QUASI-TRANSONIC FLOW PAST DELTA WINGS*

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Abstract. Linearized supersonic theory for flows past delta wings must be corrected for the cases of wing sweepback angles close to the Mach angle. Results of calculations based on a local *transonic* expansion valid near the leading edge are presented.

1. Introduction. Consider supersonic flow past a thin wing whose leading edge is swept close to the Mach angle. The undisturbed upstream flow is in the x direction with speed U as shown in Fig. 1. The planform of the wing is as shown in Fig. 2.

As pointed out by Fraenkel and Watson [2], linear theory is incorrect at points near the Mach cone and near the leading edge. The reason is that the velocity normal to the leading edge is nearly sonic and *transonic* (nonlinear) effects dominate there. Experimental studies (see Love [5]) also indicate the failure of linear theory for wings in that range of

FIG. 1

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FIG. 2

sweepback angles. This leads to a search for a local solution to be valid near the leading edge. This local solution should match with that of linear theory away from the leading edge.

Linear theory is obtained from the expansion

$$\Phi = U \{ x + \delta \phi(x, y, z; M_\infty) + \delta^2 \phi_2(x, y, z; M_\infty) + \cdots \},$$

where

M_∞ = Mach number upstream,

U = velocity in the x direction upstream,

δ = thickness ratio of the wing,

Φ = velocity potential.

The exact full equation for Φ is (see [4])

$$a^2 \nabla^2 \Phi = \Phi_x^2 \Phi_{xx} + \Phi_y^2 \Phi_{yy} + \Phi_z^2 \Phi_{zz} + 2(\Phi_x \Phi_y \Phi_{xy} + \Phi_y \Phi_z \Phi_{yz} + \Phi_z \Phi_x \Phi_{zx}),$$

where

$$\frac{a^2}{U^2} = \frac{1}{M_\infty^2} + \frac{\gamma - 1}{2} \left(1 - \frac{|\nabla \Phi|^2}{U^2} \right)$$

and

γ = ratio of specific heats.

The resulting linear equation for the first correction is

$$(M_\infty^2 - 1) \phi_{xx} - (\phi_{yy} + \phi_{zz}) = 0 \quad (1)$$

with the boundary condition of tangent flow on the wing

$$\phi_y(x, 0+, z) = \frac{\partial F}{\partial x}(x, z), \quad (2)$$

where $y = \delta F(x, z)$ is the equation of the upper surface of the wing.

For the case of a wing symmetric about the $y = 0$ plane (thickness problem), the solution to Eqs. (1) and (2) may be represented by a source distribution in the plane of the wing $y = 0$ as

$$\phi(x, y, z) = -\frac{1}{\pi} \iint \frac{\phi_y(x', 0+, z')}{\sqrt{(x - x')^2 - \beta^2 y^2 - \beta^2 (z - z')^2}} dx' dz', \quad (3)$$

where the integral is over the region in the (x, z) -plane bounded by the leading edge and the retrograde Mach cone.

This integral can be worked out analytically for certain cases (see Lagerstrom [3] and Puckett [8]) and will provide the far-field condition to the solution near the leading edge.

The local transonic expansion near the leading edge as given by Cole [1] is outlined below.

2. Quasi-transonic flow. Consider the cases when

$$F(x, z) = a(z)(x - k|z|)^\kappa + \dots \quad 0 < \kappa \leq 1,$$

near the leading edge, with k close to $\beta = \sqrt{M_\infty^2 - 1}$.

The local expansion is

$$\Phi = U\{x + \delta^{4\sigma}\varphi(\xi, \eta, \tau; \beta) + \dots\}, \quad (4)$$

where

$$\xi = \frac{x - \beta z}{\delta^{2\sigma}}, \quad \eta = \frac{y}{\delta^\sigma}, \quad \tau = z, \quad k = \beta + \delta^{2\sigma}\mu, \quad (5)$$

and $\sigma = 1/(5 - 2\kappa)$. Here ξ , η , and τ are local coordinates measuring distance downstream, vertical distance from the leading edge, and spanwise location, respectively, and μ is a constant independent of δ which measures the deviation of the edge from sonic. The analysis can also be generalized to a curved leading edge.

On inserting expansion (4) into the exact full potential equation, we collect the dominant terms. The resulting equation for φ is exactly that for nonsteady two-dimensional transonic flow

$$(\gamma + 1)M_\infty^4 \varphi_\xi \varphi_{\xi\xi} - \varphi_{\eta\eta} + 2\beta \varphi_{\xi\tau} = 0 \quad (6)$$

and the boundary condition is

$$\varphi_\eta(\xi, 0 +, \tau) = \kappa a(\tau)(\xi - \mu\tau)^{\kappa-1} \quad \text{for } \xi > \mu\tau. \quad (7)$$

To this we add a far-field condition provided by matching with the linear theory solution.

3. Symmetric delta wing. In what follows we consider the case of a nonlifting symmetric delta wing with wedge cross section. Here $a = 1/2$, $\kappa = 1$, and expansion (4) becomes

$$\Phi = U\{x + \delta^{4/3}\varphi(\xi, \eta, \tau) + \dots\} \quad (8)$$

with

$$\xi = (x - \beta z)/\delta^{2/3}, \quad \eta = y/\delta^{1/3}, \quad \tau = z, \quad (9)$$

while $k = \beta - \delta^{2/3}\mu$ and the boundary condition (7) is

$$\varphi_\eta(\xi, 0+, \tau) = \frac{1}{2} \quad \text{for } \xi > \mu\tau. \quad (10)$$

The far-field condition is obtained from (3). For the case $\mu = 0$

$$\phi = \frac{-1}{\pi\beta} \sqrt{x^2 - \beta^2(y^2 + z^2)} + \frac{y}{\pi} \cos^{-1} \frac{\beta y}{\sqrt{x^2 - \beta^2 z^2}}$$

and the far field is

$$\varphi \rightarrow -\frac{1}{\beta\pi} \sqrt{2\beta\xi\tau - \beta^2\eta^2} + \frac{\eta}{\pi} \cos^{-1} \left(\frac{\beta\eta}{\sqrt{2\beta\xi\tau}} \right), \quad (11)$$

which applies downstream. In fact, the far field turns out to be the same for all values of μ . Shock conditions are derived from the conservation equation

$$((\gamma + 1)M_\infty^4 u^2/2)_\xi - v_\eta + (2\beta u)_\tau = 0,$$

where $u = \varphi_\xi$, $v = \varphi_\eta$.

This particular flow is conical and by introducing conical variables

$$\varphi(\xi, \eta, \tau) = \beta\tau\psi(R, \vartheta),$$

where

$$R = \xi/\beta\tau - \eta^2/2\tau^2, \quad \vartheta = \eta/\tau,$$

we transform (6), (10), and (11) to the equation considered by Fraenkel and Watson:

$$\left(\frac{\gamma + 1}{\beta^2} M_\infty^4 \psi_R - 2R \right) \psi_{RR} - \psi_{\vartheta\vartheta} + \psi_R = 0 \quad (12)$$

with boundary condition

$$\psi_\vartheta(R, 0) = \frac{1}{2\beta} \quad \text{for } R > \frac{\mu}{\beta} \quad (13)$$

and far field

$$\psi \rightarrow -\frac{1}{\beta\pi} \sqrt{2R} + \frac{\vartheta}{\beta\pi} \cos^{-1} \left(\frac{\vartheta}{\sqrt{\vartheta^2 + 2R}} \right), \quad (14)$$

which applies away from the leading edge.

In conservation form we have

$$\left(\frac{\gamma + 1}{\beta^2} M_\infty^4 \frac{\psi_R^2}{2} - 2R\psi_R + 3\psi \right)_R - \psi_{\vartheta\vartheta} = 0. \quad (15)$$

We only need to solve for $\vartheta \geq 0$ since the flow is symmetric and $\psi_\vartheta(R, 0) = 0$ applies for $R < (\mu/\beta)$.

Equation (12) is of mixed type: it is hyperbolic where $[(\gamma + 1)/\beta^2]M_\infty^4\psi_R - 2R > 0$ and elliptic where $[(\gamma + 1)/\beta^2]M_\infty^4\psi_R - 2R < 0$. It is easy to see that it is elliptic in the

far field. The shock is detached for sweepback angles such that

$$\mu > \mu_a = -\frac{3M_\infty^2}{8\beta} [(\gamma + 1)M_\infty]^{2/3}$$

and its location is unknown until the solution has been found.

An estimate of the drag correction due to quasi-transonic effects can be made as follows. The pressure coefficient is defined by

$$c_p = (p - p_\infty) / (\rho_\infty U^2 / 2),$$

where p is pressure and p_∞ and ρ_∞ are the undisturbed pressure and density upstream. For linear theory it is given by

$$c_{p\text{LIN}} = -2\delta\phi_x$$

and for quasi-transonic theory by

$$c_{pQT} = -2\delta^{2/3}\varphi_\xi.$$

The drag coefficient of the whole wing is defined by

$$c_D = D / \left\{ \frac{\rho_\infty U^2}{2} (\text{area}) \right\},$$

where D is total drag. Then to calculate a correction we consider the difference between linearized and quasi-transonic drag coefficients near the leading edge. Consider

$$c_{D\text{LIN}} = -\frac{4\delta^2}{\tan \Lambda} \int \int \phi_x dx dz$$

and

$$c_{DQT} = -\frac{4\delta^{5/3}}{\tan \Lambda} \int \int \varphi_\xi dx dz,$$

where Λ is the sweepback angle and the first integral is carried out near the leading edge. It follows that the correction is given by

$$\Delta c_D = c_{DQT} - c_{D\text{LIN}} = \frac{4\delta^2}{\tan \Lambda} \left\{ \int \int \phi_x dx dz - \delta^{-1/3} \int \int \varphi_\xi dx dz \right\}. \quad (16)$$

Let $n = k/\beta$, $t = kz/x$ ($t = 1$ on the leading edge). It is possible to work out the integral (3) and obtain that (see Puckett [8]) for $y = 0$:

(a) If $n < 1$ then

$$\begin{aligned} \phi_x &= -\frac{1}{2\beta\sqrt{1-n^2}} \quad \text{ahead of the Mach wave,} \\ \phi_x &= -\frac{1}{2\beta\sqrt{1-n^2}} \left\{ 1 - \frac{2}{\pi} \sin^{-1} \left(n \sqrt{\frac{1-(t/n)^2}{1-t^2}} \right) \right\} \quad \text{behind the Mach wave.} \end{aligned} \quad (17)$$

(b) If $n = 1$ then

$$\phi_x = -\frac{1}{\pi\beta} \frac{x}{\sqrt{x^2 - \beta^2 z^2}}. \quad (18)$$

(c) If $n > 1$ then

$$\phi_x = -\frac{1}{\pi\beta\sqrt{n^2 - 1}} \cosh^{-1} \left(n \sqrt{\frac{1 - (t/n)^2}{1 - t^2}} \right). \quad (19)$$

The drag coefficient as given by linear theory is (see Puckett [8])

$$c_{D_{\text{LIN}}} = \frac{2\delta^2}{\pi\beta} \left[\frac{\log n}{\sqrt{n^2 - 1}} + \sin^{-1} \frac{1}{n} \right] \quad \text{for } n > 1,$$

$$c_{D_{\text{LIN}}} = \frac{\delta^2}{\beta} \quad \text{for } n \leq 1.$$

From (17)–(19) and the matching condition between quasi-transonic and linear theories, we obtain the formulas for the estimated drag correction:

(a) $n < 1$, $\mu < 0$

$$\begin{aligned} \int \int \phi_x dx dz &= \int_0^{1/k} \left[\int_{kz}^{\beta z} \frac{-1}{2\beta\sqrt{1 - n^2}} dx + \int_{\beta z}^1 \frac{-1}{\beta\pi\sqrt{1 - n^2}} \cos^{-1} \left(n \sqrt{\frac{1 - (t/n)^2}{1 - t^2}} \right) dx \right] dz \\ &= -\frac{1}{4k^2} \sqrt{\frac{\beta - k}{\beta + k}} + \int_0^{1/k} \int_{\beta z}^1 \frac{-1}{\pi\sqrt{\beta^2 - k^2}} \cos^{-1} \left(\frac{k}{\beta} \sqrt{\frac{x^2 - \beta^2 z^2}{x^2 - k^2 z^2}} \right) dx dz. \end{aligned}$$

From (5) we obtain

$$\begin{aligned} x^2 - \beta^2 z^2 &= \delta^{2/3} 2\beta\xi\tau + \dots, & k/\beta &= 1 + \delta^{2/3}\mu/\beta, \\ x^2 - k^2 z^2 &= \delta^{2/3} 2\beta\tau(\xi - \mu\tau) + \dots, & \beta^2 - k^2 &= -2\delta^{2/3}\mu\beta + \dots, \end{aligned}$$

and upon a change of variables to ξ and τ we transform the integral as

$$\int \int \phi_x dx dz = -\delta^{1/3} \left[\frac{1}{4\beta^2} \sqrt{\frac{-\mu}{2\beta}} + \frac{1}{\pi\sqrt{-2\mu\beta}} \int_0^{1/\beta} \int_{\xi>0} \cos^{-1} \sqrt{\frac{\xi}{\xi - \mu\tau}} d\xi d\tau \right] + \dots,$$

and another change of variables to $R = \xi/\beta\tau$ gives

$$\int \int \phi_x dx dz = -\delta^{1/3} \left[\frac{1}{4\beta^2} \sqrt{\frac{-\mu}{2\beta}} + \frac{1}{2\pi\beta\sqrt{-2\mu\beta}} \int_{R>0} \cos^{-1} \sqrt{\frac{R}{R - \mu/\beta}} dR \right] + \dots.$$

The second integral in Eq. (16) is

$$\begin{aligned} \int \int \varphi_\xi dx dz &= \delta^{2/3} \int_0^{1/k} \left[\int_{\mu\tau}^0 \varphi_\xi d\xi + \int_{\xi>0} \varphi_\xi d\xi \right] d\tau \\ &= \frac{\delta^{2/3}}{2\beta} \left[\psi(0, 0) - \psi\left(\frac{\mu}{\beta}, 0\right) + \int_{R>0} \psi_R dR \right] + \dots. \end{aligned}$$

Taking into account the matching condition we obtain

$$\begin{aligned}\Delta c_D &= 4\delta^{7/3} \left[\frac{1}{2} \psi \left(\frac{\mu}{\beta}, 0 \right) - \frac{1}{4\beta} \sqrt{-\mu/2\beta} - \frac{1}{2\pi\sqrt{-2\mu\beta}} \frac{\pi\mu}{2\beta} \right] \\ &= 2\delta^{7/3} \psi \left(\frac{\mu}{\beta}, 0 \right).\end{aligned}$$

(b) $n = 1, \mu = 0$

$$\begin{aligned}\int \int \phi_x dx dz &= \int_0^{1/\beta} \int_{\beta z}^1 \left[\frac{-1}{\pi\beta} \frac{x}{\sqrt{x^2 - \beta^2 z^2}} \right] dx dz \\ &= \int_0^{1/\beta} \int_{\xi > 0} \left[\frac{-1}{\pi\beta} \delta^{1/3} \frac{\beta\tau}{\sqrt{2\beta\xi\tau}} \right] d\xi d\tau + \dots\end{aligned}$$

It follows that

$$\begin{aligned}\Delta c_D &= -4\delta^{7/3}\beta \int_0^{1/\beta} \int_{\xi > 0} \left[\varphi_\xi(\xi, 0, \tau) + \frac{1}{\pi} \sqrt{\frac{\tau}{2\beta\xi}} \right] d\xi d\tau \\ &= -2\delta^{7/3} \int_{R > 0} \left[\psi_R(R, 0) + \frac{1}{\beta\pi\sqrt{2R}} \right] dR \\ &= 2\delta^{7/3} \psi(0, 0).\end{aligned}$$

(c) $n > 1, \mu > 0$

$$\begin{aligned}\int \int \phi_x dx dz &= \int_0^{1/k} \int_{kz}^1 \frac{-1}{\pi\beta\sqrt{n^2 - 1}} \cosh^{-1} \left(n \sqrt{\frac{x^2 - \beta^2 z^2}{x^2 - k^2 z^2}} \right) dx dz \\ &= -\delta^{1/3} \frac{1}{2\pi\beta\sqrt{2\mu\beta}} \int_{R > \mu/\beta} \cosh^{-1} \sqrt{\frac{R}{R - \mu/\beta}} dR, \\ \int \int \varphi_\xi dx dz &= \frac{\delta^{2/3}}{2\beta} \int_{R > \mu/\beta} \psi_R(R, 0) dR.\end{aligned}$$

Consequently,

$$\Delta c_D = 2\delta^{7/3} \left[\psi(\mu/\beta, 0) + \frac{1}{\pi\beta} \sqrt{\frac{\mu}{2\beta}} \right].$$

We summarize these results as

$$\begin{aligned}\Delta c_D &= 2\delta^{7/3} \psi \left(\frac{\mu}{\beta}, 0 \right) \quad \text{for } \mu \leq 0, \\ \Delta c_D &= 2\delta^{7/3} \left\{ \psi \left(\frac{\mu}{\beta}, 0 \right) + \frac{1}{\pi\beta} \sqrt{\frac{\mu}{2\beta}} \right\} \quad \text{for } \mu > 0.\end{aligned}$$

Numerical calculations were performed to find the potential at the nose and to compare the results with those of linear theory. A drag correction of the order of 15% resulted for $M_\infty = 1.414$ and a wing 10% thick.

4. Numerical results. We used a finite-difference successive relaxation scheme similar to that of Murman and Cole (see Murman and Cole [6] and Murman [7]) to calculate the solution to (12)–(14). Several runs were made for different Mach numbers and sweepback angles, for supersonic ($\mu < 0$), subsonic ($\mu > 0$), and sonic ($\mu = 0$) wings.

We present below the results for $M_\infty = 1.414$. The shock appears to have the right asymptotic behavior in the far field, namely, asymptotic to the Mach wave away from the wing (Fig. 3).

The plots in Figs. 4 and 5 show the pressure field for different values of the sweepback angle (Λ) and depict the shock location. Figures 6 and 7 show the pressure coefficient on the wing ($\vartheta = 0$) as calculated numerically. A weak singularity in c_p is expected at the leading edge when the shock is detached. This is not shown on the graph.

FIG. 3. Shock and Mach wave: $\mu = 0$.

FIG. 4. Pressure field for $\mu = 0.1$ FIG. 5. Pressure field for $\mu = -1.0$

Linear theory predicts a singular behavior of the drag coefficient at the value of Λ equal to the Mach angle. The results obtained from the correction due to quasi-transonic effects show no such behavior (see Fig. 8) and the corrections become smaller as the sweepback angle increases or decreases away from the Mach angle. Figure 8 shows a plot of c_D against $m = \tan \Lambda / \tan \theta_M$, where $\theta_M = \text{Mach angle} = \sin^{-1}(1/M_\infty)$.

MACH=1.414

THETA=0

MU=0.1

FIG. 6. Pressure coefficient on the wing ($\vartheta = 0$).

MACH=1.414 THETA=0 MU=-1.0

FIG. 7. Pressure coefficient on the wing ($\vartheta = 0$).

FIG. 8. Drag coefficient as a function of m . Solid line, linear theory; $\square$, quasi-transonic theory.

5. Final remarks. The numerical results obtained support the statements of Fraenkel and Watson [2] regarding the drag correction due to the nonlinear character of the flow near the leading edge. Although it is of $O(\delta^{7/3})$, which is small in comparison to the $O(\delta^2)$ drag of linear theory, in practice it is not negligible and provides at least qualitative agreement with experimental drag values. Notice also that the correction comes from a $O(\delta^{4/3})$ term in the expansion of the velocity potential near the leading edge, which is comparatively larger than the second-order correction $O(\delta^2)$ in the linear theory expansion.

The theory and calculations presented here can be extended to different wing shapes as well as lifting wings, with only minor changes, as long as the flow remains conical. For nonconical flows Eq. (6) would have to be solved and a different numerical scheme would have to be used.

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