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SINGULAR ASYMPTOTICS ANALYSIS FOR THE SINGULARITY AT A HOLE NEAR A BOUNDARY*

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Abstract. The hoop stress at a hole near the edge of a plate loaded in tension is analyzed as the hole approaches the boundary. The solution given by Mindlin in terms of a series is converted to an integral to which singular asymptotics are applied to obtain the singularity in the limit.

Introduction. The stress distribution around a hole near the edge of a plate in tension was first studied by Jeffery [1] in 1920 and subsequently, in 1948, by Mindlin [2], who corrected Jeffery's result.

The solution for the hoop stress at the hole is given by Mindlin in a rather explicit form as an infinite series. The expression is sufficiently complicated, however, that certain interesting questions about the behavior of the solution have been impossible to answer so far. One such problem is the behavior of the stress field as the distance of the hole from the boundary edge approaches zero. This is the problem that we address in this paper.

Mindlin's solution to the problem of the stress field in a wide plate under simple tension T parallel to a straight edge and containing a hole near the edge (Fig. 1) is obtained in terms of bipolar coordinates α and β, which are described briefly in the Appendix.

The boundary of the hole is simply one of the α =constant curves, say $\alpha = \alpha_1$. From the geometry we have $a \cdot \coth \alpha_1 = d + R$ and $a \cdot \cosh \alpha_1 = R$ (where a is the parameter of the transformation to bipolar coordinates, R is the radius of the hole, and d is its distance to the edge), so that $\cosh \alpha_1 = 1 + d/r$, which, by series expansion for small α_1, gives $\alpha_1^2 \sim d/R$, and hence $1/\alpha_1 \sim (\frac{R}{2d})^{1/2}$.

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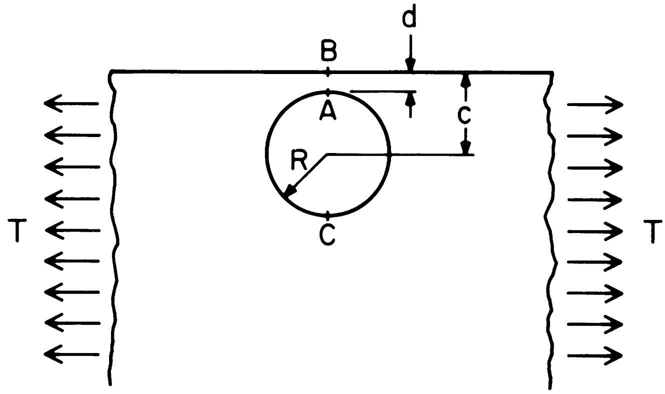

FIG. 1.

The solution of Mindlin for the stress at point A is given by the following expression, evaluated at $\beta = \pi$,

$$2T\left[1 - \frac{2\sinh^2\alpha_1 \sin^2\beta}{\cosh\alpha_1 - \cos\beta}\right] + 2T(\cosh\alpha_1 - \cos\beta)$$
$$\cdot\left\{\frac{1}{2}\operatorname{csch}\alpha_1 + 2e^{-2\alpha_1}\cos\beta + \sum_{n=2}^{\infty} N_n \cos n\beta\right\}$$

where

$$N_n = \frac{n^2\sinh\alpha_1\cosh n\alpha_1 - n\sinh n\alpha_1\cosh\alpha_1}{\sinh^2 n\alpha_1 - n^2\sinh^2\alpha_1} - 2n(n\sinh\alpha - \cosh\alpha_1)e^{-n\alpha_1}.$$

As the hole approaches the boundary, $d \to 0$ and therefore $\alpha_1 \to 0$.

This is the limit we will study in the rest of the paper. Using the general methods of Sec. 1, we will show in Sec. 2 that the stress field has an asymptotic expansion in powers of α to all orders, the leading term being $O(\alpha_1^{-1})$.

To understand the meaning of the limit $\alpha_1 \to 0$, observe that it can be interpreted either as $d \to 0$ while keeping the radius R constant, or as $R \to \infty$ while d is kept constant. As $R \to \infty$, a point with a fixed value for the parameter β on the boundary of the hole approaches the x-axis asymptotically, with the exception of the value $\beta = 0$. In the latter case the point remains in the position furthest from the boundary of the material. Thus we may expect that for all values of β in $(0, 2\pi)$ we will have similar behavior as $\alpha_1 \to 0$. We also observe, however, that the distance of a point with a fixed value of β from the point A becomes negligible in comparison with the circumference of the hole boundary as $\alpha_1 \to 0$. Thus, in the limit $\alpha_1 \to 0$ with β fixed, the hold is approaching the boundary of the material, but at the same time the point with bipolar coordinates (α_1, β) on the boundary of the hole is approaching the point A that is closest to the boundary. The value $\beta = \pi$ of course corresponds to the point A itself, which is the only point that does not move on the boundary as we take the limit. For this reason, values of β different from π have no particular meaning in our limit.

According to the results of Sec. 2, the hoop stress at the point A ($\beta = \pi$) is

$$\sim 4T\alpha_1^{-1} + O(1) \sim 2\sqrt{2}T\sqrt{\frac{R}{d}} + O(1)$$

as $\alpha_1 \to 0$. For the point C ($\beta = 0$) the coefficient of α_1^{-1} becomes zero. This limit agrees with the estimate obtained by Duan *et al.* [3] by a different method and under some intuitive assumptions.

The rest of the paper is technical. A large part (Sec. 1) is devoted to some general singular asymptotics techniques we have tried to develop to deal with our rather special problem. For us, these results alone were well worth the exercise!

1. Some problems in singular asymptotics. The problem of estimating the stress field on the boundary of a hole near the boundary of the material is essentially a problem in singular asymptotics. One encounters some difficulties, however, which can be formulated in rather general terms and seem likely to arise in many different situations. We begin by describing these problems and how we resolved them. The asymptotics of the stress field of the hole can then be dealt with by straightforward application of these ideas.

A. *Classes of functions with singular distributional expansions.* Let us review some of the standard theory of singular asymptotics of integrals. Let $h(x, y) \in C^\infty(\bar{R}_+^- \times \bar{R}_+^-)$ and assume that it is of compact support in x. Suppose further that for all $x > 0$, $y > h$,

$$|\partial_x^k h(x, y)| \le H_k(y)y^k,$$

for some $H_k(y)$ satisfying $\int_0^1 H_k(1/t)\,dt < \infty$. We say that h is a function of class A. Then

$$\int_0^\infty h(x, s/x)\,dx = \sum_{k=0}^m A_k[h]s^k + \sum_{k=1}^m B_k[h]_s^k \ln s + O(s^{m+1-\varepsilon}) \tag{1.1}$$

as $s \to 0$, for any $\varepsilon > 0$ and $m > 0$. The coefficients $A_k[h]$, $B_k[h]$ are linear functionals of h. A thorough discussion of these expansions can be found in [4, 5, and 6].

There is a larger class of functions for which the expansion (1.1) holds, by reduction to functions of class A. We say that a function $h(x, s)$ is of extended class A if there exists a function $h^1(x, y, s) \in C^\infty(\bar{R}_+^- \times \bar{R}_+^- \times \bar{R}_+^-)$ such that

$$h(x, s) = h^1(x, s/x, s)$$

and

(1) $\partial_s^j h^1(x, y, 0)$ is of class A for each j.

(2) $|\partial_s^j h^1(x, y, s)| \le H_j(y)$ for all x, y, $0 \le s \le 1$, where $\int_0^1 H_j(1/t)\,dr < \infty$.

We can easily see that a function of extended class A has an expansion of the form (1.1). In fact we can write

$$h^1(x, y, s) = \sum_{j=0}^m \frac{s^j}{j!}\partial_s^j h^1(x, y, 0) + \frac{1}{(m+1)!}\partial_s^{m+1} h^1(x, y, \sigma)s^{m+1},$$

where $0 < \sigma < s$. In the integral $\int_0^\infty h(x, s)\,dx$, the remainder gives $O(s^{m+1})$ while each term in the first part can be expanded as in (1.1) because $\partial_s^j h^1(x, y, 0)$ is of class A.

Example 1.a.1. Let $h(x, y)$ be such that, for all x, y,

$$|\partial_x^k h(x, y)| \leq H_k(y) y^{k+a},$$

where $\alpha \geq 0$ and $\int_0^1 H_k(1/t)\,dt < \infty$. Then for any integer $m > a$, the function $p(x, s) = x^m h(x, s/x)$ is of extended type A. To see this, let $\chi(y) \in C^\infty[0, \infty)$, $\chi = 1$ for a y near 0, and let $\chi_1 = 1 - \chi$. Write

$$x^m h(x, s/x) = x^m \chi(s/x) h(x, s/x) + s^m (s/x)^{-m} \chi_1(s/x) h(s, x/s).$$

Choosing $h^1(x, y, s) = x^m \chi(y) h(x, y) + s^m y^{-m} \chi_1(y) h(x, y)$, we see that $p(x, s)$ is of extended class A.

We will need a smaller class of functions for which the expansion (1.1) holds. These will be referred to as functions of class B and are defined as those functions $h(x, y) \in C^\infty([0, \beta) \times [0, \beta))$ which are of compact support in x and satisfy

$$|\partial_x^k \partial_y^m h(x, y)| \leq H_{k,m}(y) y^{k-m}$$

for all x, $y \geq 0$ and some $H_{k,m}$ with $\int_0^1 H_{k,m}(1/t)\,dt < \infty$. Thus h is of class B if and only if $(y\partial_y)^m h$ is of class A for all m. We say that a function $h(x, s)$ is of extended class B if there exists a function $h^1(x, y, s)$ such that $h(x, s) = h^1(x, s/x, s)$ and $\partial_s^m h^1(x, y, s)$ is of class B for every m. From Example 1.a.1 above we see that if $h(x, y)$ is of class B, the function $p(x, s) = (x\partial_x)^k (y\partial_y)^m h(x, s/x)$ is of extended class B for any k and m. Finally we define h to be of (extended) class A or B of order n by requiring only $h \in C^n$ and that the respective conditions only hold for derivatives of order $\leq n$. For these classes (1.1) holds if $m < n$.

Our description of functions of class B can be sharpened if we use the following lemma.

LEMMA 1.a.1. Suppose $f \in C^\infty[0, \beta)$ and $|f^{(m)}(y)| \leq H_m(y) y^{k-m}$ for some k and H_m with $\int_0^1 H_m(1/t)\,dt < \infty$. Then

$$|f^{(m)}(y)| \leq C_m(y) y^{k-m+1},$$

where $C_m(y)$ is linear in H_j, $j \leq m$, and $C_m(y) = O(1)$ as $y \to \infty$.

Proof. Let $g(y) = y^{-k-2} f(y)$. Then it is easy to see that

$$|g^{(m)}(y)| \leq C_m \sum_{j=0}^{m} y^{-j-2} H_{m-j}(y) y^{-m+j} \equiv H_m^1(y) y^{-2-m},$$

where $\int_0^1 H_j^1(1/t)\,dt < \infty$. Now for $y' > y > 1$

$$|g^{(m)}(y) - g^{(m)}(y')| = \left| \int_y^{y'} g^{(m+1)}(r)\,dr \right| \leq \int_y^{y'} H_m^1(r) r^{-m-3}\,dr$$
$$\leq y^{-m-1} \int_0^1 H_m^1\left(\frac{1}{t}\right)\,dt.$$

Thus $g^{(m)}(y)$ is Cauchy as $y \to \infty$, and therefore has a limit. We must have $\lim_{y\to\infty} g^{(m)}(y) = 0$, because otherwise $H_m(1/t)$ which is $\geq (1/t)^{2+m} |g^{(m)}(1/t)|$

would not be integrable near $t = 0$. Thus we can write

$$|g^{(m)}(y)| = \left|\int_{\infty}^{y} g^{(m+1)}(r)\,dr\right| \le y^{-m-1}\int_0^1 H^1\left(\frac{1}{t}\right)dt$$

from which our estimate follows easily.

This lemma gives an improved pointwise estimate of functions of class B, but the bound satisfies weaker integrability conditions.

B. *Singular integrands.* Let us show first how these results can be applied to the case where $h(x, y)$ is a smooth function only for $x > 0$, i.e., not necessarily up to $x = 0$. We need of course some additional assumptions. We might suppose, for example, that, even though $h(x, y)$ may not be differentiable at $x = 0$, by applying enough successive regularizations to h, we can obtain a function which is as many times differentiable as desired, where by a regularization of a function $f(x)$ at $x = 0$ we mean in this case the function

$$f_{(1)}(x) = (1/x)\partial_0^x f(x')\,dx'. \tag{1.2}$$

To see the significance of this kind of regularization for singular problems, consider first an integral of the form

$$I(s) = \int_0^\infty h(x, s/x)\phi(x)\,dx, \tag{1.3}$$

where $h(x, y)$ is of class B but $\phi(x)$ is only assumed to be a bounded function of compact support. This integral can be rewritten in terms of the regularization of ϕ:

$$I(s) = \int_0^\infty (y\partial_y - x\partial_x)h(x, y)|_{y=s/x}\phi_{(1)}(x)\,dx. \tag{1.4}$$

Indeed

$$\begin{aligned} I(s) &= \int_0^\infty h(x, s/x)\partial_x[x\phi_{(1)}(x)]\,dx \\ &= h(x, s/x)x\phi_{(1)}(x)|_0^\infty - \int_0^\infty [\partial_x h(x, s/x)]x\phi_{(1)}(x)\,dx. \end{aligned}$$

The boundary term at ∞ vanishes because $h(x, y)$ is of compact support in x. The one at 0, however, vanishes if $h(x, y)$ is of class B, by Lemma 1.a.1, but not necessarily if it is of class A. Equation (1.4) is now clear in view of the fact that

$$\begin{aligned} x\partial_x h(x, s/x) &= x(\partial_x h(x, y) - (s/x^2)\partial_y h(x, y))|_{y=s/x} \\ &= \left(x\partial_x - \frac{s}{x}\partial_y\right)h(x, y)|_{y=s/x}. \end{aligned} \tag{1.5}$$

Hence, according to (1.4), $I(s)$ can be written in the form $\int_0^\infty h_1(x, s/x)\phi_{(1)}(x)\,dx$, or, inductively, $\int_0^\infty h_k(x, s/x)\phi_{(k)}(x)\,dx$, where $h_k(x, y) = (-x\partial_x + y\partial_y)^k h(x, y)$. Now we know that, for each k, $h_k(x, y)$ is of extended class B if $h(x, y)$ is. If, in addition, $\phi_{(k)} \in C^n[0, \infty)$, the latter integral can be expanded in the form (1.1), for any $m < n$, because $h_k(x, y)\phi_{(k)}(x)$ is of extended class B. Thus the integral (1.3) has an expansion of the form (1.1) for any m if for any given n there exists a k such that $\phi_{(k)} \in C^n[0, \infty)$.

This is the result we will need but it is easy to make a more general statement. For a function $h(x,y)$ let $h_{(k)}(x,y)$ denote regularization k times with respect to x, i.e., let

$$h_{(1)} = (1/x)\int_0^x h(x',y)\,dx'$$

and inductively $h_{(k)} = (h_{(k-1)})_{(1)}$. Suppose that for each k, $h_{(k)}$ is in class B of order 0. Then

$$\int_0^\infty h(x,s/x)\,dx = \int_0^\infty (y\partial_y)^k h_{(k)}(x,y)|_{y=s/x}\,dx.$$

As a result, the integral $\int_0^\infty h(x,s/x)\,dx$ has an expansion of the form (1.1) for any m, if for every n there exists a k such that $h_{(k)}$ is of class B of order n.

Examples. (1) For $\phi(x) = x^k$, we have

$$(x^k)_{(1)} = (1/x)\int_0^x x'^k\,dx' = \frac{1}{k+1}x^k.$$

(2) If $\phi \in C^k[0,\infty)$ and $\phi^{(j)}(0) = 0$ for $j < k$ we have $\phi_{(1)} \in C^k[0,\infty)$ and $\phi(0)^{(j)}(0) = 0$ for $j < k$. In fact

$$\phi_{(1)}^{(j)}(x) = \sum_{i=0}^{j}\binom{j}{i}(1/x)^{(i)}\left(\int_0^x \phi(x')\,dx'\right)^{(j-i)}$$
$$= (1/x)^{(j)}\int_0^x \phi(x')\,dx' + \sum_{i=0}^{j+1}\binom{j}{i}(-1)i!x^{-i-1}\phi^{(j-i-1)}(x)$$

and we have $|\phi^{(j)}(x)| \le Cx^{k-m+1}$ for $m \le k$ by Taylor's theorem, so that $|\phi_{(1)}^{(j)}(x)| \le C'x^{k-j+1}$ for $j \le k$.

(3) If now $\phi \in C^k[0,\infty)$ we find $\phi_{(1)} \in C^k[0,\infty)$ and $\phi_{(1)}^{(j)}(0) = (1/(j+1))\phi^{(j)}(0)$ for $j \le k$. In fact, $\phi(x) = \phi^1(x) + \sum_0^k (1/j!)\phi^{(j)}(0)x^j$, where $\phi^{1(j)}(0) = 0$ for $j \le k$. The result follows from (1) and (2).

(4) Let $\phi_k(x) = x^k e^{i\alpha/x}$, $k \ge 1$, $\alpha \ne 0$, which is in $C^{k-1}[0,\infty)$. We have $\phi_k^{(j)}(0) = 0$ for $j < k$. By a simple integration by parts we find

$$(\phi_k)_{(1)}(x) = i\alpha^{-1}\phi_{k+1}(x) - i\alpha^{-1}(k+2)(\phi_{k+1})_{(1)}(x).$$

Thus

$$(\phi_k)_{(1)} \in C^k[0,\infty) \quad\text{and}\quad (\phi_k)_{(1)}^{(j)}(0) = 0 \quad \text{for } j \le k,$$

and

$$(\phi_k)_{(m)} \in C^{k+m-1}[0,\infty), \qquad (\phi_k)_{(m)}^{(j)}(0) = 0 \quad \text{for } j \le k+m-1.$$

(5) Finally, let $\phi(x) = f(e^{i/x})e^{i\beta/x}$ where $f(z)$ is analytic for $|z| < 1+\varepsilon$, some $\varepsilon > 0$. We have

$$\phi(x) = \sum_{n=0}^{\infty} f_n e^{i(n+\beta)/x},$$

where $\limsup_{n\to\infty} f_n^{1/n} < 1$. Because of the absolute and uniform convergence of the series we can use (4) to regularize ϕ term by term. We find that, if $\beta \ne -1,-2,\dots$,

$$\phi_{(k)}^{(j)}(0) = 0 \quad \text{for } j < k, \tag{1.6}$$

while, if $\beta = -n$, $\phi_{(k)}(0) = f_n$ and (1.6) holds if $j \ne 0$.

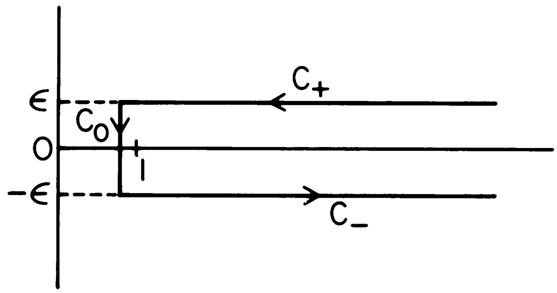

FIG. 2

C. *Singular asymptotics of infinite series.* While dealing with the asymptotics of series, rather than definite integrals of functions, seems to be difficult in general, some cases can be tackled by rewriting the series as an integral. A case in point is a series of the form

$$\sum_{n=1}^{\infty} H(sn, 1/n),$$

where we might assume that $H(\omega, 1/\xi)$ is an analytic function in (ω, ξ) for $\delta \operatorname{Re} \omega > |\operatorname{Im} \omega|$, $\varepsilon \operatorname{Re} \xi > |\operatorname{Im} \delta|$ for some δ, $\varepsilon > 0$. Suppose that

$$H(sn, 1/n) \leq C_s |n|^{-1-\varepsilon} \quad \text{for } s \text{ near } 0 \quad \text{and} \quad \varepsilon \operatorname{Re} n > |\operatorname{Im} n|. \tag{1.7}$$

Then we can write

$$\sum_{n=1}^{\infty} H(sn, 1/n) = \frac{1}{2i} \int_C dz H(sz, 1/z) \cot \pi z, \tag{1.8}$$

where C is the contour shown in Fig. 2 (with $\varepsilon > 0$ sufficiently small). Since $\cot \pi z$ has simple poles at $z = 0, \pm 1, \pm 2, \ldots$ with residue $1/\pi$, to check this formula it suffices to show that the integral converges absolutely and that if we cut off the contour as in Fig. 3, the integral over the vertical segment $C_N = \{z \operatorname{Re} z = N + 1/2, |\operatorname{Im} z| < \varepsilon\}$ goes to 0 as the integer $N \to \infty$. We estimate

$$|\cot \pi z| = \frac{1 + e^{-2i\pi z}}{1 - e^{-2i\pi z}} \leq (1 + e^{2 \operatorname{Im} z\pi})[1 + e^{4\pi \operatorname{Im} z} - 2e^{2\pi \operatorname{Im} z} \cos 2\pi \operatorname{Re} z]^{-1/2}.$$

For given $\operatorname{Im} z \neq 0$ we have

$$|\cot \pi z| \leq (1 + e^{2\pi |\operatorname{Im} z|})|1 - e^{2\pi \operatorname{Im} z}|^{-1/2}$$

which is a finite constant. For $\operatorname{Re} z = N + \frac{1}{2}$ we obtain $\cos 2\pi \operatorname{Re} z = \cos \pi = -1$ and $|\cot \pi z| \leq 1$. These simple estimates prove (1.8).

When we attempt to expand (1.8) asymptotically as $s \to 0$, it becomes apparent that the expansion will be singular at $z = \infty$. To see whether singular asymptotics can be applied we look at the part of the integral near $z = \infty$; any part for which z is bounded can be treated by Taylor expansion in s. Thus we choose $\chi \in C_0^\infty[0, \infty)$, $\chi = 1$ on $[0, 1]$, let $\chi_1 = 1 - \chi$; using χ_1, the large z part on the $C_\pm$ pieces of the contour can be isolated as

$$I_\pm = (2\pi i)^{-1} \int_{C_\pm} \chi_1(\operatorname{Re} z) H(sz, 1/z) \cot \pi z \, dz.$$

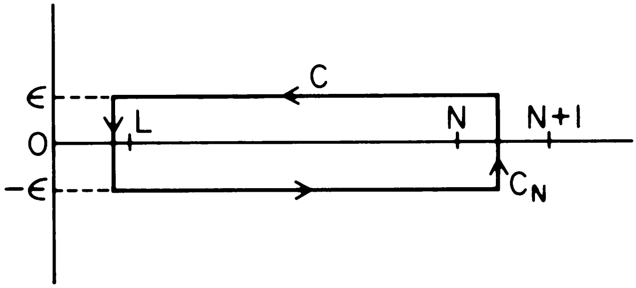

FIG. 3.

Change variables to $x = (\operatorname{Re} z)^{-1}$. Then

$$I_\pm = \mp(2\pi i)^{-1}\int_0^\infty \chi_1(x)x^{-2}H(\pm is\varepsilon + s/x, x/[1\pm i\varepsilon x])\cot\pi(\pm i\varepsilon + 1/x)\,dx. \quad (1.9)$$

The function $\phi_\pm(x) = \cot\pi(\pm i\varepsilon + 1/x)$ is of the form $f_\pm(e^{\mp i/x})$ with $f_\pm(z)$ analytic in $|z| < r$, $r > 1$, if $f_\pm$ are taken as the functions

$$f_-(z) = i(z^2 + e^{2\pi\varepsilon})/(z^2 - e^{2\pi\varepsilon}),$$
$$f_+(z) = i(1 + e^{-2\pi\varepsilon}z^2)/(1 - e^{-2\pi\varepsilon}z^2).$$

Thus, if $H(y, x)$ is a function of class B we can use the results of Example (5) to treat the integrals (1.9) by regularizing $\phi_\pm(x)$ as explained in part B of this section.

The same analysis can be applied to the series

$$\sum_{n=1}^{\infty} H(sn, 1/n)e^{i\beta n}.$$

We will have to use, in this case,

$$\phi_\pm(x) = e^{i\beta(\pm i\varepsilon + 1/x)}\cot\pi(\pm i\varepsilon + 1/x) \quad (1.10)$$

which has also been treated in Example (5) above.

Example 1.c.1. We will apply these ideas to a very simple example, which can actually be worked out much more easily by summing the series in closed form. Exactly because there is an alternative method, however, we will be able to compute certain integrals that we will need later but seem difficult to evaluate directly. Let $S(\alpha, \beta) = \sum_{n=2}^{\infty} e^{-\alpha n}\cos\beta n$. Summing the series we obtain

$$S(\alpha, \beta) = (e^{-2\alpha}\cos 2\beta - e^{-3\alpha}\cos\beta)/(1 + e^{-2\alpha} - 2e^{-\alpha}\cos\beta).$$

From this we get

$$S(\alpha, 0) = e^{-2\alpha}/(1 - e^{-\alpha}) = (1/\alpha) + O(1).$$

On the other hand, if $\cos\beta \neq 1$,

$$S(\alpha, \beta) = S(0, \beta) + O(\alpha) = (-\cos\beta + \tfrac{1}{2}) + O(\alpha).$$

We now compare with our alternative method. We have in this case

$$H(\omega, \zeta) = e^{-\omega}.$$

As in the general case, we use χ, χ_1 to separate the parts of the integral (1.8) for bounded z and for $z \to \infty$. The bounded z part is identified as

$$\frac{1}{2i}\int_{C_0} dz H(\alpha z, 1/x)\cos\beta z \cot\pi z = \frac{1}{2i}\int \cos\beta z \cot\pi z\, dx\, \chi(\operatorname{Re} z) + O(\alpha).$$

The unbounded z parts are

$$I_\pm = \mp(2i)^{-1}\alpha^{-2}\int_0^\infty h_\pm(x,\alpha)\phi_\pm(x)\chi_1(1/x),$$

where, in this case, $\phi_\pm(x) = \cos\beta z \cot\pi z|_{z=\pm i\varepsilon+1/x}$ and

$$h^1_\pm(x,y,\alpha) = y^2 e^{\mp i\varepsilon\alpha - y}. \tag{1.11}$$

Since we want to evaluate to $O(\alpha^0)$, any regularization $(\phi_\pm)_{(k)}$ with $k > 3$, say $k = 4$, will do. Consider first the case $\beta \neq 2\pi n$, $n = \pm 1, \pm 2, \ldots$. We have $(\phi_\pm)^{(j)}_{(4)} = 0$ for $j < 4$. Thus the asymptotics of $I_\pm$ can be computed by a simple Taylor series in α. If

$$h_\pm(x,y,\alpha) = \sum_{k=0}^{n} h_{\pm,k}(x,\alpha)y^k + O(y^{n+1})$$

we have

$$(y\partial_y)^m h_\pm(x,y,\alpha) = \sum_{k=0}^{n} k^m h_{\pm,k}(x,\alpha)y^k + O(y^{n+1}).$$

For $h^1_\pm$ in (1.11) we have

$$(y\partial_y)^4 h^1_\pm(x,y,\alpha) = 2^4 y^2 + O(y^3) + O(\alpha).$$

Thus

$$I_\pm = \mp(2i)^{-1}\int_0^\infty dx\, \chi_1(1/x) 2^4 \phi_{\pm(4)}(x) + O(\alpha).$$

Comparing with the result we obtain by directly summing the series, we find

$$\int_{C_0\chi(\operatorname{Re} z)\ldots} + I_+ + I_- = (-\cos\beta + \tfrac{1}{2})$$

or, in other words,

$$(2/i)^{-1}\left\{\int_{C_0}\cos\beta z\cot\pi z\chi(\operatorname{Re} z)\,dx + \int_0^\infty dx\,\chi_1(1/x)2^4[\phi_-(x) - \phi_+(x)]\right\} = (-\cos\beta + \tfrac{1}{2}) \tag{1.12}$$

for $\cos\beta \neq 0$.

Consider now the case $\cos\beta = 0$. The bounded z part of the integral can be ignored in this case, because it gives an $O(1)$ contribution. The $I_\pm$ integrals can be estimated using an explicit formula for some of the coefficients in the expansion (1.1), applied to the function

$$h(x,y) = (x\partial_x - y\partial y)[h^1_\pm(x,y,0)\chi_1(1/x)]\phi_{\pm(1)}(x),$$

with $h^1_\pm$ given by (1.11). Note that we only need a single regularization of $\phi_\pm$ for an expansion to $O(\alpha)$. According to the results of [4], we have $A_0 = 0$ and $B_1 = 0$. For A_1 we have the formula

$$A_1 = -\int_0^\infty d\xi \ln\xi\, \partial_\xi[\xi R_2(0, 1/\xi)]\phi_{\pm(1)}(0),$$

where R_2 is the remainder in the Taylor expansion of h with respect to y after two terms. In this case, $R_2 = h$ and $\phi_{\pm(1)}(0) = \pm i$, so that

$$A_1 = -\int_0^\infty d\xi\, \partial_\xi[\xi^{-1}(2-\xi^{-1})e^{-1/\xi}](\pm i) = \mp i$$

for the two values of $\pm$. From this we obtain $S(\alpha, \beta) = (1/\alpha) + O(1)$, in agreement with the result obtained by directly summing the series.

2. Asymptotics of the stress field on the boundary of a hole in a material. The stress field we would like to estimate is given by $2TS(\alpha_1, \beta)$, where

$$S(\alpha, \beta) = 1 - \frac{2\sinh^2\alpha \sin^2\beta}{(\cosh\alpha - \cos\beta)^2} + (\cosh\alpha - \cos\beta) \times \left\{ \sum_{n=2}^{\infty} (N_n(\alpha) + N_n'(\alpha)) \cos\beta n + \frac{1}{2}\operatorname{csch}\alpha + 2e^{-2\alpha}\cos\beta \right\},$$

where

$$N_n = [n^2 \sinh\alpha \cosh n\alpha - n\sinh n\alpha \cosh\alpha]/[\sinh^2 n\alpha - n^2\sinh^2\alpha],$$
$$N_n' = -2n(n\sinh\alpha\cosh\alpha)e^{-n\alpha}.$$

The meaning of the parameters is described in the Introduction. The asymptotic limit is $\alpha \to 0+$. Only the value $\beta = \pi$ has a clear meaning in this limit. However, we can work it out for any β.

The N_n' part can be summed in closed form. The N_n part is what prompted us to develop the machinery of the last section. To see how it actually applies observe that

$$N_n(\alpha) = -\left(\frac{1}{\alpha}\cosh\alpha\right)\eta(\omega, \alpha),$$

where $\omega = n\alpha$,

$$\begin{aligned} \eta(\omega, \alpha) &= \cosh\omega[f(\omega) - f(\alpha)]/[g(\omega) - g(\alpha)], \\ f(\omega) &= \tanh\omega/\omega = 1 - \omega^2/3 + \cdots, \\ g(\omega) &= \sinh^2\omega/\omega^2 = 1 + \omega^2/3 + \cdots. \end{aligned}$$

We encounter a new problem here, because $\eta(\omega, \alpha)$ is not manifestly of extended class B, even though it is written in a very simple form. That is to say, we cannot expand $\eta(\omega, \alpha)$ in a Taylor series in α and then treat each term using singular asymptotics. The trick is to use

$$H(\omega, 1/\xi) = \eta(\omega, \omega/\xi).$$

We have $\eta(\alpha n, \alpha) = H(\alpha n, 1/n)$ and $H(\omega, \zeta)$ is of class B ($(x, y) = (\xi, \omega)$), as we easily check. The procedure of Example 1.c.1 can be used. Retaining the same notation otherwise, we will have, instead of (1.11),

$$h^1_{\pm}(x, y, \alpha) = y^2 H(\omega, \zeta)|_{\omega=\pm i\alpha\varepsilon+y, \zeta=x/(\pm i\varepsilon x+1)}.$$

It turns out that to $O(1)$ this problem is identical to Example 1.c.1. In fact, for $\cos\beta \neq 1$,

$$(y\partial_y)^4 h^1_{\pm}(x, y, \alpha) = 2^4 \frac{f''(0)}{g''(0)} y^2 + O(y^3) + O(\alpha).$$

Thus, up to the factor $f''(0)/g''(0)$ the result is the same as in Example 1.c.1. Since this factor is -1, we have

$$\sum \eta(\alpha n, \alpha) = (\cos\beta + \tfrac{1}{2}) + O(1).$$

Thus

$$\sum N_n \cos\beta n = -\alpha^{-1}(\cos\beta + \tfrac{1}{2}) + O(1).$$

$\sum N'_n \cos\beta n$ is obtained by a combination of the α- and β-derivatives of Example 1.c.1 and it is $O(1)$. Thus we obtain

$$S(\alpha, \beta) = -(1 - \cos\beta)\cos\beta\alpha^{-1} + O(1)$$

for $\cos\beta \neq 1$.

From the mathematical point of view, it is interesting that a full expansion of $S(\alpha, \beta)$ can be written down to all orders in α. To expand the integral representing $S(\alpha, \beta)$ to a given order in α, we need to regularize $\phi_{\pm}$ to a sufficiently high order k. For $\cos\beta \neq 0$, the derivatives of $\phi_{\pm}$ of order less than k at 0 will be 0. From the formulas in [3] we see that the coefficients of the logarithmic terms are linear functions of these derivatives, and therefore will be 0 in this case. Thus the stress field has an expansion of the form

$$\sum_{k=-1}^{\infty} \alpha_1^k C_k(\beta),$$

where we have already computed

$$C_{-1}(\beta) = -2T(1 - \cos\beta)\cos\beta.$$

Consider now the case $\beta = 0$. The asymptotics of $\sum N_n$ can be obtained as a linear combination of the first and second derivatives of the $\beta = 0$ result of Example 1.c.1, with coefficients $2\cosh\alpha$ and $-2\sinh\alpha$, respectively. We then see that $\sum N'_\alpha$ is $O(\alpha_1^{-2})$. In the expression for the stress field, however, this is multiplied by $1-\cosh\alpha$, so that its final contribution is $O(1)$. To estimate the term $\sum N_n$ we mimick the procedure at the end of Example 1.c.1. We then find $\sum \eta(\alpha n, \alpha) = O(\alpha^{-1})$, although the coefficient of the leading term seems to be difficult to evaluate. The corresponding contribution to the stress field is then again $O(1)$. The remaining terms in $S(\alpha, \beta)$ are also $O(1)$. Thus the stress field on the boundary of the hole is $O(1)$ as the hole approaches the boundary for $\beta = 0$. This is not surprising because $\beta = 0$ corresponds to a point which never approaches the boundary in our limit.

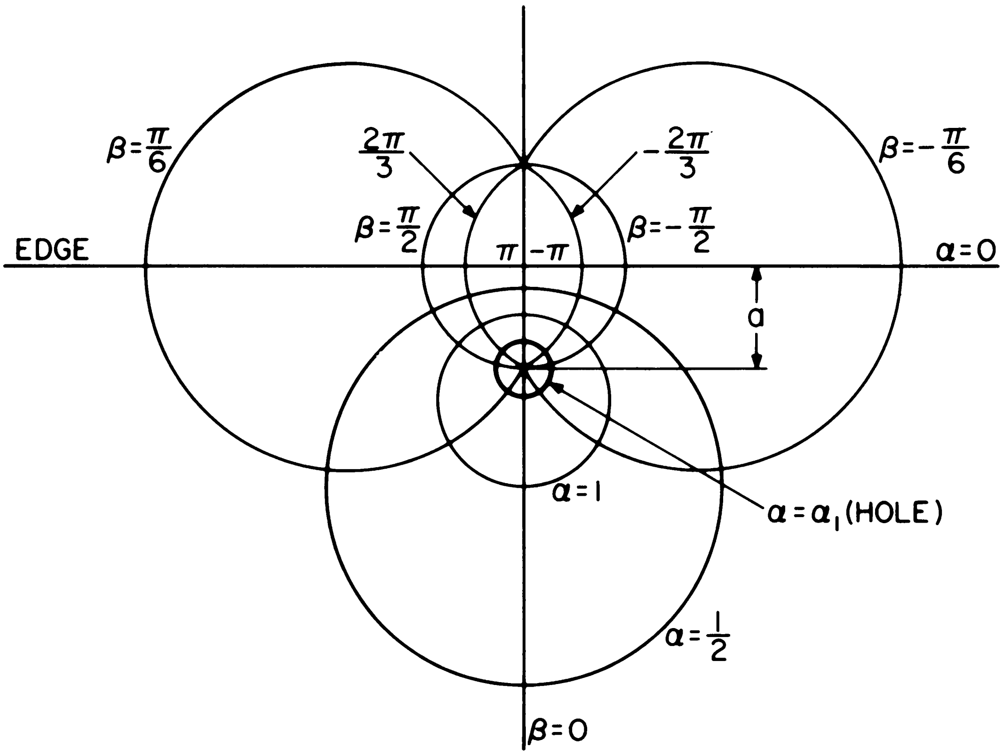

FIG. 4.

Appendix. In this appendix we give a few basic facts about the transformation to bipolar coordinates used by Jeffery and Mindlin in the derivation of the formula for the stress field that we study in this paper. A more detailed discussion and the elementary proofs of the statements that follow can be readily found in [2].

The two bipolar coordinates α and β are defined by the conformal transformation [1, 2]

$$x + iy = ia \coth \tfrac{1}{2}(\alpha + i\beta),$$

where a is an arbitrary positive constant that defines a scale in the problem. Referring to Fig. 4, this transformation maps the strip $-\pi < \beta < \pi$ in the $\alpha + i\beta$ plane onto the $x + iy$ plane with the straight segment connecting the points $P = (0, -a)$ and $P = (0, a)$ removed. The singular points P and P' correspond to $\alpha = \infty$ and $\alpha = -\infty$, respectively, while the segment $x = 0$, $-a < y < a$ is obtained for $\beta = \pi$ or $-\pi$. A simple calculation shows that, if we write

$$x + i(y + a) = r_1 e^{i\phi 1},$$
$$x + i(y - a) = r_2 e^{i\phi 2},$$

we have

$$\alpha = \ln(r_1/r_2), \qquad \beta = \phi_1 - \phi_2.$$

The curves of constant α and constant β are then seen to be circles, as shown in Fig. 4. The latter pass through the points P and P' so that their centers lie on the

x-axis, while the former are orthogonal to them and their centers lie on the y-axis. The β coordinate has a discontinuity of 2π at $x = 0$ for $-a < y < a$.

These coordinates are obviously convenient for the problem at hand, because the boundary of the hole and the edge of the material are represented as $\alpha = \alpha_1$ and $\alpha = 0$, respectively.

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