THE ROOTS OF $J_0(z) - iJ_1(z) = 0$ *

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Abstract. Synolakis [1] has proved that the equation

$$J_0(z) - iJ_1(z) = 0$$

has no zeros in the half plane $\text{Im } z > 0$. In this note a table of the first thirty roots, correct to $O(10^{-6})$, is presented and an asymptotic formula, which is correct to better than one tenth of one percent for the smallest zero, is derived.

1. Introduction. The roots of the equation

$$f(z) \equiv J_0(z) - iJ_1(z) = 0, \quad (1)$$

in which $J_n(z)$ denotes the Bessel Function of the first kind of order n , are of interest to the specialist in the problem of water wave runup on a sloping beach (Carrier [2], Synolakis [3]). Synolakis [1] has shown that the equation has no roots in the upper half plane. In the case of the solitary wave propagating over water of constant depth and running up a sloping beach, this feature enables integrals representing leading approximations to important physical quantities to be evaluated without a detailed knowledge of the roots of Eq. (1). In the present note a table of the first thirty roots, correct to $O(10^{-6})$, will be presented and an asymptotic formula, which predicts all subsequent roots to this accuracy, or better, will be derived.

2. Extracting the roots. Write $z = i\zeta$ in Eq. (1). Then

$$f(i\zeta) \equiv I_0(\zeta) + I_1(\zeta) = 0, \quad (2)$$

where $I_n(\zeta)$ denotes the modified Bessel function of the first kind of order n . From the ascending series for $I_n(\zeta)$, $n = 0, 1$, it is easy to show that when ζ_j is a root of Eq. (2) so too is $\bar{\zeta}_j$, the complex conjugate of ζ_j . As the roots of Eq. (1) lie in the

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third and fourth quadrants of the z -plane, those of Eq. (2) lie in the second and third quadrants of the ζ -plane. In what follows, it will be assumed that $\text{Im}(\zeta) < 0$.

When $|\zeta| \gg 1$ and $-\frac{3}{2}\pi < \arg \zeta < \frac{1}{2}\pi$, $I_n(\zeta)$ is accurately represented by a suitably truncated version of the asymptotic expansion [4],

$$I_n(\zeta) \sim \frac{e^\zeta}{(2\pi\zeta)^{1/2}} \left[1 + \sum_{r=1}^{\infty} (-1)^r \frac{[4n^2 - 1^2][4n^2 - 3^2] \cdots [4n^2 - (2r-1)^2]}{r! 2^{3r} \zeta^r} \right] \\ + \frac{e^{-[n+1/2]\pi i} e^{-\zeta}}{(2\pi\zeta)^{1/2}} \left[1 + \sum_{r=1}^{\infty} \frac{[4n^2 - 1^2][4n^2 - 3^2] \cdots [4n^2 - (2r-1)^2]}{r! 2^{3r} \zeta^r} \right]. \quad (3)$$

The second series is negligible only when $|\arg \zeta| < \pi/2$, which condition is not satisfied by the roots of Eq. (2). To find approximations to the roots, Eq. (3) is used to derive an approximate equation which is valid when $|\zeta| \gg 1$. From this equation an asymptotic formula for the roots of Eq. (2) is derived. Improved accuracy is achieved by use of Newton–Raphson iteration. For the first few roots the $I_n(\zeta)$, $n = 0, 1$, are calculated from their ascending series, otherwise Eq. (3), with each series truncated at its fourth term, is used.

3. The asymptotic formulae. When each series in Eq. (3) is truncated at its third term, Eq. (2) is approximated by

$$4\zeta e^{2\zeta} \left[1 - \frac{1}{8\zeta} - \frac{3}{128\zeta^2} - \frac{15}{1024\zeta^3} \right] = -i \left[1 - \frac{3}{8\zeta} + \frac{45}{128\zeta^2} - \frac{525}{1024\zeta^3} \right]. \quad (4)$$

From Eq. (4), it follows, on taking the logarithm of both sides and noting that $|\zeta| \gg 1$, that

$$2\zeta = -\ln 4\zeta + \left[2p - \frac{1}{2} \right] \pi i - \frac{1}{4\zeta} + \frac{5}{16\zeta^2} - \frac{73}{192\zeta^3} + O \left[\frac{1}{\zeta^4} \right], \quad (5)$$

where p can be any integer, including zero.

An approximate solution of Eq. (5) is given by

$$\zeta = r_A e^{i\theta_A},$$

where r_A and θ_A satisfy the equations:

$$r_A = j\pi \left[1 + \{1 + 4\alpha^2\} \frac{1}{8r_A^2} + \{-7 + 120\alpha - 24\alpha^2 + 144\alpha^4\} \frac{1}{384r_A^4} \right] \\ - \frac{\alpha}{2r_A} - \frac{\alpha^3}{3r_A^3} - \frac{5}{64r_A^3} + O \left[\frac{\alpha^2}{r_A^4} \right], \quad (6)$$

$$\theta_A = -\frac{1}{2}\pi - \frac{\alpha}{r_A} - (15 - 12\alpha + 16\alpha^3) \frac{1}{96r_A^3} + O \left[\frac{\alpha^2}{r_A^4} \right], \quad (7)$$

in which j is any (large) positive integer and

$$\alpha = \frac{1}{2} \ln 4r_A.$$

Since $r_A \gg \alpha > 1$, Eq. (7) leads to

$$r_A = j\pi + \{1 - 4\alpha_A(1 - \alpha_A)\} \frac{1}{8j\pi} + \{-61 + 264\alpha_A - 360\alpha_A^2 + 256\alpha_A^3 - 48\alpha_A^4\} \frac{1}{(384j^3\pi^3)} + O\left[\frac{\alpha_A^2}{j^4\pi^4}\right] \quad (8)$$

where

$$\alpha_A = \frac{1}{2} \ln(4j\pi).$$

Finally, use of Eq. (8) in Eq. (7) gives

$$\theta_A = -\frac{1}{2}\pi - \frac{\alpha_A}{j\pi} - \frac{1}{96j^3\pi^3}[21 - 48\alpha_A + 72\alpha_A^2 - 32\alpha_A^3] + O\left[\frac{\alpha_A^2}{j^4\pi^4}\right]. \quad (9)$$

Equations (8) and (9) give an approximation to the j th third quadrant root of Eq. (2), when $j\pi \gg 1$.

The j th second quadrant root is given by

$$\zeta = r_A e^{-i\theta_1}.$$

The roots of Eq. (1) are

$$z = r_A e^{i[\pi/2 \pm \theta_1]}.$$

Table 1 presents the numerical and the asymptotic approximations to the *fourth quadrant* roots of

$$J_0(z) - iJ_1(z) = 0.$$

The third quadrant roots are obtained by reflecting the fourth quadrant roots in the negative y -axis. The roots of

$$I_0(\zeta) + I_1(\zeta) = 0$$

are obtained from

$$\zeta = -iz.$$

As is clear from the table, the asymptotic approximation, which is formally valid when $j\pi \gg 1$, is acceptable for $j = 1$!

n	x_n	y_n	x_n^A	y_n^A
1	2.980382	-1.279603	2.978	-1.282
2	6.175153	-1.618717	6.1747	-1.6190
3	9.341961	-1.818873	9.3418	-1.8190
4	12.498507	-1.961460	12.4984	-1.9615
5	15.650104	-2.072310	15.65006	-2.07233
6	18.798911	-2.163011	18.79888	-2.16302
7	21.945980	-2.239773	21.94596	-2.23978
8	25.091886	-2.306313	25.09187	-2.30632
9	28.236973	-2.365036	28.236964	-2.365038
10	31.381461	-2.417587	31.381454	-2.417588
11	34.525496	-2.465140	34.525490	-2.465141
12	37.669178	-2.508564	37.669174	-2.508564
13	40.812583	-2.548518	40.812579	-2.548518
14	43.955762	-2.585516	43.955759	-2.585516
15	47.098757	-2.619966	47.098754	-2.619966
16	50.241599	-2.652195	50.241597	-2.652195
17	53.384312	-2.682473	53.384310	-2.682473
18	56.526916	-2.711023	56.526915	-2.711023
19	59.669427	-2.738031	59.669425	-2.738031
20	62.811857	-2.763655	62.811856	-2.763655
21	65.954216	-2.788030	65.954215	-2.788030
22	69.096515	-2.811272	69.096514	-2.811272
23	72.238759	-2.833482	72.238758	-2.833482
24	75.380955	-2.854748	75.380954	-2.854748
25	78.523108	-2.875146	78.523107	-2.875146
26	81.665223	-2.894744	81.665223	-2.894744
27	84.807304	-2.913604	84.807303	-2.913604
28	87.949354	-2.931778	87.949353	-2.931778
29	91.091376	-2.949315	91.091375	-2.949315
30	94.233372	-2.966258	94.233372	-2.966258

TABLE 1. The fourth quadrant roots of $J_0(z) - iJ_1(z) = 0$.

$z_n = x_n + iy_n$ is the root correct to $O(10^{-6})$.

$z_n^A = x_n^A + iy_n^A$ is the asymptotic approximation given by Eqs. (8)

and (9). For $n \geq 30$, z_n^A is correct to $O(10^{-6})$, or better.

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