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ON HELICITY FLUCTUATIONS IN TURBULENCE*

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Abstract. The role that helicity fluctuations play in relationship to small-scale intermittency and coherent structures in turbulent flows is examined from a theoretical standpoint. It has been argued recently that regions of large turbulent activity are associated with regions of small local helicity and, hence, that such a correlation exists. In this paper, it will be proven that fluctuations in the local helicity are not Galilean invariant and, consequently, it is possible to have completely different local helicity fluctuations for a given fluctuating velocity field and its associated turbulence statistics. It is thus highly doubtful that local helicity fluctuations can be of any fundamental value in correlating small-scale intermittency or coherent structures.

1. Introduction. In recent years, there has been an increasing interest in the role that helical structures play in turbulent flows, [1, 2]. Levich and his co-workers [2–5] have argued that helicity fluctuations may be related to small-scale intermittency and coherent structures in turbulence. In fact, they have conjectured that coherent structures have the characteristic that their global helicity is essentially nonzero during the time of their existence. More recently, Pelz et al. [6] and Orszag [7] calculated local helicity fluctuations (based on the cosine of the angle between the velocity and vorticity vectors) in a direct numerical simulation of the Navier–Stokes equations for turbulent channel flow. While the results of these studies were interesting (e.g., they were suggestive of a Beltrami-type flow in the interior of the channel which could be useful in modeling coherent structures in turbulence), no conclusive support could be provided concerning these conjectures about the relationship between regions of large turbulent activity and helicity fluctuations. If anything, some suspicions were cast on such a connection since the cosine of the angle between the fluctuating velocity and fluctuating vorticity gave rise to a flat probability distribution across the channel.

The purpose of the present paper is to explore in more detail the role of helicity fluctuations in turbulence from a theoretical standpoint. It will be demonstrated that local

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helicity fluctuations are not Galilean invariant. Consequently, it is possible to have completely different helicity fluctuations for a given fluctuating velocity and pressure field which form a unique set of turbulence statistics. On this basis, it is highly doubtful that there can be a direct correlation between local helicity fluctuations and small-scale intermittency. Likewise, it will be shown that any such direct correlation with coherent structures is also doubtful. The significance of these results in light of some of the recent computations dealing with local helicity fluctuations will be discussed in more detail.

2. The properties of local helicity fluctuations. The turbulent flow of a homogeneous and incompressible viscous fluid will be considered which is governed by the Navier–Stokes equations and continuity equation

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u}, \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0. \tag{2}$$

In Eq. (1), ρ is the density of the fluid, μ is the dynamic viscosity of the fluid, $\mathbf{u}$ is the velocity field, and p is the pressure field (for simplicity, it has been assumed that the body forces are conservative). As in the usual studies of the turbulence, the velocity $\mathbf{u}$ and pressure p will be decomposed into mean and fluctuating parts as follows:

$$\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}', \qquad p = \bar{p} + p', \tag{3}$$

where, in general, an overbar represents an ensemble average. For statistically steady or homogeneous turbulence, time averages or spatial averages can be substituted, respectively. Here, the mean velocity $\bar{\mathbf{u}}$ and fluctuating velocity $\mathbf{u}'$ are solutions of the momentum equations [8]:

$$\rho\left(\frac{\partial \bar{\mathbf{u}}}{\partial t} + \bar{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}}\right) = -\nabla\bar{p} + \mu\nabla^2\bar{\mathbf{u}} + \nabla\cdot\boldsymbol{\tau}, \tag{4}$$

$$\rho\left(\frac{\partial \mathbf{u}'}{\partial t} + \bar{\mathbf{u}}\cdot\nabla\mathbf{u}'\right) = -\rho\mathbf{u}'\cdot\nabla\mathbf{u}' - \rho\mathbf{u}'\cdot\nabla\bar{\mathbf{u}} - \nabla p' + \mu\nabla^2\mathbf{u}' - \nabla\cdot\boldsymbol{\tau}, \tag{5}$$

where

$$\nabla\cdot\bar{\mathbf{u}} = 0, \tag{6}$$

$$\nabla\cdot\mathbf{u}' = 0, \tag{7}$$

and $\boldsymbol{\tau} = -\rho\overline{\mathbf{u}'\mathbf{u}'}$ is the Reynolds stress tensor. Of course, Eqs. (4)–(7) are a rigorous consequence of the Navier–Stokes equations and the continuity equation given by Eqs. (1) and (2).

The local helicity is defined as the quantity

$$h = \mathbf{u}\cdot\boldsymbol{\omega}, \tag{8}$$

where

$$\boldsymbol{\omega} = \nabla\times\mathbf{u} \tag{9}$$

is the vorticity vector. The vorticity $\boldsymbol{\omega}$ can be split into a mean and fluctuating part as follows:

$$\boldsymbol{\omega} = \bar{\boldsymbol{\omega}} + \boldsymbol{\omega}', \tag{10}$$

where

$$\bar{\boldsymbol{\omega}} = \nabla \times \bar{\mathbf{u}}, \qquad \boldsymbol{\omega}' = \nabla \times \mathbf{u}' \tag{11}$$

Likewise, the local helicity can be similarly decomposed:

$$h = \bar{h} + h', \tag{12}$$

where

$$\bar{h} = \bar{\mathbf{u}} \cdot \bar{\boldsymbol{\omega}} + \overline{\mathbf{u}' \cdot \boldsymbol{\omega}'}, \tag{13}$$

$$h' = \bar{\mathbf{u}} \cdot \boldsymbol{\omega}' + \mathbf{u}' \cdot \bar{\boldsymbol{\omega}} + \mathbf{u}' \cdot \boldsymbol{\omega}' - \overline{\mathbf{u}' \cdot \boldsymbol{\omega}'}. \tag{14}$$

Now it will be proven that although $\mathbf{u}'$ (as well as $\bar{\boldsymbol{\omega}}, \boldsymbol{\omega}'$, and p') is Galilean invariant, neither h, $\bar{h}$, nor h' has this property. A Galilean transformation is defined as follows:

$$\mathbf{x}^* = \mathbf{x} + \mathbf{V}t + \mathbf{C}, \tag{15}$$

where $\mathbf{V}$ and $\mathbf{C}$ are any *constant* vectors. If $\mathbf{x}$ is an inertial framing, then $\mathbf{x}^*$ will constitute the class of inertial frames of reference whose motions differ by an arbitrary constant translational velocity $\mathbf{V}$. By differentiating Eq. (15), it is a simple matter to show that

$$\mathbf{u}^* = \mathbf{u} + \mathbf{V} \tag{16}$$

under a Galilean transformation. Since the mean of a constant vector is the *same* constant vector, it is clear that

$$\bar{\mathbf{u}}^* = \bar{\mathbf{u}} + \mathbf{V}; \tag{17}$$

hence

$$\mathbf{u}'^* = \mathbf{u}' \tag{18}$$

under a Galilean transformation. By taking the curl of Eqs. (17) and (18), it follows that

$$\bar{\boldsymbol{\omega}}^* = \bar{\boldsymbol{\omega}}, \tag{19}$$

$$\boldsymbol{\omega}'^* = \boldsymbol{\omega}'; \tag{20}$$

hence, the vorticity vectors $\bar{\boldsymbol{\omega}}$ and $\boldsymbol{\omega}'$ are Galilean invariant as well as $\mathbf{u}'$. Furthermore, we must have

$$p'^* = p' \tag{21}$$

under a Galilean transformation, since the concept of force is frame independent. However, as a direct consequence of the Galilean transformations (16)–(20), it is a simple matter to show that

$$h^* = h + \mathbf{V} \cdot \boldsymbol{\omega}, \tag{22}$$

$$\bar{h}^* = \bar{h} + \mathbf{V} \cdot \bar{\boldsymbol{\omega}}, \tag{23}$$

$$h'^* = h' + \mathbf{V} \cdot \boldsymbol{\omega}'. \tag{24}$$

Therefore, it is clear that neither the local helicity nor its mean and fluctuating parts are Galilean invariant.

It will now be demonstrated that since $\mathbf{u}'$ is Galilean invariant, while neither $\bar{h}$ nor h' has this property, serious doubts are cast on the usefulness of such helicity fluctuations in correlating small-scale intermittency or coherent structures. The Galilean transformations (16)–(18) can be physically realized by conducting a turbulence experiment in the laboratory frame of reference and then repeating the *same* identical experiment in a frame of reference which is translating with some constant velocity $\mathbf{V}$ relative to the laboratory framing (see Fig. 1). Since the equations of motion (1) and (2) are Galilean invariant, it follows that the velocity fields obtained from these two experiments are related as follows:

$$\mathbf{u}_{(b)} = \mathbf{u}_{(a)} + \mathbf{V}, \qquad \bar{\mathbf{u}}_{(b)} = \bar{\mathbf{u}}_{(a)} + \mathbf{V}. \tag{25}$$

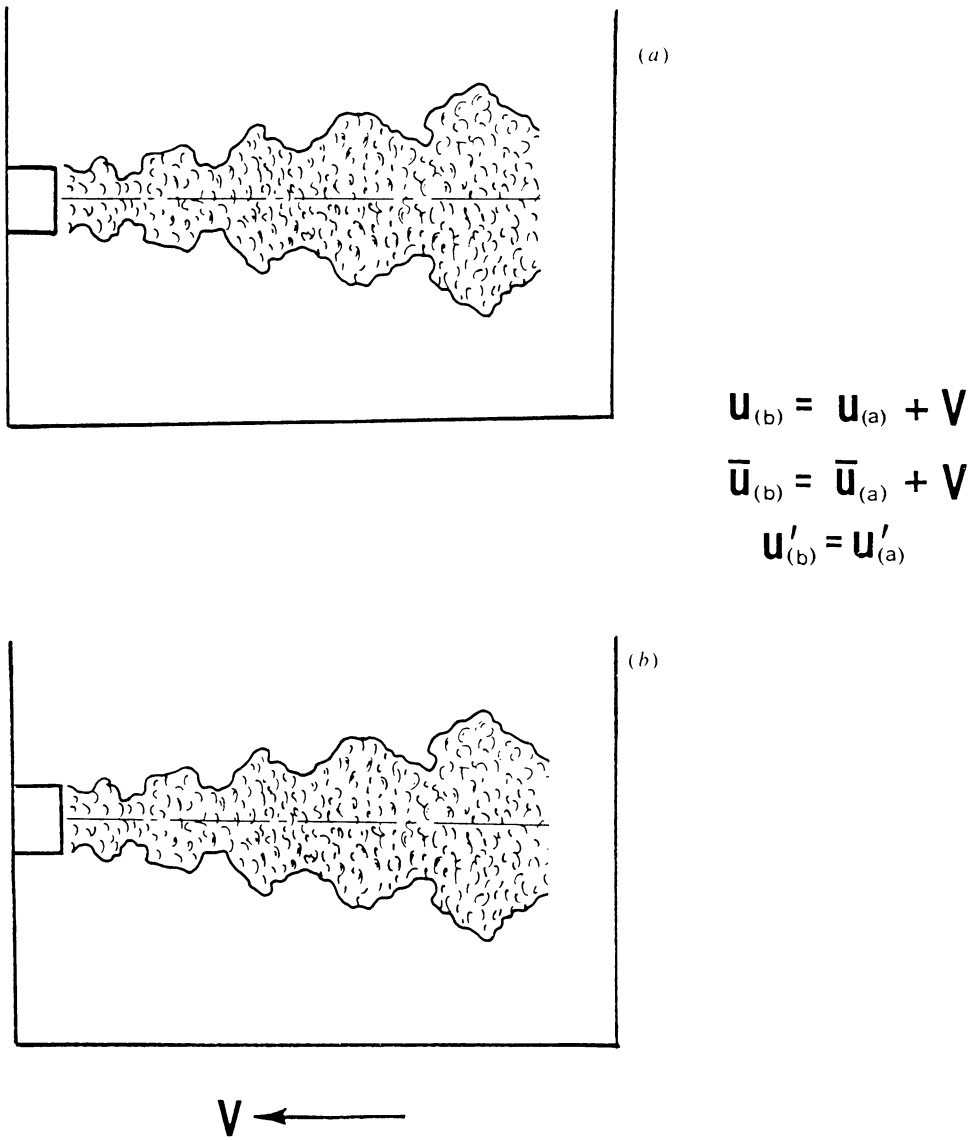

FIGURE 1. Characteristic turbulence experiment:
(a) in the laboratory frame of reference
(b) in a frame of reference translating with a constant velocity **V** relative to the laboratory framing.

Hence, the fluctuating velocities and pressures are the same, i.e.,

$$\mathbf{u}'_{(b)} = \mathbf{u}'_{(a)}, \qquad p'_{(b)} = p'_{(a)}. \tag{26}$$

However, as a direct consequence of (23) and (24), the mean and fluctuating helicities for the two cases are completely different, i.e.,

$$\bar{h}_{(b)} = \bar{h}_{(a)} + \mathbf{V} \cdot \bar{\boldsymbol{\omega}}, \tag{27}$$

$$h'_{(b)} = h'_{(a)} + \mathbf{V} \cdot \boldsymbol{\omega}', \tag{28}$$

where $\mathbf{V}$ is arbitrary.

Small-scale intermittency as well as coherent structures in turbulence are effects which can be characterized by the nature of the fluctuating velocity and pressure fields. For instance, small-scale intermittency is often characterized by the structure of the dissipation rate and kinetic energy of turbulence, respectively, given by [8]:

$$2\nu \overline{\frac{\partial u'_k}{\partial x_l} \frac{\partial u'_k}{\partial x_l}}, \qquad \frac{1}{2}\rho \overline{\mathbf{u}' \cdot \mathbf{u}'}$$

(where $\nu \equiv \mu/\rho$ is the kinematic viscosity and the Einstein summation convention applies to repeated indices). These quantities are determined uniquely by the fluctuating velocity. Since there can be completely different local helicity fluctuations for a given fluctuating velocity and pressure field, it is highly unlikely that this measure can be of any fundamental importance in characterizing such effects. This fact can be dramatically demonstrated if one considers the normalized helicity (i.e., the cosine of the angle between the velocity and vorticity vector) given by

$$h_N = \frac{\mathbf{u} \cdot \boldsymbol{\omega}}{|\mathbf{u}||\boldsymbol{\omega}|}, \tag{29}$$

which forms the basis of the calculations conducted by Pelz et al. [6]. Under a Galilean transformation (which has no effect on the fluctuating velocity field), we have

$$h_N^* = \frac{(\mathbf{u} + \mathbf{V}) \cdot \boldsymbol{\omega}}{|\mathbf{u} + \mathbf{V}||\boldsymbol{\omega}|}. \tag{30}$$

If $\mathbf{V}$, which is arbitrary, is chosen such that

$$|\mathbf{V}| \gg |\mathbf{u}|, \tag{31}$$

it follows that

$$h_N^* \approx \frac{\boldsymbol{\lambda} \cdot \boldsymbol{\omega}}{|\boldsymbol{\omega}|} = \cos(\boldsymbol{\lambda}, \boldsymbol{\omega}), \tag{32}$$

where $\boldsymbol{\lambda} \equiv \mathbf{V}/|\mathbf{V}|$ is an arbitrary unit vector. Hence, for a given fluctuating velocity, the normalized helicity can be the angle between the vorticity vector and *any* arbitrary direction.

It should be noted at this point that in turbulent flows where the mean velocity $\bar{\mathbf{u}}$ vanishes, local helicity fluctuations reduce to the form

$$h = \mathbf{u}' \cdot \boldsymbol{\omega}', \tag{33}$$

which (as a result of (18)) is Galilean invariant. Hence, it could be argued that this problem might be resolved by using the quantity $\mathbf{u}' \cdot \boldsymbol{\omega}'$ instead of $\mathbf{u} \cdot \boldsymbol{\omega}$ in correlating inhomogeneous turbulent flows. However, this quantity does *not* account for interactions between the mean and fluctuating velocity fields, which are known to play an important role in characterizing coherent structures. Furthermore, the recent computations of Pelz et. al. [6] indicated that the normalized helicity measure based on $\mathbf{u}' \cdot \boldsymbol{\omega}'$ was *not* of any use in correlating turbulence activity in channel flow. While there are some questions about the accuracy of their calculations (only 32 Fourier modes were used), at the minimum it is clear that serious reservations can be raised about the suitability of the measure $\mathbf{u}' \cdot \boldsymbol{\omega}'$ in characterizing coherent structures or small-scale intermittency in turbulent flows with nonzero mean velocities. Of course, as demonstrated above, fluctuations in the local helicity $\mathbf{u} \cdot \boldsymbol{\omega}$ are not of any fundamental value in correlating turbulence activity where there is a nonzero mean flow because of the problems which arise from its lack of Galilean invariance.

3. Conclusion. It has been proven that fluctuations in the local helicity $\mathbf{u} \cdot \boldsymbol{\omega}$ are not Galilean invariant. As a direct consequence of this lack of invariance, it was demonstrated that it is possible to have completely different helicity fluctuations for a given fluctuating velocity and pressure field as well as their associated turbulence statistics. Hence, it is extremely unlikely that these helicity fluctuations can be correlated with turbulence activity such as coherent structures and small-scale intermittency since these effects are characterized by the nature of the velocity and pressure field fluctuations. The only possible exception to this conclusion is the special case of vanishing mean flow for which local helicity fluctuations (as well as various other measures of helicity) reduce to the invariant measure $\mathbf{u}' \cdot \boldsymbol{\omega}'$. Such flows, however, are not of much value in gaining a better understanding of coherent structures or small-scale intermittency in turbulence and, thus, were excluded from consideration in the calculations of Pelz et al. [6]. If measures of helicity are to shed any important new light on the structure of turbulence, they must be formulated in such a way that they are suitable for the description of inhomogeneous turbulent flows.

The results of this study clearly demonstrate that fluctuations in the local helicity $\mathbf{u} \cdot \boldsymbol{\omega}$ should be abandoned in favor of alternative measures of helicity which are Galilean invariant. Several such measures of helicity will be the subject of another paper. However, substantial future research from both a theoretical and computational standpoint is needed before any definitive conclusions can be drawn concerning the relationship between helical structures and small-scale intermittency or coherent structures in turbulence.

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