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HEAT BATHS ARE LIMITED SOURCES OF WORK*

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1. Introduction. In [4], Ricou has presented an axiomatization of thermodynamics in which the first and second laws are statements about the behaviour of thermal systems not in cyclic, or even approximately cyclic, processes but in processes that are genuinely noncyclic. Ricou's statement of a first law, or weak first law as he calls it, is based on the observation that "once a machine capable of delivering mechanical work is built, it will be unable to deliver arbitrarily large amounts of work unless it is connected to an external power source" [4, p. 88].

Within Ricou's theory, the construction of a machine is interpreted as the setting up of a thermal system in some initial configuration, or state, by making it undergo an initial process; the operation of the machine is interpreted as making the system undergo a further process which follows on from the initial one. As Ricou explains, such considerations can be made precise by using Serrin's "follow relation."

The present article is prompted by the same observation that underlies Ricou's weak first law but pursues a different course. We shall consider a deformable body which starts in a known initial state, and which is immersed in a heat bath whose temperature varies with time in a known way. We shall ask if, by adjusting the pressure on the body, we can make the body perform an arbitrarily large amount of work, i.e., we shall ask whether, for a given body, a heat bath is an unlimited source of work or only a limited source. It turns out that the answer depends upon how accurately we model the behaviour of the body. Thus, if we adopt the crudest model and assume that the temperature throughout the body always coincides with that of the heat bath, we shall be forced to conclude that a heat bath is an unlimited source of work. Such an assumption about the temperature is, in fact, the one that is in force throughout most of classical thermodynamics. On the other hand, the conclusion has to be reversed once the possibility is admitted that the temperature within the body may be spatially inhomogeneous, and it becomes necessary to take account of heat conduction within the body. According to this more accurate model, a heat bath is only a limited source of work: the amount of work that the body can do is bounded above by a number which depends on the temperature of the heat bath, the initial state of the body, and the material properties of the body, but not on what pressure is applied to the body.

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Similar conclusions are known to be true for cyclic processes [1, 2, 3]; the extension to noncyclic processes requires a new argument to take account of the, virtually arbitrary, initial state of the body.

2. A bound on the work done. Our arguments are based on the same special theory as underlies [3]. The theory is doubly approximate in that it has been linearised and ignores inertia; however, the theory does incorporate both heat conduction and thermomechanical coupling.

We consider an isotropic, but possibly inhomogeneous, thermoelastic body which occupies a slab

$$\{(x, y, z): A \leq x \leq B,\ -\infty < y < \infty,\ -\infty < z < \infty\},$$

and which undergoes a motion in which points are displaced in directions parallel to the x-axis of a system of rectangular Cartesian coordinates. The temperature $\theta(x,t)$ and the displacement $u(x,t)$ satisfy an energy equation

$$(k(x)\theta_x)_x = c(x)\theta_t + \theta_0\mu(x)u_{xt},$$

and, if inertia be neglected, a momentum equation

$$\sigma_x = 0$$

in which

$$\sigma = \beta(x)u_x - \mu(x)(\theta - \theta_0)$$

is the xx-component of stress. In these equations, the positive constant θ_0 is the temperature of a stress-free reference state of the body, $k(x)$ is the thermal conductivity, $c(x)$ is the specific heat at constant strain, $\mu(x)$ is the stress-temperature modulus, and $\beta(x)$ is an elastic modulus. Each of k, c, μ, and β, is assumed to be positive throughout the interval $[A, B]$.

Now let $[t_1, t_2]$ be any time interval. We suppose that the body is immersed in a heat bath whose temperature is $\tau(t)$, and that the faces $x = A$ and $x = B$ are subjected to a pressure $p(t)$, i.e., the boundary conditions

$$\begin{aligned} \theta(A,t) = \theta(B,t) = \tau(t) \qquad (t_1 \leq t \leq t_2), \\ \sigma(A,t) = \sigma(B,t) = -p(t) \qquad (t_1 \leq t \leq t_2), \end{aligned} \tag{2.1}$$

are presumed to be in force. We regard τ as a datum over which we have no control, but suppose that we can control p.

According to the momentum equation and the boundary condition that σ satisfies, the stress throughout the body is $\sigma(x,t) \equiv -p(t)$, and it follows from the constitutive relation for σ that

$$u_x = \frac{\mu(x)}{\beta(x)}(\theta - \theta_0) - \frac{1}{\beta(x)}p(t).$$

On differentiating with respect to t, and substituting for u_{xt} in the energy equation, we see that the temperature satisfies the parabolic equation

$$(k(x)\theta_x)_x = \left(c(x) + \theta_0\frac{\mu(x)^2}{\beta(x)}\right)\theta_t - \theta_0\frac{\mu(x)}{\beta(x)}\dot{p}(t). \tag{2.2}$$

For our purpose, the initial state of the body may be taken to be the pair (Θ, Π), where

$$\Theta(x) = \theta(x, t_1) \qquad (A \le x \le B) \tag{2.3}$$

is the temperature field at the instant t_1, and

$$\Pi = p(t_1) \tag{2.4}$$

is the pressure at that instant.

The rate at which the body performs work is

$$W(t) = p(t)u_t(B, t) - p(t)u_t(A, t).$$

Hence

$$\begin{aligned} W &= p \int_A^B u_{xt}\, dx \\ &= p \int_A^B \left(\frac{\mu}{\beta}\theta_t - \frac{1}{\beta}\dot{p} \right) dx \\ &= p \int_A^B \frac{\mu}{\beta}\theta_t\, dx - p\dot{p} \int_A^B \frac{1}{\beta}\, dx, \end{aligned}$$

and the work done by the body in the time interval $[t_1, t_2]$ is

$$\int_{t_1}^{t_2} W\, dt = \int_{t_1}^{t_2} \int_A^B p\frac{\mu}{\beta}\theta_t\, dx\, dt + \frac{1}{2}(\Pi^2 - p(t_2)^2) \int_A^B \frac{1}{\beta}\, dx. \tag{2.5}$$

Our aim is to establish that *there is a finite constant* $\Delta = \Delta(\tau, \Theta, \Pi, k, c, \mu, \beta, \theta_0)$ *such that*

$$\int_{t_1}^{t_2} W\, dt \le \Delta. \tag{2.6}$$

The significant feature of this result is that the bound Δ depends upon the heat bath temperature, the initial state, and the material properties of the body, but is independent of how the pressure p is varied on the interval $(t_1, t_2]$.

3. What the classical approach predicts. Before we attempt to prove the inequality (2.6), we pause to note that the classical approach to the calculation of the work done ignores the fact that the temperature must satisfy the parabolic equation (2.2) and the initial condition (2.3), and assumes it to be sufficient to make the approximation $\theta(x, t) \equiv \tau(t)$ throughout the body. Thus, according to the classical approach, the work done by the body is

$$\int_{t_1}^{t_2} \int_A^B p\frac{\mu}{\beta}\dot{\tau}\, dx\, dt + \frac{1}{2}(\Pi^2 - p(t_2)^2) \int_A^B \frac{1}{\beta}\, dx. \tag{3.1}$$

That the classical approach yields a conclusion which is contrary to (2.6) is shown by the remark that *if* τ *is any nonconstant and continuous function on* $[t_1, t_2]$, *and if* Π *and* E *are any numbers, it is possible to choose* p *in such a way that* $p(t_1) = \Pi$ *and the expression* (3.1) *takes the value* E.

The nonconstancy of τ is essential here, for if τ is constant (3.1) cannot exceed

$$\frac{1}{2}\Pi^2 \int_A^B \frac{1}{\beta}\, dx.$$

In order to substantiate the remark, we introduce the mean heat bath temperature

$$\bar{\tau} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \tau \, dt$$

and make the choice

$$p(t) = \Pi - M \int_{t_1}^{t} (\tau(s) - \bar{\tau}) \, ds$$

where M is the constant

$$\left[\frac{E}{\int_A^B \frac{\mu}{\beta} \, dx} - \Pi(\tau(t_2) - \tau(t_1)) \right] \Big/ \int_{t_1}^{t_2} (\tau - \bar{\tau})^2 \, dt.$$

Then $p(t_1) = p(t_2) = \Pi$, $\dot{p} = -M(\tau - \bar{\tau})$, and the expression (3.1) reduces to

$$\int_{t_1}^{t_2} \int_A^B p \frac{\mu}{\beta} \dot{\tau} \, dx \, dt = \int_A^B \frac{\mu}{\beta} \, dx \int_{t_1}^{t_2} p \dot{\tau} \, dt.$$

An integration by parts now tells us that this last is

$$\begin{aligned}
&\int_A^B \frac{\mu}{\beta} \, dx \left(p(t_2)\tau(t_2) - p(t_1)\tau(t_1) - \int_{t_1}^{t_2} \dot{p} \tau \, dt \right) \\
&\quad = \int_A^B \frac{\mu}{\beta} \, dx \left(\Pi(\tau(t_2) - \tau(t_1)) + M \int_{t_1}^{t_2} (\tau - \bar{\tau}) \tau \, dt \right) \\
&\quad = \int_A^B \frac{\mu}{\beta} \, dx \left(\Pi(\tau(t_2) - \tau(t_1)) + M \int_{t_1}^{t_2} [(\tau - \bar{\tau})^2 + (\tau - \bar{\tau})\bar{\tau}] \, dt \right) \\
&\quad = \int_A^B \frac{\mu}{\beta} \, dx \left(\Pi(\tau(t_2) - \tau(t_1)) + M \int_{t_1}^{t_2} (\tau - \bar{\tau})^2 \, dt \right)
\end{aligned}$$

and, therefore, takes the required value E, by virtue of the choice of M.

4. Verification of the bound on the work done. With a view to verifying the bound (2.6), we decompose the temperature field into a sum $\theta = \phi + \psi$, where ϕ is the temperature field which would result if the pressure were held at its initial value Π throughout the interval $[t_1, t_2]$. In other words, ϕ satisfies the conditions

$$(k\phi_x)_x = \left(c + \theta_0 \frac{\mu^2}{\beta} \right) \phi_t, \tag{4.1}$$

$$\phi(A, t) = \phi(B, t) = \tau(t) \qquad (t_1 \le t \le t_2), \tag{4.2}$$

$$\phi(x, t_1) = \Theta(x) \qquad (A \le x \le B), \tag{4.3}$$

while, as we see from (2.1), (2.2), and (2.3), ψ must satisfy the conditions

$$(k\psi_x)_x = \left(c + \theta_0 \frac{\mu^2}{\beta} \right) \psi_t - \theta_0 \frac{\mu}{\beta} \dot{p}, \tag{4.4}$$

$$\psi(A, t) = \psi(B, t) = 0 \qquad (t_1 \le t \le t_2), \tag{4.5}$$

$$\psi(x, t_1) = 0 \qquad (A \le x \le B). \tag{4.6}$$

If we return to the formula (2.5) for the work done, and carry out an integration by parts, we deduce that

$$
\begin{aligned}
\int_{t_1}^{t_2} W\,dt = &- \int_{t_1}^{t_2}\int_A^B \dot{p}\frac{\mu}{\beta}\theta\,dx\,dt + p(t_2)\int_A^B \frac{\mu(x)}{\beta(x)}\theta(x,t_2)\,dx \\
&- \Pi\int_A^B \frac{\mu}{\beta}\Theta\,dx + \frac{1}{2}(\Pi^2 - p(t_2)^2)\int_A^B \frac{1}{\beta}\,dx \\
= &- \int_{t_1}^{t_2}\int_A^B \dot{p}\frac{\mu}{\beta}(\phi+\psi)\,dx\,dt + p(t_2)\int_A^B \frac{\mu(x)}{\beta(x)}(\phi(x,t_2)+\psi(x,t_2))\,dx \\
&- \Pi\int_A^B \frac{\mu}{\beta}\Theta\,dx + \frac{1}{2}(\Pi^2 - p(t_2)^2)\int_A^B \frac{1}{\beta}\,dx.
\end{aligned}
\tag{4.7}
$$

The expression (4.7) depends upon the pressure p in a rather complicated way, since ψ depends upon p; it will be necessary to cast (4.7) into a more convenient form. To start we observe that the equation

$$
(\psi k \psi_x)_x = k\psi_x^2 + \left(c + \theta_0\frac{\mu^2}{\beta}\right)\psi\psi_t - \theta_0\frac{\mu}{\beta}\dot{p}\psi
$$

is a consequence of (4.4). On integrating with respect to x, and appealing to the boundary conditions (4.5), we see that

$$
0 = \int_A^B k\psi_x^2\,dx + \frac{1}{2}\dot{\overline{\int_A^B \left(c+\theta_0\frac{\mu^2}{\beta}\right)\psi^2\,dx}} - \theta_0\int_A^B \dot{p}\frac{\mu}{\beta}\psi\,dx,
$$

and, on making a further integration with respect to t and using the initial condition (4.6), we arrive at the conclusion that

$$
\theta_0\int_{t_1}^{t_2}\int_A^B \dot{p}\frac{\mu}{\beta}\psi\,dx\,dt = \int_{t_1}^{t_2}\int_A^B k\psi_x^2\,dx\,dt + \frac{1}{2}\int_A^B \left(c(x)+\theta_0\frac{\mu(x)^2}{\beta(x)}\right)\psi(x,t_2)^2\,dx.
\tag{4.8}
$$

Next, we multiply (4.4) through by $\phi - \tau$ and obtain the equation

$$
((\phi-\tau)k\psi_x)_x = \phi_x k\psi_x + \left(c+\theta_0\frac{\mu^2}{\beta}\right)(\phi-\tau)\psi_t - \theta_0\frac{\mu}{\beta}\dot{p}(\phi-\tau).
$$

By virtue of the boundary conditions (4.2), we have

$$
\begin{aligned}
0 = &\int_A^B \phi_x k\psi_x\,dx + \int_A^B \left(c+\theta_0\frac{\mu^2}{\beta}\right)(\phi-\tau)\psi_t\,dx \\
&- \theta_0\int_A^B \dot{p}\frac{\mu}{\beta}\phi\,dx + \theta_0\dot{p}\tau\int_A^B \frac{\mu}{\beta}\,dx \\
= &\int_A^B \phi_x k\psi_x\,dx + \dot{\overline{\int_A^B \left(c+\theta_0\frac{\mu^2}{\beta}\right)(\phi-\tau)\psi\,dx}} \\
&- \int_A^B \left(c+\theta_0\frac{\mu^2}{\beta}\right)(\phi_t-\dot{\tau})\psi\,dx - \theta_0\int_A^B \dot{p}\frac{\mu}{\beta}\phi\,dx \\
&+ \theta_0\dot{p}\tau\int_A^B \frac{\mu}{\beta}\,dx.
\end{aligned}
$$

On integrating with respect to t, and rearranging the resulting equation, we deduce that

$$\theta_0\int_{t_1}^{t_2}\int_A^B \dot{p}\frac{\mu}{\beta}\phi\,dx\,dt = \int_{t_1}^{t_2}\int_A^B \phi_x k\psi_x\,dx\,dt - \int_{t_1}^{t_2}\int_A^B \left(c+\theta_0\frac{\mu^2}{\beta}\right)(\phi_t-\dot{\tau})\psi\,dx\,dt$$
$$+\theta_0\int_A^B \frac{\mu}{\beta}\,dx\int_{t_1}^{t_2}\dot{p}\tau\,dt$$
$$+\int_A^B\left(c(x)+\theta_0\frac{\mu(x)^2}{\beta(x)}\right)(\phi(x,t_2)-\tau(t_2))\psi(x,t_2)\,dx. \tag{4.9}$$

The third term on the right-hand side of (4.9) has now to be modified by a further artifice. Let the positive number λ be the smallest eigenvalue, and let $\xi(x)$ be a corresponding eigenfunction, of the Sturm-Liouville problem

$$(k\xi')' = -\lambda\left(c+\theta_0\frac{\mu^2}{\beta}\right)\xi, \qquad \xi(A)=\xi(B)=0.$$

As is well known, ξ is of fixed sign in the open interval (A,B) and, therefore, the integral

$$\int_A^B \xi\frac{\mu}{\beta}\,dx$$

cannot vanish. Then

$$(\xi k\psi_x - \psi k\xi')_x = \xi(k\psi_x)_x - \psi(k\xi')'$$
$$= \xi\left(c+\theta_0\frac{\mu^2}{\beta}\right)\psi_t - \theta_0\xi\frac{\mu}{\beta}\dot{p} + \lambda\xi\left(c+\theta_0\frac{\mu^2}{\beta}\right)\psi,$$

and, on integrating with respect to x, we see that

$$0 = \overline{\int_A^B \xi\left(c+\theta_0\frac{\mu^2}{\beta}\right)\psi\,dx}^{\,\boldsymbol{\cdot}} - \theta_0\dot{p}\int_A^B \xi\frac{\mu}{\beta}\,dx + \lambda\int_A^B \xi\left(c+\theta_0\frac{\mu^2}{\beta}\right)\psi\,dx.$$

Thus, if we multiply through by τ and rearrange the resulting expression, we have

$$0 = \overline{\tau\int_A^B \xi\left(c+\theta_0\frac{\mu^2}{\beta}\right)\psi\,dx}^{\,\boldsymbol{\cdot}} - \dot{\tau}\int_A^B \xi\left(c+\theta_0\frac{\mu^2}{\beta}\right)\psi\,dx$$
$$-\theta_0\dot{p}\tau\int_A^B \xi\frac{\mu}{\beta}\,dx + \lambda\tau\int_A^B \xi\left(c+\theta_0\frac{\mu^2}{\beta}\right)\psi\,dx.$$

An integration with respect to t now enables us to conclude that

$$0 = \tau(t_2)\int_A^B \xi(x)\left(c(x)+\theta_0\frac{\mu(x)^2}{\beta(x)}\right)\psi(x,t_2)\,dx$$
$$+\int_{t_1}^{t_2}\int_A^B (\lambda\tau-\dot{\tau})\xi\left(c+\theta_0\frac{\mu^2}{\beta}\right)\psi\,dx\,dt - \theta_0\int_A^B \xi\frac{\mu}{\beta}\,dx\int_{t_1}^{t_2}\dot{p}\tau\,dt,$$

and, therefore, that

$$\theta_0 \int_{t_1}^{t_2} \dot{p}\tau \, dt = \frac{1}{\int_A^B \xi \frac{\mu}{\beta} \, dx} \left[\tau(t_2) \int_A^B \xi(x) \left(c(x) + \theta_0 \frac{\mu(x)^2}{\beta(x)} \right) \psi(x, t_2) \, dx \right.$$
$$\left. + \int_{t_1}^{t_2} \int_A^B (\lambda\tau - \dot{\tau})\xi \left(c + \theta_0 \frac{\mu^2}{\beta} \right) \psi \, dx \, dt \right].$$

If we substitute this last expression into the right-hand side of (4.9) we see that

$$\theta_0 \int_{t_1}^{t_2} \int_A^B \dot{p} \frac{\mu}{\beta} \phi \, dx \, dt = \int_{t_1}^{t_2} \int_A^B \phi_x k \psi_x \, dx \, dt - \int_{t_1}^{t_2} \int_A^B \left(c + \theta_0 \frac{\mu^2}{\beta} \right) (\phi_t - \dot{\tau}) \psi \, dx \, dt$$
$$+ \frac{\int_A^B \frac{\mu}{\beta} \, dx}{\int_A^B \xi \frac{\mu}{\beta} \, dx} \left[\tau(t_2) \int_A^B \xi(x) \left(c(x) + \theta_0 \frac{\mu(x)^2}{\beta(x)} \right) \psi(x, t_2) \, dx \right.$$
$$\left. + \int_{t_1}^{t_2} \int_A^B (\lambda\tau - \dot{\tau})\xi \left(c + \theta_0 \frac{\mu^2}{\beta} \right) \psi \, dx \, dt \right]$$
$$+ \int_A^B \left(c(x) + \theta_0 \frac{\mu(x)^2}{\beta(x)} \right) (\phi(x, t_2) - \tau(t_2)) \psi(x, t_2) \, dx. \tag{4.10}$$

On substituting from (4.10) and (4.8) into (4.7), for the integrals

$$\int_{t_1}^{t_2} \int_A^B \dot{p} \frac{\mu}{\beta} \phi \, dx \, dt, \qquad \int_{t_1}^{t_2} \int_A^B \dot{p} \frac{\mu}{\beta} \psi \, dx \, dt,$$

we arrive at a formula

$$\int_{t_1}^{t_2} W \, dt = X + Y - \Pi \int_A^B \frac{\mu}{\beta} \Theta \, dx + \frac{1}{2} \Pi^2 \int_A^B \frac{1}{\beta} \, dx,$$

in which

$$X = -\frac{1}{\theta_0} \int_{t_1}^{t_2} \int_A^B k \psi_x^2 \, dx \, dt - \frac{1}{\theta_0} \int_{t_1}^{t_2} \int_A^B \phi_x k \psi_x \, dx \, dt + \frac{1}{\theta_0} \int_{t_1}^{t_2} \int_A^B f \psi \, dx \, dt,$$

$$Y = -\int_A^B \left[\frac{1}{2\theta_0} \left(c(x) + \theta_0 \frac{\mu(x)^2}{\beta(x)} \right) \psi(x, t_2)^2 - \frac{\mu(x)}{\beta(x)} \psi(x, t_2) p(t_2) \right.$$
$$\left. + \frac{1}{2\beta(x)} p(t_2)^2 + g(x) \psi(x, t_2) + h(x) p(t_2) \right] dx,$$

$$f = \left(c + \theta_0 \frac{\mu^2}{\beta} \right) (\phi_t - \dot{\tau} - \nu(\lambda\tau - \dot{\tau})\xi),$$

$$g(x) = \frac{1}{\theta_0} \left(c(x) + \theta_0 \frac{\mu(x)^2}{\beta(x)} \right) (\nu \xi(x) \tau(t_2) + \phi(x, t_2) - \tau(t_2)),$$

$$h(x) = -\frac{\mu(x)}{\beta(x)} \phi(x, t_2),$$

$$\nu = \int_A^B \frac{\mu}{\beta} \, dx \Big/ \int_A^B \xi \frac{\mu}{\beta} \, dx.$$

It is important to observe that the pressure appears in these equations in only two ways—either directly, as a displayed symbol in the definition of Y, or indirectly, through ψ. Each of ϕ, f, g, and h, is independent of the pressure.

To complete the proof of the inequality (2.6) we shall estimate X and Y separately, and show that there are constants $\Delta_1 = \Delta_1(\tau, \Theta, k, c, \mu, \beta, \theta_0)$ and $\Delta_2 = \Delta_2(\tau, \Theta, k, c, \mu, \beta, \theta_0)$, both independent of the initial pressure Π, such that $X \le \Delta_1$ and $Y \le \Delta_2$. It will then follow, as required, that

$$\int_{t_1}^{t_2} W\,dt \le \Delta_1 + \Delta_2 - \Pi \int_A^B \frac{\mu}{\beta}\Theta\,dx + \frac{1}{2}\Pi^2 \int_A^B \frac{1}{\beta}\,dx = \Delta.$$

In order to estimate X we appeal to the boundary conditions (4.5) that ψ satisfies, and to the variational characterisation of the eigenvalue λ, to obtain the inequality

$$\lambda \int_A^B \left(c + \theta_0 \frac{\mu^2}{\beta}\right) \psi^2\,dx \le \int_A^B k\psi_x^2\,dx.$$

This last and Schwarz's inequality then imply that

$$\begin{aligned}
X \le & -\frac{1}{\theta_0}\int_{t_1}^{t_2}\int_A^B k\psi_x^2\,dx\,dt + \frac{1}{\theta_0}\left(\int_{t_1}^{t_2}\int_A^B k\phi_x^2\,dx\,dt\right)^{1/2}\left(\int_{t_1}^{t_2}\int_A^B k\psi_x^2\,dx\,dt\right)^{1/2} \\
& + \frac{1}{\theta_0}\left(\int_{t_1}^{t_2}\int_A^B \frac{f^2}{c+\theta_0\frac{\mu^2}{\beta}}\,dx\,dt\right)^{1/2}\left(\int_{t_1}^{t_2}\int_A^B \left(c+\theta_0\frac{\mu^2}{\beta}\right)\psi^2\,dx\,dt\right)^{1/2} \\
\le & -\frac{1}{\theta_0}\int_{t_1}^{t_2}\int_A^B k\psi_x^2\,dx\,dt \\
& + \frac{1}{\theta_0}\left[\left(\int_{t_1}^{t_2}\int_A^B k\phi_x^2\,dx\,dt\right)^{1/2} + \left(\frac{1}{\lambda}\int_{t_1}^{t_2}\int_A^B \frac{f^2}{c+\theta_0\frac{\mu^2}{\beta}}\,dx\,dt\right)^{1/2}\right] \\
& \cdot\left(\int_{t_1}^{t_2}\int_A^B k\psi_x^2\,dx\,dt\right)^{1/2}
\end{aligned}$$

and, in view of the inequality $-x^2 + mx \le m^2/4$, that $X \le \Delta_1$, where

$$\Delta_1 = \frac{1}{4\theta_0}\left[\left(\int_{t_1}^{t_2}\int_A^B k\phi_x^2\,dx\,dt\right)^{1/2} + \left(\frac{1}{\lambda}\int_{t_1}^{t_2}\int_A^B \frac{f^2}{c+\theta_0\frac{\mu^2}{\beta}}\,dx\,dt\right)^{1/2}\right]^2.$$

In order to estimate Y it is enough to examine how the integrand, which is enclosed within square brackets in the equation that defines Y, depends upon ψ and p. (For the moment, we ignore the dependence of ψ upon p.) Indeed, consider the function of two variables

$$(\psi, p) \mapsto \frac{1}{2\theta_0}\left(c + \theta_0\frac{\mu^2}{\beta}\right)\psi^2 - \frac{\mu}{\beta}\psi p + \frac{1}{2\beta}p^2 + g\psi + hp.$$

Since the coefficient of ψ^2 is positive, and so is the discriminant

$$\frac{1}{\theta_0}\left(c + \theta_0\frac{\mu^2}{\beta}\right)\cdot\frac{1}{\beta} - \left(\frac{\mu}{\beta}\right)^2 = \frac{c}{\theta_0\beta},$$

the function has just one stationary point in the (ψ, p)-plane, and that point is a minimum. The minimum value proves to be

$$-\frac{\theta_0\beta}{c}\left(\frac{1}{2\beta}g^2+\frac{\mu}{\beta}gh+\frac{1}{2\theta_0}\left(c+\theta_0\frac{\mu^2}{\beta}\right)h^2\right)$$

and, thus, $Y \leq \Delta_2$, where

$$\Delta_2=\theta_0\int_A^B\frac{\beta}{c}\left(\frac{1}{2\beta}g^2+\frac{\mu}{\beta}gh+\frac{1}{2\theta_0}\left(c+\theta_0\frac{\mu}{\beta}\right)h^2\right)dx.$$

Hence the proof of (2.6) is complete.

Our considerations raise a question which we shall not attempt to answer here; the question asks "What are the implications for the coefficients k, c, μ, and β, if we reverse the line of argument and postulate that an inequality (2.6) is valid, for some constant $\Delta(\tau, \Theta, \Pi, k, c, \mu, \beta, \theta_0)$?"

5. The performance of a positive amount of work. It would appear to be difficult to produce an explicit expression for the optimum value of the constant Δ, and even if such an expression were available it would almost certainly be a complicated one. The arguments of §4, however, enable us to show that, in certain circumstances, it is possible to choose the pressure p in such a way that the body performs a positive amount of work for which a simple formula is available. For simplicity, we treat only the case in which the initial pressure $\Pi = 0$. The formula is couched in terms of the field ϕ which, it will be recalled, is the temperature which would result if the pressure were held constant throughout the interval $[t_1, t_2]$. It will be necessary to introduce a positive number λ^*, which is the least eigenvalue associated with the Sturm-Liouville problem

$$(k\zeta')'=-\lambda^*\frac{\mu}{\beta}\zeta,\qquad \zeta(A)=\zeta(B)=0.$$

What will be proved is that *if* $\Pi = 0$ *it is possible to choose* p *in such a way that*

$$\int_{t_1}^{t_2}W\,dt\geq\frac{\lambda^*}{4\theta_0\int_A^B\frac{\mu}{\beta}\,dx}\int_{t_1}^{t_2}\left(\int_A^B\frac{\mu}{\beta}\phi\,dx-m\right)^2dt,\tag{5.1}$$

where m *is the mean value*

$$\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\int_A^B\frac{\mu}{\beta}\phi\,dx\,dt.$$

The appropriate choice of the pressure turns out to be

$$p(t)=-\frac{\lambda^*}{2\theta_0\int_A^B\frac{\mu}{\beta}\,dx}\int_{t_1}^{t}\left(\int_A^B\frac{\mu(x)}{\beta(x)}\phi(x,s)\,dx-m\right)ds.\tag{5.2}$$

This choice ensures that $p(t_1) = 0 = p(t_2)$ and, hence, Eq. (4.7) tells us that

$$\int_{t_1}^{t_2}W\,dt=-\int_{t_1}^{t_2}\int_A^B\dot{p}\frac{\mu}{\beta}(\phi+\psi)\,dx\,dt.\tag{5.3}$$

The boundary conditions (4.5) that ψ satisfies, and the variational characterisation of the eigenvalue λ^*, imply the inequality

$$\lambda^* \int_A^B \frac{\mu}{\beta}\psi^2\,dx \le \int_A^B k\psi_x^2\,dx$$

and it follows, with the aid of the Schwarz inequality, that

$$\begin{aligned}\lambda^* \left(\int_{t_1}^{t_2}\int_A^B \dot{p}\frac{\mu}{\beta}\psi\,dx\,dt\right)^2 &\le \lambda^* \int_{t_1}^{t_2}\int_A^B \dot{p}^2\frac{\mu}{\beta}\,dx\,dt \cdot \int_{t_1}^{t_2}\int_A^B \frac{\mu}{\beta}\psi^2\,dx\,dt \\ &\le \int_A^B \frac{\mu}{\beta}\,dx \cdot \int_{t_1}^{t_2}\dot{p}^2\,dt \cdot \int_{t_1}^{t_2}\int_A^B k\psi_x^2\,dx\,dt.\end{aligned} \tag{5.4}$$

On combining (5.4) with (4.8), and discarding the (nonnegative) second term on the right-hand side of (4.8), we see that

$$\begin{aligned}&\theta_0 \int_A^B \frac{\mu}{\beta}\,dx \cdot \int_{t_1}^{t_2}\dot{p}^2\,dt \cdot \int_{t_1}^{t_2}\int_A^B \dot{p}\frac{\mu}{\beta}\psi\,dx\,dt \\ &\quad\ge \int_A^B \frac{\mu}{\beta}\,dx \cdot \int_{t_1}^{t_2}\dot{p}^2\,dt \cdot \int_{t_1}^{t_2}\int_A^B k\psi_x^2\,dx\,dt \\ &\quad\ge \lambda^* \left(\int_{t_1}^{t_2}\int_A^B \dot{p}\frac{\mu}{\beta}\psi\,dx\,dt\right)^2\end{aligned}$$

and so we have

$$\int_{t_1}^{t_2}\int_A^B \dot{p}\frac{\mu}{\beta}\psi\,dx\,dt \le \frac{\theta_0}{\lambda^*}\int_A^B \frac{\mu}{\beta}\,dx \cdot \int_{t_1}^{t_2}\dot{p}^2\,dt.$$

This last inequality and (5.3) now yield the lower bound

$$\int_{t_1}^{t_2} W\,dt \ge -\int_{t_1}^{t_2} \dot{p}\left(\int_A^B \frac{\mu}{\beta}\phi\,dx\right)dt - \frac{\theta_0}{\lambda^*}\int_A^B \frac{\mu}{\beta}\,dx \cdot \int_{t_1}^{t_2}\dot{p}^2\,dt,$$

and when p is chosen as in (5.2) this reduces to (5.1), as required.

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