) &= w(x)D(x), \\ r_1(t) &= g_1(t), \quad r_2(t) = g_2(t)/w(h).\end{aligned}\quad (2.5)$$

The solution of the boundary value problem (2.4) admits the integral representation (see Appendix)

$$\begin{aligned}r(x, t) &= \int_0^t r_1(\tau) \hat{D}(0) \left. \frac{\partial K(x, y, t - \tau)}{\partial y} \right|_{y=0} d\tau \\ &\quad - \int_0^t r_2(\tau) \hat{D}(h) \left. \frac{\partial K(x, y, t - \tau)}{\partial y} \right|_{y=h} d\tau\end{aligned}\quad (2.6)$$

where $K(x, y, t)$ is the Green's function satisfying

$$\begin{aligned}\frac{\partial K(x, y, t)}{\partial t} - \frac{1}{w(x)} \frac{\partial}{\partial x} \left[\hat{D}(x) \frac{\partial K(x, y, t)}{\partial x} \right] &= \frac{1}{w(y)} \delta(x - y) \delta(t), \\ 0 &< x, y < h, \quad t \geq 0 \\ \lim_{t \rightarrow 0^+} K(x, y, t) &= \frac{1}{w(y)} \delta(x - y), \quad 0 < x, y < h,\end{aligned}\quad (2.7)$$

$$K(0, y, t) = 0 = K(h, y, t), \quad t > 0,$$

$$K(x, y, t) = K(y, x, t).$$

Now consider:

Case I. $g_1(t) \not\equiv 0$, $g_2(t) = 0$, $t > 0$. Then,

$$r(x, t) = \hat{D}(0) \int_0^t r_1(\tau) \left. \frac{\partial K}{\partial y}(x, y, t - \tau) \right|_{y=0} d\tau, \quad t > 0, 0 \leq x \leq h,$$

and so

$$\left. \frac{\partial r(x, t)}{\partial x} \right|_{x=h} = \hat{D}(0) \int_0^t r_1(\tau) \left. \frac{\partial^2 K}{\partial x \partial y}(x, y, t - \tau) \right|_{x=h, y=0} d\tau, \quad t > 0.$$

Accordingly, the flux at $x = h$ is given by

$$\begin{aligned}j_1(h, t) &= -\hat{D}(h) \left. \frac{\partial r}{\partial x} \right|_{x=h} \\ &= -\hat{D}(0) \hat{D}(h) \int_0^t r_1(\tau) \left. \frac{\partial^2 K(x, y, t - \tau)}{\partial x \partial y} \right|_{x=h, y=0} d\tau, \quad t > 0.\end{aligned}\quad (2.8)$$

Case II. $g_1(t) = 0$, $g_2(t) \not\equiv 0$, $t > 0$. In this case

$$r(x, t) = -\hat{D}(h) \int_0^t r_2(\tau) \left. \frac{\partial K(x, y, t - \tau)}{\partial y} \right|_{y=h} d\tau, \quad t > 0, 0 \leq x \leq h,$$

so that

$$\begin{aligned}j_2(0, t) &= -\hat{D}(0) \left. \frac{\partial r}{\partial x} \right|_{x=0} \\ &= \hat{D}(0) \hat{D}(h) \int_0^t r_2(\tau) \left. \frac{\partial^2 K}{\partial y \partial x}(x, y, t - \tau) \right|_{x=0, y=h} d\tau\end{aligned}\quad (2.9)$$

Hence, if

$$\frac{c_1(0, t)}{c_2(h, t)} = \frac{g_1(t)}{g_2(t)} = r_0 = \text{constant}, \quad (2.10)$$

then, in view of relations (2.5)_{2,3},

$$\begin{aligned} j_1(h, t) &= -r_0 w(h) \hat{D}(0) \hat{D}(h) \int_0^t r_2(\tau) \left. \frac{\partial^2 K(x, y, t - \tau)}{\partial x \partial y} \right|_{x=h, y=0} d\tau \\ &= -r_0 w(h) \hat{D}(0) \hat{D}(h) \int_0^t r_2(\tau) \left. \frac{\partial^2 K(x, y, t - \tau)}{\partial y \partial x} \right|_{x=0, y=h} d\tau \\ &= -r w(h) j_2(0, t), \end{aligned}$$

on use of the symmetry property

$$K(x, y, t) = K(y, x, t).$$

Thus, we obtain the result

$$\frac{-j_1(h, t)}{j_2(0, t)} = \frac{c_1(0, t)}{c_2(h, t)} \exp \left[\int_0^h \frac{v(x)}{D(x)} dx \right], \quad (2.11)$$

where $c_1(0, t)/c_2(h, t) = r_0$.

It is noted that integral relations analogous to (2.6) and hence results similar to (2.11) hold for equations which are obtained by replacing $\partial c/\partial t$ by $\partial^2 c/\partial t^2$ or $-\partial^2 c/\partial t^2$ in (2.1)₁. The physical interpretation of the ratio theorem changes under such replacements. Thus, in the hyperbolic case, we obtain constancy of a ratio of boundary stresses rather than boundary fluxes.

The possibility of extending these results to other geometries is also suggested. Indeed, a ratio theorem for conditions of cylindrical symmetry has already been indicated elsewhere [6].

3. A ratio theorem for a pair of nonlinear boundary value problems. In [7] it was established that the nonlinear Burgers' hierarchy

$$u_t + \left[\left\{ \sum_{i=0}^M \alpha_i \Phi_{i,t} + \sum_{j=0}^N \beta_j \Phi_j \right\} / \sum_{i=0}^M \alpha_i \Phi_i \right]_x = 0, \quad (3.1)$$

where the Φ_j are given recursively by

$$\begin{aligned} \Phi_j &= \Phi_{j-1,x} + \Phi_{j-1} u, \quad j = 1, 2, \dots, \\ \Phi_0 &= 1, \end{aligned} \quad (3.2)$$

is reduced to the linear canonical form

$$\sum_{i=0}^M \alpha_i \frac{\partial^{i+1} \bar{c}}{\partial x^i \partial t} + \sum_{j=0}^N \beta_j \frac{\partial^j \bar{c}}{\partial x^j} = 0 \quad (3.3)$$

by the Bäcklund transformation

$$\begin{aligned} \bar{c}_x &= u \bar{c}, \\ \bar{c}_t &= - \left[\left\{ \sum_{i=0}^M \alpha_i \Phi_{i,t} + \sum_{j=0}^N \beta_j \Phi_j \right\} / \sum_{i=0}^M \alpha_i \Phi_i \right] \bar{c}. \end{aligned} \quad (3.4)$$

The Cole-Hopf transformation, which takes Burgers' equations to the classical 1 + 1 heat equation, is readily retrieved as a special case of this reduction.

Here, application of the above result shows that under the transformation

$$\begin{aligned}\bar{c}_x &= u\bar{c}, \\ \bar{c}_t &= -[(-D(x)u + v(x))_x + (-D(x)u + v(x))u]\bar{c},\end{aligned}\quad (3.5)$$

the boundary value problem

$$\left. \begin{aligned}\frac{\partial \bar{c}}{\partial t} + \frac{\partial}{\partial x} \left[-D(x) \frac{\partial \bar{c}}{\partial x} + v(x)\bar{c} \right] &= 0, \quad 0 < x < h, t > 0, \\ \bar{c} &= \exp \left[-\int_0^t \bar{g}_I(\sigma) d\sigma \right] \quad \text{on } x = 0, t > 0, \\ \bar{c} &= 1 \quad \text{on } x = h, t > 0, \\ \bar{c} &= 1 \quad \text{at } t = 0,\end{aligned}\right\} \quad (3.6)$$

is associated with the *nonlinear* boundary value problem

$$\left. \begin{aligned}u_t + [(-D(x)u + v(x))_x + (-D(x)u + v(x))u]_x &= 0, \quad 0 < x < h, t > 0 \\ (-D(x)u + v(x))_x + (-D(x)u + v(x))u &= \bar{g}_I(t), \quad \text{on } x = 0, t > 0 \\ (-D(x)u + v(x))_x + (-D(x)u + v(x))u &= 0, \quad \text{on } x = h, t > 0 \\ u &= 0 \quad \text{at } t = 0,\end{aligned}\right\} \text{I} \quad (3.7)$$

for a class of generalized Burgers' equations. The boundary conditions correspond to a prescribed flux $\bar{g}_I(t)$ at $x = 0$ with insulation at the boundary $x = h$.

If we set $\bar{c} - 1 = c$ in the original boundary value problem (3.6), we obtain if $v = \text{constant}$

$$\left. \begin{aligned}\frac{\partial c}{\partial t} + \frac{\partial}{\partial x} \left[-D(x) \frac{\partial c}{\partial x} + vc \right] &= 0, \quad 0 < x < h, t > 0, \\ c &= \exp \left[-\int_0^t \bar{g}_I(\sigma) d\sigma \right] - 1 := \Phi_I(t) \quad \text{on } x = 0, t > 0, \\ c &= 0 \quad \text{on } x = h, t > 0, \\ c &= 0 \quad \text{at } t = 0.\end{aligned}\right\} \text{I} \quad (3.8)$$

In a similar manner, the linear boundary value problem

$$\left. \begin{aligned}\frac{\partial c}{\partial t} + \frac{\partial}{\partial x} \left[-D(x) \frac{\partial c}{\partial x} + vc \right] &= 0, \quad 0 < x < h, t > 0, \\ c &= 0 \quad \text{on } x = 0, t > 0, \\ c &= \exp \left[-\int_0^t \bar{g}_{II}(\sigma) d\sigma \right] - 1 := \Phi_{II}(t) \quad \text{on } x = h, t > 0, \\ c &= 0 \quad \text{at } t = 0,\end{aligned}\right\} \text{II} \quad (3.9)$$

under the Bäcklund transformation (3.5) with $\bar{c} = 1 + c$ corresponds to the *nonlinear* boundary value problem

$$\left. \begin{aligned}u_t + [(-D(x)u + v)_x + (-D(x)u + v)u]_x &= 0, \quad 0 < x < h, t > 0, \\ (-D(x)u + v)_x + (-D(x)u + v)u &= 0 \quad \text{on } x = 0, t > 0, \\ (-D(x)u + v)_x + (-D(x)u + v)u &= \bar{g}_{II}(t) \quad \text{on } x = h, t > 0, \\ u &= 0 \quad \text{at } t = 0.\end{aligned}\right\} \text{II} \quad (3.10)$$

Under the Bäcklund transformation (3.5)

$$\begin{aligned} j_I(h, t) &= -D(x)c_{Ix} + vc_I|_{x=h} \\ &= -D(x)\bar{c}_{Ix}|_{x=h} = -D(x)u_I\bar{c}_I|_{x=h} = -D(h)u_I(h, t), \\ j_{II}(0, t) &= -D(x)c_{IIx} + vc_{II}|_{x=0} \\ &= -D(x)\bar{c}_{IIx}|_{x=0} = -D(x)u_{II}\bar{c}_{II}|_{x=0} = -D(0)u_{II}(0, t). \end{aligned}$$

Thus, the extended Ussing result (2.11) shows that the companion *density ratio* relation

$$\frac{-u_I(h, t)}{u_{II}(0, t)} = r_0 \frac{D(0)}{D(h)} \exp\left[\int_0^h \frac{v}{D(x)} dx\right], \quad t > 0, \quad (3.11)$$

for the nonlinear boundary value problems I and II is *time-independent* provided the boundary fluxes $J_I(0, t) = \bar{g}_I(t)$ and $J_{II}(h, t) = \bar{g}_{II}(t)$ are chosen so that the ratio

$$\begin{aligned} r_0 &= \left[\exp\left[-\int_0^t J_I(0, \sigma) d\sigma\right] - 1 \right] / \left[\exp\left[-\int_0^t J_{II}(h, \sigma) d\sigma\right] - 1 \right] \\ &= \Phi_I(t)/\Phi_{II}(t), \quad t > 0 \end{aligned} \quad (3.12)$$

is constant. It is noted that the condition (3.12) is satisfied, in particular, if $\bar{g}_I(t) = \bar{g}_{II}(t)$.

4. A ratio theorem generated by a reciprocal Bäcklund transformation. Here, a reciprocal Bäcklund transformation is applied to the result of the previous section to construct a ratio theorem for a second class of important nonlinear boundary value problems.

In what follows, use is made of a result established in [8] which states that the conservation law

$$\frac{\partial}{\partial t} \left\{ T \left(\frac{\partial}{\partial x}; \frac{\partial}{\partial t}; \bar{u} \right) \right\} + \frac{\partial}{\partial x} \left\{ F \left(\frac{\partial}{\partial x}; \frac{\partial}{\partial t}; \bar{u} \right) \right\} = 0 \quad (4.1)$$

is transformed to the reciprocally associated conservation law

$$\frac{\partial}{\partial t'} \left\{ T' \left(\frac{\partial}{\partial x'}; \frac{\partial}{\partial t'}; \bar{u} \right) \right\} + \frac{\partial}{\partial x'} \left\{ F' \left(\frac{\partial}{\partial x'}; \frac{\partial}{\partial t'}; \bar{u} \right) \right\} = 0 \quad (4.2)$$

by the reciprocal transformation

$$\left. \begin{aligned} dx' &= T dx - F dt, \quad t' = t, \\ T' &= \frac{1}{T(\mathbb{D}'; \partial'; \bar{u})}, \\ F' &= -\frac{F(\mathbb{D}'; \partial'; \bar{u})}{T(\mathbb{D}'; \partial'; \bar{u})}, \end{aligned} \right\} \quad R(4.3)$$

where

$$\mathbb{D}' := \frac{\partial}{\partial x} = \frac{1}{T'} \frac{\partial}{\partial x'}, \quad (4.4)$$

$$\partial' := \frac{\partial}{\partial t} = \frac{F'}{T'} \frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}, \quad (4.5)$$

and we have adopted the notation

$$\begin{aligned} \Phi \left(\frac{\partial}{\partial x}; \frac{\partial}{\partial t}; \bar{u} \right) &:= (\bar{u}, \bar{u}_x, \bar{u}_{xx}, \dots; \bar{u}_t, \bar{u}_{tt}, \dots), \\ \Psi \left(\frac{\partial}{\partial x'}; \frac{\partial}{\partial t'}; \bar{u} \right) &:= \Psi(\bar{u}, \bar{u}_{x'}, \bar{u}_{x'x'}, \dots; \bar{u}_{t'}, \bar{u}_{t't'}, \dots). \end{aligned}$$

The reciprocal nature of the transformation resides in the involutory property $R^2 = I$.

The result of Sec. 3 is now specialized to the case when the nonlinear evolution equation in I and II reduces to the conventional Burgers' equation with a convective term, namely

$$u_t + [-d_0 u_x + (-d_0 u + v_0)u]_x = 0, \quad d_0 > 0. \quad (4.6)$$

If we set $u = \bar{u} - 1$, then the boundary value problem I becomes

$$\left. \begin{aligned} \bar{u}_t + [-d_0 \bar{u}_x + (-d_0(\bar{u} - 1) + v_0)(\bar{u} - 1)]_x &= 0, \quad 0 < x < h, \quad t > 0, \\ -d_0 \bar{u}_x + (-d_0(\bar{u} - 1) + v_0)(\bar{u} - 1) &= \bar{g}_l(t) \quad \text{on } x = 0, \quad t > 0 \\ -d_0 \bar{u}_x + (-d_0(\bar{u} - 1) + v_0)(\bar{u} - 1) &= 0 \quad \text{on } x = h, \quad t > 0 \\ \bar{u} &= 1 \quad \text{at } t = 0. \end{aligned} \right\} \quad (4.7)$$

We now introduce the reciprocal Bäcklund transformation

$$\left. \begin{aligned} dx' &= \bar{u} dx - [-d_0 \bar{u}_x + (-d_0(\bar{u} - 1) + v_0)(\bar{u} - 1)] dt \\ &= T dx - F dt, \\ t' &= t, \\ \bar{u}' &= 1/\bar{u}. \end{aligned} \right\} R \quad (4.8)$$

The boundary value problem (4.7) becomes

$$\left. \begin{aligned} \frac{\partial \bar{u}'}{\partial t'} - \frac{\partial}{\partial x'} \left[-d_0 \bar{u}' \mathbb{D}' \left(\frac{1}{\bar{u}'} \right) + (-d_0(1 - \bar{u}') + v_0 \bar{u}') \left(\frac{1}{\bar{u}'} - 1 \right) \right] &= 0, \\ -d_0 \mathbb{D}' \left(\frac{1}{\bar{u}'} \right) + \frac{1}{\bar{u}'} (-d_0(1 - \bar{u}') + v_0 \bar{u}') \left(\frac{1}{\bar{u}'} - 1 \right) &= \bar{g}_l(t) \quad \text{on } x' = \Phi_0, \\ -d_0 \mathbb{D}' \left(\frac{1}{\bar{u}'} \right) + \frac{1}{\bar{u}'} (-d_0(1 - \bar{u}') + v_0 \bar{u}') \left(\frac{1}{\bar{u}'} - 1 \right) &= 0 \quad \text{on } x' = \Phi_h, \\ \bar{u}' &= 1 \quad \text{at } t' = 0, \end{aligned} \right\} \quad (4.9)$$

where

$$\mathbb{D}' := \frac{\partial}{\partial x} = \frac{1}{T'} \frac{\partial}{\partial x'} = \frac{1}{\bar{u}'} \frac{\partial}{\partial x'}$$

and

$$x' = \Phi_0 \text{ corresponds to } x = 0,$$

$$x' = \Phi_h \text{ corresponds to } x = h,$$

under R .

Now, under R

$$\begin{aligned} \frac{\partial x'}{\partial x} &= T, \quad \frac{\partial x'}{\partial t} = -F = -\int_0^x \frac{\partial F}{\partial x} dx - F|_{x=0} \\ &= \int_0^x \frac{\partial T}{\partial t} dx - F|_{x=0}, \end{aligned}$$

so that

$$x'(x, t) = \int_0^x \bar{u} dx - K_l(t) + K_l(0) \quad (4.10)$$

where $\dot{K}_I(t) = \bar{g}_I(t)$ and we have set $x'(0, 0) = 0$. Thus, $x = 0$ corresponds to the moving boundary $x' = -K_I(t') + K_I(0)$ under R . On the other hand, $x = h$ corresponds to

$$x' = \int_0^h \bar{u} dx - K_I(t) + K_I(0).$$

But

$$[F]_0^h = \int_0^h \frac{\partial F}{\partial x} dx = - \int_0^h \frac{\partial T}{\partial t} dx,$$

so that

$$\int_0^h \frac{\partial T}{\partial t} dx = \bar{g}_I(t),$$

whence, on use of the initial condition $\bar{u}|_{t=0} = 1$, it is seen that

$$\int_0^h \bar{u} dx = K_I(t) - K_I(0) + h. \quad (4.11)$$

Accordingly, the boundary $x = h$ corresponds to $x' = h$ under R .

Thus, we obtain the reciprocally associated nonlinear boundary value problem

$$\left. \begin{aligned} \frac{\partial \bar{u}'}{\partial t'} - \frac{\partial}{\partial x'} \left[\frac{d_0}{\bar{u}'^2} \frac{\partial \bar{u}'}{\partial x'} - \frac{d_0}{\bar{u}'} - (d_0 + v_0) \bar{u}' \right] &= 0, \\ \frac{d_0}{\bar{u}'^2} \frac{\partial \bar{u}'}{\partial x'} - \frac{d_0}{\bar{u}'} - (d_0 + v_0) \bar{u}' &= f + \bar{g}_I(t') \bar{u}' \quad \text{on } x' = -K_I(t') + K_I(0), \quad t' > 0, \\ \frac{d_0}{\bar{u}'^2} \frac{\partial \bar{u}'}{\partial x'} - \frac{d_0}{\bar{u}'} - (d_0 + v_0) \bar{u}' &= f \quad \text{on } x' = h, \quad t' > 0, \\ u' &= 1 \quad \text{at } t' = 0, \end{aligned} \right\} \text{I}' \quad (4.12)$$

where $f = -v_0 - 2d_0$.

In a similar manner, the boundary value problem

$$\left. \begin{aligned} \bar{u}_t + [-d_0 \bar{u}_x + (-d_0(\bar{u} - 1) + v_0)(\bar{u} - 1)]_x &= 0 \quad 0 < x < h, \quad t > 0, \\ -d_0 \bar{u}_x + (-d_0(\bar{u} - 1) + v_0)(\bar{u} - 1) &= 0 \quad \text{on } x = 0, \quad t > 0, \\ -d_0 \bar{u}_x + (-d_0(\bar{u} - 1) + v_0)(\bar{u} - 1) &= \bar{g}_{II}(t) \quad \text{on } x = h, \quad t > 0, \\ \bar{u} &= 1 \quad \text{at } t = 0, \end{aligned} \right\} \quad (4.13)$$

becomes, under R ,

$$\left. \begin{aligned} \frac{\partial \bar{u}'}{\partial t'} - \frac{\partial}{\partial x'} \left[\frac{d_0}{\bar{u}'^2} \frac{\partial \bar{u}'}{\partial x'} - \frac{d_0}{\bar{u}'} - (d_0 + v_0) \bar{u}' \right] &= 0, \\ \frac{d_0}{\bar{u}'^2} \frac{\partial \bar{u}'}{\partial x'} - \frac{d_0}{\bar{u}'} - (d_0 + v_0) \bar{u}' &= f \quad \text{on } x' = 0, \quad t' > 0 \\ \frac{d_0}{\bar{u}'^2} \frac{\partial \bar{u}'}{\partial x'} - \frac{d_0}{\bar{u}'} - (d_0 + v_0) \bar{u}' &= f + \bar{g}_{II}(t') \bar{u}' \quad \text{on } x' = h - K_{II}(t') + K_{II}(0), \quad t' > 0, \\ \bar{u}' &= 1 \quad \text{at } t' = 0, \end{aligned} \right\} \text{II}' \quad (4.14)$$

where $\dot{K}_{II}(t) = \bar{g}_{II}(t)$. The density ratio theorem of the previous section now shows that the ratio

$$-[1/\bar{u}'_I|_{x'=h} - 1]/[1/\bar{u}'_{II}|_{x'=0} - 1] = r_0 \exp[v_0 h/d_0] \quad (4.15)$$

is *constant* provided that the functions $\bar{g}_I(t')$ and $\bar{g}_{II}(t')$ which arise in the nonlinear boundary value problems I' and II' are restricted by the condition (3.12).

It is noted that nonlinear evolution equations of the type occurring in the above reciprocal boundary value problems I' and II' arise in the analysis of two-phase flow in a porous medium [9, 10].

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Appendix. Consider the standard Sturm-Liouville problem

$$L_x \phi \equiv \frac{1}{w(x)} \frac{d}{dx} \left[\hat{D}(x) \frac{d\phi(x)}{dx} \right] = -\lambda \phi(x), \quad 0 < x < h, \quad \left. \begin{aligned} \phi(0) = 0 = \phi(h), \end{aligned} \right\} \quad (A.1)$$

with the countable set of real eigenfunctions ϕ_n , $n = 1, 2, \dots$, normalized so that

$$\int_0^h w(x) \phi_m(x) \phi_n(x) dx = \delta_{mn} \quad (A.2)$$

and with associated positive eigenvalues λ_n .

It is readily checked that the system (2.7) is satisfied by the Green's function

$$K(x, y, t) = \begin{cases} \sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n(x) \phi_n(y), & t > 0, \\ 0, & t < 0. \end{cases} \quad (A.3)$$

We now turn to the boundary value problem (2.4). Generalization of a standard method used for the classical heat equation shows on use of (2.4)₁ that

$$\int_{\delta}^{t-\varepsilon} \int_0^h w(y) K(x, y, t-\tau) [r_{\tau}(y, \tau) - L_y r(y, \tau)] dy d\tau = 0, \quad (A.4)$$

where $\varepsilon > 0$, $\delta > 0$, and $t - \varepsilon > \delta$.

On integration by parts, we obtain

$$\begin{aligned} & \int_0^h w(y) \left\{ [K(x, y, t-\tau) r(y, \tau)]_{\delta}^{t-\varepsilon} + \int_{\delta}^{t-\varepsilon} K_t(x, y, t-\tau) r(y, \tau) d\tau \right\} dy \\ & - \int_{\delta}^{t-\varepsilon} [K(x, y, t-\tau) \hat{D}(y) r_y(y, \tau) - K_y(x, y, t-\tau) \hat{D}(y) r(y, \tau)]_0^h \\ & + \int_0^h w(y) L_y K(x, y, t-\tau) r(y, \tau) dy \Big\} d\tau = 0; \end{aligned}$$

that is, on use of (2.7)₁

$$\int_0^h w(y) \left\{ \left[K(x, y, t - \tau) r(y, \tau) \right]_{\delta}^{t-\epsilon} dy - \int_{\delta}^{t-\epsilon} \left[K(x, y, t - \tau) \hat{D}(y) r_y(y, \tau) - K_y(x, y, t - \tau) \hat{D}(y) r(y, \tau) \right]_0^h d\tau \right\} = 0. \quad (\text{A.5})$$

If we now let δ and ϵ tend to zero, then the initial condition (2.4)₄ together with (2.7)₂ show that the first term in (A.5) reduces to $r(x, t)$. Thus, on use of (2.7)_{3,4} it is seen that (A.5) yields

$$r(x, t) = \int_0^t r(0, \tau) \hat{D}(0) \left. \frac{\partial K(x, y, t - \tau)}{\partial y} \right|_{y=0} d\tau - \int_0^t r(h, \tau) \hat{D}(h) \left. \frac{\partial K(x, y, t - \tau)}{\partial y} \right|_{y=h} d\tau, \quad (\text{A.6})$$

as required.

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