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THE NONUNIQUENESS OF THE FLOW OF A VISCOELASTIC FLUID OVER A STRETCHING SHEET*

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1. The background. K. R. Rajagopal, T. Y. Na, and A. S. Gupta have studied the flow of a viscoelastic fluid over a stretching sheet [1]. By using a perturbation analysis for boundary layer flows of non-Newtonian fluids, they describe the motion as a fourth-order nonlinear equation. The boundary layer model they developed consists of the system

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$\frac{u\partial u}{\partial x}+\frac{v\partial u}{\partial y}=\frac{\nu\partial^2 u}{\partial y^2}-k\left[\frac{\partial(u\partial^2 u/\partial y^2)}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{\partial^2 v}{\partial y^2}+\frac{v\partial^3 u}{\partial y^3}\right], \tag{2}$$

with the boundary conditions

$$u=cx,\ c>0,\ v=0,\quad \text{at } y=0;\qquad u\to 0\quad \text{as } y\to\infty, \tag{3}$$

where k is the Non-Newtonian parameter, a positive constant that is only related to the material constants and the kinematic viscosity. By the transformation $\eta = (c/\nu)^{1/2}y$, $k_1 = kc/\nu$, $u = cxf'$, $v = -(\nu c)^{1/2}f$, Rajagopal, etc., transformed (1), (2), (3) into the fourth-order equation

$$(f')^2-ff''=f'''-k_1[2f'f'''-(f'')^2-ff''''], \tag{4}$$

$$f'(0)=1,\qquad f(0)=0,\qquad f'(\infty)=0. \tag{5}$$

It has been proved simultaneously by B. McLeod and K. R. Rajagopal [2] and by W. C. Troy, etc., [3] that (4) and (5) has a unique solution $g(\eta) = 1 - e^{-\eta}$ when $k_1 = 0$. W. C. Troy, etc., also found [3], when $0 < k_1 < 1$, that the solution to (4), (5) is given by

$$g(\eta)=\sqrt{1-k_1}(1-e^{-\eta/\sqrt{1-k_1}}). \tag{6}$$

Many mathematicians believe, according to physical experiment and the computer estimates, that the solution (6) is also unique. The goal of this short note is to show that the solution to (4) and (5) is not necessarily unique.

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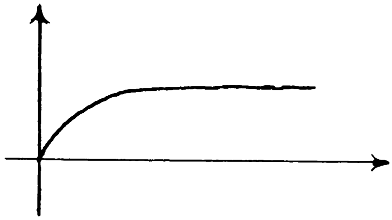

FIG. 1.

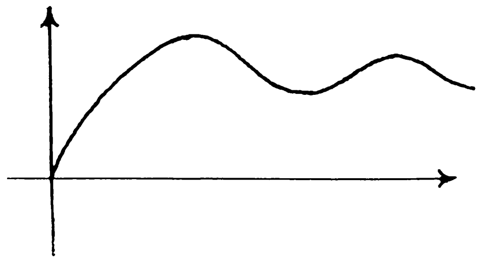

FIG. 2

2. The nonuniqueness. By some analysis and by careful computer estimates, we may see that the solution to (4) with boundary condition (5) has the following behavior:

Figure 1 corresponds to Troy's solution (6). Figure 2 motivates us into looking for a solution like

$$f(\eta) = a(1 - e^{-b\eta} \cos \omega\eta), \tag{7}$$

where the coefficients a, b, and ω are to be determined. We elaborate on $k_1 = 1/2$. Now the equation (4) is

$$(f')^2 - ff'' = f''' - f'f''' + (1/2)[ff'''' + (f'')^2]. \tag{4'}$$

By using (5) we can determine $a \neq 0$ and $b = 1/a$. By comparison of coefficients in (4′) we end up with the following equations:

$$\omega^2 - b^2 - 1 = 0,$$
$$2ab\omega^2 - \omega^2 - 2ab^3 + 3b^2 - 2ab = 0,$$
$$-a\omega^4 + (6ab^2 - 6b + 2a)\omega^2 - ab^4 + 2b^3 - 2ab^2 = 0.$$

The solution to these equations is $a = \sqrt{2}$, $b = 1/\sqrt{2}$, $\omega = \sqrt{3/2}$. Thus we found another solution to (4) and (5):

$$f(\eta) = \sqrt{2}[1 - e^{-\eta/\sqrt{2}} \cos(\sqrt{3/2} \cdot \eta)].$$

Together with the solution

$$g(\eta) = (1 - e^{-\sqrt{2}\cdot\eta})/\sqrt{2},$$

we have proved that the solution is not unique.

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