A PROPERTY OF PERIODIC SOLUTIONS OF PARABOLIC EQUATIONS WHICH HAS IMPLICATIONS FOR THERMOMECHANICS*

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1. Statement of an inequality. The aim of this note is to establish a property of solutions of the parabolic equation

$$(k(x)\theta_x)_x + b(x)\dot{p}(t) = a(x)\theta_t, \quad (1.1)$$

in which $A \leq x \leq B$, $-\infty < t < \infty$, and the coefficients k , a , and b are positive.

Our concern is with what happens when θ satisfies the boundary conditions

$$\theta(A, t) = \theta(B, t) = \tau(t) \quad (-\infty < t < \infty), \quad (1.2)$$

and θ , τ , and p are periodic in t , with a common period T , i.e., $\theta(x, t + T) = \theta(x, t)$, $\tau(t + T) = \tau(t)$, and $p(t + T) = p(t)$.

It will be shown that there are positive constants c_1 and c_2 , depending only upon the coefficients k , a , and b , such that

$$-\int_0^T \int_A^B b \dot{p} \theta \, dx \, dt \leq (c_1 + c_2 T^2) \int_0^T \tau^2 \, dt. \quad (1.3)$$

It is the fact that the right-hand side of (1.3) is independent of p which is important here.

One consequence of (1.3) is that

$$-\int_0^T \int_A^B b \dot{p} \theta \, dx \, dt \leq 0,$$

whenever τ is constant.

2. Implications for thermomechanics. In order to understand the significance of the inequality (1.3), let us recall that classical thermodynamics draws heavily upon arguments about the work done by a deformable body operating in a cycle; in the course of such arguments, though, it is tacitly assumed that the effects of heat conduction and inertia can be ignored. What (1.3) tells us is that a certain conclusion of classical theory has to be amended in a rather striking way once heat conduction is taken into account.

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To see this, consider an isotropic, but possibly inhomogeneous, thermoelastic body which occupies the slab

$$\{(x, y, z) : A \leq x \leq B, -\infty < y < \infty, -\infty < z < \infty\}$$

and which undergoes a motion in which points are displaced in directions parallel to the x -axis of a system of rectangular Cartesian coordinates. According to the linearized theory of thermoelasticity [1], the temperature $\theta(x, t)$ and the displacement $u(x, t)$ satisfy an energy equation

$$(k(x)\theta_x)_x = c(x)\theta_t + \theta_0\mu(x)u_{xt}, \quad (2.1)$$

and, if inertia be ignored, a momentum equation

$$\sigma_x = 0, \quad (2.2)$$

in which

$$\sigma = \beta(x)u_x - \mu(x)(\theta - \theta_0) \quad (2.3)$$

is the xx -component of stress. In these equations, the positive constant θ_0 is the temperature of a stress-free reference state of the body, $k(x)$ is the thermal conductivity, $c(x)$ is the specific heat at constant strain, $\mu(x)$ is the stress-temperature modulus, and $\beta(x)$ is an elastic modulus. Each of k , c , μ , and β , is assumed to be positive.

The faces $x = A$ and $x = B$ are required to be subject to a variable pressure $p(t)$, i.e., the boundary conditions

$$\begin{aligned} \theta(A, t) &= \theta(B, t) = \tau(t), \\ \sigma(A, t) &= \sigma(B, t) = -p(t), \end{aligned}$$

are presumed to be in force; the first of these is (1.2).

According to (2.2), the stress throughout the body is $\sigma(x, t) = -p(t)$, and it follows from (2.3) that

$$u_x = \frac{\mu(x)}{\beta(x)}(\theta - \theta_0) - \frac{1}{\beta(x)}p(t).$$

On differentiating with respect to t , and substituting for u_{xt} in (2.1), we see that θ satisfies an equation of the form (1.1), with $a = c + \theta_0\mu^2/\beta$ and $b = \theta_0\mu/\beta$.

Suppose now that θ , τ , and p have period T in their dependence upon t . The rate at which the body performs work on its exterior is

$$\begin{aligned} W(t) &= p(t)u_t(B, t) - p(t)u_t(A, t) \\ &= p \int_A^B u_{xt} dx \\ &= p \int_A^B \left(\frac{\mu}{\beta} \theta_t - \frac{1}{\beta} \dot{p} \right) dx \\ &= p \int_A^B \frac{\mu}{\beta} \theta_t dx - p \dot{p} \int_A^B \frac{1}{\beta} dx \\ &= -\dot{p} \int_A^B \frac{\mu}{\beta} \theta dx + p \int_A^B \frac{\mu}{\beta} \theta dx - \frac{1}{2} p^2 \int_A^B \frac{1}{\beta} dx, \end{aligned}$$

and, on integrating with respect to t and appealing to periodicity, we see that the work done by the body in a period is

$$\int_0^T W dt = - \int_0^T \int_A^B \dot{p} \frac{\mu}{\beta} \theta dx dt = - \frac{1}{\theta_0} \int_0^T \int_A^B b \dot{p} \theta dx dt.$$

Hence, by virtue of (1.3), the work done by the body in a period must satisfy the estimate

$$\int_0^T W dt \leq \frac{1}{\theta_0} (c_1 + c_2 T^2) \int_0^T \bar{\tau}^2 dt.$$

In short, for a given periodic environmental temperature τ , there is a finite upper bound on the work that the body can do in a period, no matter how the periodic pressure p is adjusted; moreover, it is impossible to extract a positive amount of work from an environment at constant temperature.

In [2] I have previously obtained results of this type in the special case in which τ and p are trigonometric functions. In [3] I have proved related results in the context of a somewhat different theory of thermoelastic behavior.

Classical theory, by contrast, adopts the homogeneous field approximation, i.e., the fact that θ is a solution of the parabolic equation (1.1) is ignored, and it is assumed that it will do to make the approximation $\theta(x, t) \equiv \tau(t)$ throughout the body. If we make this approximation, however, we are led to approximate the work done by the body in a period by the expression

$$-\frac{1}{\theta_0} \int_0^T \int_A^B b \dot{p} \tau dx dt = -\frac{1}{\theta_0} \int_A^B b dx \int_0^T \dot{p} \tau dt.$$

Now let τ be any nonconstant periodic environmental temperature, let

$$\bar{\tau} = \frac{1}{T} \int_0^T \tau(t) dt$$

be its mean, let E be any number, and choose the pressure to be

$$p(t) = p_0 - \frac{\theta_0 E \int_0^t (\tau(s) - \bar{\tau}) ds}{\int_A^B b dx \int_0^T (\tau(s) - \bar{\tau})^2 ds}$$

where p_0 is any constant. Then, as is readily verified, p is periodic and

$$\begin{aligned} -\frac{1}{\theta_0} \int_A^B b dx \int_0^T \dot{p} \tau dt &= E \int_0^T (\tau - \bar{\tau}) \tau dt \bigg/ \int_0^T (\tau - \bar{\tau})^2 dt \\ &= E \int_0^T [(\tau - \bar{\tau})^2 + (\tau - \bar{\tau}) \bar{\tau}] dt \bigg/ \int_0^T (\tau - \bar{\tau})^2 dt \\ &= E. \end{aligned}$$

Thus, classical theory predicts that, for a given nonconstant periodic environmental temperature, the periodic pressure can be adjusted so that the work done by the body in a period takes any assigned value. Our results tell us, though, that once we take account of heat conduction within the body we can no longer draw the same conclusion.

3. Proof of the inequality. In order to prove (1.3) we introduce an auxiliary function $\phi(x, t)$, which is defined as the periodic solution of the equation

$$(k(x)\phi_x)_x + a(x)\phi_t = b(x)\dot{p}(t) \quad (3.1)$$

which satisfies the homogeneous boundary conditions

$$\phi(A, t) = \phi(B, t) = 0 \quad (-\infty < t < \infty). \quad (3.2)$$

The required ϕ can be constructed explicitly by using separation of variables and Fourier analysis, and its uniqueness can be established by either an energy integral argument or an appeal to a maximum principle.

We now argue that

$$\begin{aligned} (\phi k \theta_x - \theta k \phi_x)_x &= \phi(k \theta_x)_x - \theta(k \phi_x)_x \\ &= \phi(a \theta_t - b \dot{p}) + \theta(a \phi_t - b \dot{p}) \\ &= a(\theta \phi)_t - b \dot{p}(\theta + \phi). \end{aligned}$$

On integrating with respect to x , and appealing to the boundary conditions on θ and ϕ , we see that

$$-\tau[k(B)\phi_x(B, t) - k(A)\phi_x(A, t)] = \overline{\int_A^B a \theta \phi dx} - \dot{p} \int_A^B b(\theta + \phi) dx. \quad (3.3)$$

On the other hand, if we integrate (3.1) with respect to x we see that

$$k(B)\phi_x(B, t) - k(A)\phi_x(A, t) + \int_A^B a \phi dx = \dot{p} \int_A^B b dx$$

and, thus, when we substitute into (3.3) for the terms enclosed within square brackets we obtain the equation

$$-\tau \left[\dot{p} \int_A^B b dx - \overline{\int_A^B a \phi dx} \right] = \overline{\int_A^B a \theta \phi dx} - \dot{p} \int_A^B b(\theta + \phi) dx,$$

or, after some rearrangement of terms, the equation

$$-\dot{p} \int_A^B b \theta dx = -\dot{p} \tau \int_A^B b dx + \tau \overline{\int_A^B a \phi dx} - \tau \int_A^B a \phi dx - \overline{\int_A^B a \theta \phi dx} + \dot{p} \int_A^B b \phi dx.$$

Hence, if we integrate with respect to t , and appeal to periodicity, we arrive at the formula

$$-\int_0^T \int_A^B b \dot{p} \theta dx dt = -\int_A^B b dx \int_0^T \dot{p} \tau dt - \int_0^T \int_A^B \tau a \phi dx dt + \int_0^T \int_A^B b \dot{p} \phi dx dt. \quad (3.4)$$

The third term on the right-hand side of (3.4) will be replaced by an expression that is more convenient for our purpose. To do this we multiply (3.1) through by ϕ to deduce that

$$(\phi k \phi_x)_x + a \phi \phi_t = k \phi_x^2 + b \dot{p} \phi.$$

An integration with respect to x then yields the conclusion that

$$\frac{1}{2} \overline{\int_A^B a \phi^2 dx} = \int_A^B k \phi_x^2 dx + \dot{p} \int_A^B b \phi dx,$$

and a further integration with respect to t and an appeal to periodicity give

$$0 = \int_0^T \int_A^B k \phi_x^2 dx dt + \int_0^T \int_A^B b \dot{p} \phi dx dt.$$

Hence (3.4) can be replaced by the equation

$$-\int_0^T \int_A^B b \dot{p} \theta dx dt = -\int_A^B b dx \int_0^T \dot{p} \tau dt - \int_0^T \int_A^B \dot{\tau} a \phi dx dt - \int_0^T \int_A^B k \phi_x^2 dx dt. \quad (3.5)$$

The first term on the right-hand side of (3.5) is none other than the classical approximation to the left-hand side; it too will be replaced by a more convenient expression. To this end, let us introduce a positive constant λ and a function $\psi(x)$, which are, respectively, the smallest eigenvalue and a corresponding eigenfunction of the Sturm–Liouville problem

$$(k(x)\psi')' = -\lambda a(x)\psi, \quad \psi(A) = \psi(B) = 0.$$

Since, as is well known, $\psi(x)$ has no zeros in $A < x < B$, the integral $\int_A^B \psi b dx$ cannot vanish.

We have

$$\begin{aligned} (\psi k \phi_x - \phi k \psi')_x &= \psi (k \phi_x)_x - \phi (k \psi')' \\ &= \psi (b \dot{p} - a \phi_t) + \lambda \psi a \phi \end{aligned}$$

and, on integrating with respect to x and appealing to the boundary conditions that ϕ and ψ satisfy, we find that

$$0 = \dot{p} \int_A^B \psi b dx - \int_A^B \psi a \phi_t dx + \lambda \int_A^B \psi a \phi dx.$$

Hence

$$\dot{p} = \left(\int_A^B \psi a \phi_t dx - \lambda \int_A^B \psi a \phi dx \right) / \int_A^B \psi b dx.$$

On multiplying through by $\tau - \bar{\tau}$, where, as before, $\bar{\tau}$ is the mean value of τ , we see that

$$\begin{aligned} \dot{p}(\tau - \bar{\tau}) &= \left[(\tau - \bar{\tau}) \int_A^B \psi a \phi_t dx - \lambda (\tau - \bar{\tau}) \int_A^B \psi a \phi dx \right] / \int_A^B \psi b dx \\ &= \left[\overline{(\tau - \bar{\tau}) \int_A^B \psi a \phi dx} - (\dot{\tau} + \lambda(\tau - \bar{\tau})) \int_A^B \psi a \phi dx \right] / \int_A^B \psi b dx \end{aligned}$$

and, if we integrate with respect to t and invoke periodicity again, we conclude that

$$\begin{aligned} \int_0^T \dot{p} \tau dt &= \int_0^T \dot{p}(\tau - \bar{\tau}) dt \\ &= - \int_0^T \int_A^B (\dot{\tau} + \lambda(\tau - \bar{\tau})) \psi a \phi dx dt / \int_A^B \psi b dx. \end{aligned}$$

This last expression and (3.5) now yield the formula

$$-\int_0^T \int_A^B b \dot{p} \theta \, dx \, dt = \int_0^T \int_A^B ((\xi \psi - 1) \dot{\tau} + \lambda \xi (\tau - \bar{\tau}) \psi) a \phi \, dx \, dt - \int_0^T \int_A^B k \phi_x^2 \, dx \, dt, \quad (3.6)$$

where $\xi = \int_A^B b \, dx / \int_A^B \psi b \, dx$.

The required inequality (1.3) is a straightforward deduction from (3.6). For, the inequality

$$\lambda \int_A^B a \phi^2 \, dx \leq \int_A^B k \phi_x^2 \, dx,$$

which is a consequence of the variational characterization of the eigenvalue λ and of the boundary conditions (3.2), enables us to replace (3.6) by the estimates

$$\begin{aligned} & -\int_0^T \int_A^B b \dot{p} \theta \, dx \, dt \\ & \leq \left(\int_0^T \int_A^B a ((\xi \psi - 1) \dot{\tau} + \lambda \xi (\tau - \bar{\tau}) \psi)^2 \, dx \, dt \right)^{1/2} \left(\int_0^T \int_A^B a \phi^2 \, dx \, dt \right)^{1/2} \\ & \quad - \int_0^T \int_A^B k \phi_x^2 \, dx \, dt \\ & \leq \left(\frac{1}{\lambda} \int_0^T \int_A^B a ((\xi \psi - 1) \dot{\tau} + \lambda \xi (\tau - \bar{\tau}) \psi)^2 \, dx \, dt \right)^{1/2} \left(\int_0^T \int_A^B k \phi_x^2 \, dx \, dt \right)^{1/2} \\ & \quad - \int_0^T \int_A^B k \phi_x^2 \, dx \, dt. \end{aligned}$$

Thus, in view of the elementary inequality $m x - x^2 \leq m^2/4$, it must be that

$$\begin{aligned} & -\int_0^T \int_A^B b \dot{p} \theta \, dx \, dt \\ & \leq \frac{1}{4\lambda} \int_0^T \int_A^B a ((\xi \psi - 1) \dot{\tau} + \lambda \xi (\tau - \bar{\tau}) \psi)^2 \, dx \, dt \\ & = \frac{1}{4\lambda} \int_0^T \int_A^B a ((\xi \psi - 1)^2 \dot{\tau}^2 + 2\lambda \xi (\xi \psi - 1) \psi \dot{\tau} (\tau - \bar{\tau}) + \lambda^2 \xi^2 (\tau - \bar{\tau})^2 \psi^2) \, dx \, dt. \end{aligned}$$

However, the periodicity of τ ensures that the integral

$$\int_0^T \dot{\tau} (\tau - \bar{\tau}) \, dt = 0$$

and that Wirtinger's inequality

$$\int_0^T (\tau - \bar{\tau})^2 \, dt \leq \left(\frac{T}{2\pi} \right)^2 \int_0^T \dot{\tau}^2 \, dt$$

holds. Thus (1.3) is correct, with

$$c_1 = \frac{1}{4\lambda} \int_A^B a (\xi \psi - 1)^2 \, dx, \quad c_2 = \frac{\lambda \xi^2}{16\pi^2} \int_A^B a \psi^2 \, dx.$$

REFERENCES

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