BOUNDED AND UNBOUNDED SOLUTIONS OF THE VON KARMAN VORTEX TRAIL*

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Abstract. The general initial value problem for the linearized von Karman vortex trail is solved. In particular, the fundamental differences between von Karman's normal mode solutions and aperiodic solutions with compact support are discussed. Within the natural space ℓ_2 , it is shown that the von Karman trail at its special aspect ratio (neutrally stable case) supports unbounded solutions. Two invariants of the equations determine whether a solution is bounded or unbounded. The asymptotic behaviour of aperiodic solutions is discussed. It is found that unbounded solutions grow at the rate $O(\ln \tau)$ in ℓ_∞ while bounded solutions decay as $O(1/\sqrt{\tau})$. In addition, a Loschmidt's demon is constructed for the reversible dispersive equations which demonstrates the focussing effect in the von Karman trail.

I. Introduction. In this paper we consider disturbances of the von Karman vortex trail that have compact support. As will be seen the aperiodic disturbances lead to fundamentally different solutions than periodic ones. Thus we discuss the general initial value problem for the linearized case. In contrast to the periodic case we show the existence of unbounded solutions. Two geometrical invariants determine whether a solution is bounded or unbounded. For the class of unbounded solutions, we find that the asymptotic behaviour consists of growing wave fronts that propagate at a pair of well-defined group speeds. On the other hand, bounded solutions decay at the rate $O(1/\sqrt{\tau})$ but dispersion can focus arbitrarily small initial data into relatively large waveforms.

The von Karman vortex trail is a one-dimensional lattice of point vortices introduced by von Karman [1, 2, 3] to describe the vortex street behind cylindrical bluff bodies (see Fig. 1). The aspect ratio

$$k \equiv \frac{h}{\ell} \tag{1}$$

which is the nondimensionalized separation between the two rows in the equilibrium lattice, plays an important role in the normal mode analysis of the problem [2]. In

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FIG. 1. Von Karman Vortex Trail. The circulation is denoted by κ , the distance between vortices in each row by ℓ , and the separation between the rows by h .

Sec. II, we outline von Karman's stability analysis of which additional details and historical remarks can be found in Lamb [4].

Recently the linear theory for periodic solutions of the von Karman trail was applied to the experimental situation of quasi-periodic flows behind circular cylinders [5]. A comparison between the wave speeds for the kind of disturbances considered in this paper and experimental results for the vortex street behind circular cylinders is reported in [6, 7]. The nonlinear theory for periodic solutions was applied to several experimental situations involving vortex patterns in the wakes of oscillating aerofoils [8].

In another vein linear theory has been applied to two-dimensional arrays of vortices (generalizing the one-dimensional lattice considered here) in connection with quantized vortices that arise in rotating superfluid Helium [9, 10]. The problems treated in this paper are more tractable because of the single dimension involved. Similar but more difficult questions can be formulated for two-dimensional vortex lattices.

II. Normal mode analysis and the initial value problem. The equations governing the vortex trail are given by [4]

$$\frac{2\pi}{\kappa} \frac{d\bar{Z}_m^\pm}{dt} = i \sum_k' \frac{1}{Z_m^\pm - Z_k^\pm} \mp i \sum_\ell \frac{1}{Z_m^\pm - Z_\ell^\mp} \quad (2)$$

where $Z_m^\pm$ denote the location of the m th vortex in the upper and lower rows respectively, κ is the circulation, and the bar denotes complex conjugation. It should be noted that Eq. (2) is invariant under the transformation

$$\kappa \rightarrow -\kappa, \quad t \rightarrow -t. \quad (3)$$

If we denote perturbations from equilibrium, (Fig. 1) moving horizontally at a speed V , by

$$\begin{aligned} z_{mo}^+(t) &= m\ell + i\frac{h}{2} + Vt \\ z_{mo}^-(t) &= \left(m + \frac{1}{2}\right)\ell - i\frac{h}{2} + Vt \end{aligned} \quad (4)$$

$$z_m^\pm = x_m^\pm + iy_m^\pm$$

then, in the harmonic approximation, these are governed by

$$\frac{d}{dt} z_m^\pm = \mp \frac{i\kappa}{2\pi} \sum_k \frac{z_m^\pm - z_k^\pm}{(m-k)^2 \ell^2} \pm \frac{i\kappa}{2\pi} \sum_k \frac{z_m^\pm - z_k^\mp}{[(m-k \mp \frac{1}{2})\ell + ih]^2}. \quad (5)$$

In order to put the spectral problem as well as the general initial value problem within one framework, we introduce the generating functions

$$\begin{aligned} (X^+, Y^+) &= \sum_m (X_m^+, y_m^+) \exp(im\phi) \\ (X^-, Y^-) &= \sum_m (x_m^-, y_m^-) \exp[i(m + \frac{1}{2})\phi], \quad \phi \in [0, 2\pi] \end{aligned} \quad (6)$$

and the transformations

$$\begin{aligned} \alpha &= \frac{(X^+ + X^-)}{2}, \quad \beta = \frac{(Y^+ - Y^-)}{2} \\ \mu &= \frac{(X^+ - X^-)}{2}, \quad \nu = \frac{(Y^+ + Y^-)}{2}. \end{aligned} \quad (7)$$

When these are applied to Eq. (5), we obtain the equation

$$\frac{d}{d\tau} \begin{bmatrix} \alpha & \bar{\nu} \\ \beta & \bar{\mu} \end{bmatrix} = \mathbf{F} \begin{bmatrix} \alpha & \bar{\nu} \\ \beta & \bar{\mu} \end{bmatrix} \quad (8)$$

where

$$\mathbf{F} = \begin{bmatrix} ib & a - c \\ a + c & ib \end{bmatrix}. \quad (9)$$

Time has been normalized by $\tau = \kappa t / (2\pi \ell^2)$ and the elements of the spectral matrix $\mathbf{F}(\phi)$ are listed in the appendix.

The elements of $\mathbf{F}(\phi)$ depend on the value of the aspect ratio (1). Von Karman's classic stability results [1, 2, 3] can thus be stated in the following manner: *the eigenvalues of the spectral matrix $\mathbf{F}(\phi)$ are all pure imaginary for $\pi \in [0, 2\pi]$ if and only if the aspect ratio takes the value*

$$\kappa = \kappa_0 = \frac{1}{\pi} \sinh^{-1} 1 \simeq 0.281. \quad (10)$$

For this case, the eigenvalues are determined by

$$\pm \omega^\pm(\phi) = \pm(b(\phi) \pm K(\phi)) \quad (11)$$

where

$$K(\phi) = \sqrt{c^2 - a^2} \quad (12)$$

(cf. Fig. 2) [5]. Henceforth we consider only the von Karman trail at the ratio, κ_0 .

Solution to Eq. (8) is straightforward, and is given by

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\tau, \phi) = \exp(ib\tau) \begin{bmatrix} \cos K\tau & (a-c) \frac{\sin K\tau}{K} \\ (a+c) \frac{\sin K\tau}{K} & \cos K\tau \end{bmatrix} \begin{bmatrix} \alpha_0(\phi) \\ \beta_0(\phi) \end{bmatrix} \quad (13)$$

FIG. 2. Dispersion Relation as function of wavenumber, ϕ .

where zero subscripts denote initial data. The expression for (ν, μ) is gotten from (13) under the transformation $(\alpha, \beta, b) \rightarrow (\nu, \mu, -b)$. To obtain the solution in terms of the physical variables, (4), we must invert the generating functions, e.g.,

$$\begin{aligned} x_m^+(\tau) &= \frac{1}{2\pi} \int_0^{2\pi} X^+(\tau, \phi) \exp(-im\phi) d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} (\alpha + \mu) \exp(-im\phi) d\phi. \end{aligned} \quad (14)$$

In order to discuss the difference between normal modes and the more general aperiodic solutions, we first describe properties of the generating functions, (6), (7) in each case. Periodic solutions are transformed into delta functions, with support at isolated wavenumbers, $\phi \in [0, 2\pi]$, according to their periodicities. On the other hand, solutions with compact support become smooth functions of $\phi \in [0, 2\pi]$ under the transformations (6), (7). In the first case, the zero normal mode, i.e., the delta function at $\phi = 0$, corresponds to a uniform perturbation of the entire vortex trail. This produces a different equilibrium trail that moves rigidly at a velocity not equal to the original trail. From the form of (11) (cf. Fig. 2) and (9), we find that the spectral matrix $F(\phi)$ has semi-simple zero eigenvalues for $\phi = 0, 2\pi$. The existence of these eigenvalues means that the zero normal mode will grow linearly in time, which agrees with the above observation.

In von Karman's normal mode analysis, this zero mode growth is justifiably treated as a trivial instability and thus disregarded. We remark that the linear growth rate reported in Domm's analysis [11] is the same phenomenon and hence should not be interpreted as instability of the von Karman trail (cf. [6]). The correct nonlinear stability analysis of the periodic problem for Eq. (5) is given by Kochin [12], who in fact demonstrates instability even when (10) holds.

On the other hand, the effects of the semi-simple zero eigenvalues cannot be disregarded when we treat the general initial value problem for (5). By definition, rigid motions of the entire lattice are excluded from the class of solutions with compact support. We wish to underline this important difference between the two cases. It is to the latter class of solutions of the general initial value problem that we direct our attention.

III. Bounded and unbounded solutions. Part 1. To consider the effect of the semi-simple zeros on the general initial value problem, note that at $\phi = 0$, the solution (13) takes the form

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\tau, 0) = \begin{bmatrix} \alpha_0(0) - \pi^2 \tau \beta_0(0) \\ \beta_0(0) \end{bmatrix} \quad (15)$$

where a zero subscript denotes initial values. Equation (15) states that $\alpha(\tau, 0)$ grows linearly with τ provided the invariant $\beta(0) \neq 0$. Similar arguments for the vector $[\nu, \mu]$ show that $\mu(0)$ is the second invariant and if $\mu(0) \neq 0$, then $\nu(\tau, 0)$ grows linearly with τ .

From the definitions of the generating functions (6), (7) we deduce

$$\beta(0) = C_I \equiv \sum_m (y_m^+ - y_m^-), \quad \mu(0) = C_R \equiv \sum_m (x_m^+ - x_m^-). \quad (16)$$

These formulas give the physical meanings to the above invariants, i.e., they represent the sum of the changes in the vertical and horizontal separations between vortex rows at each *cell* or pair of opposite vortices. Within the class of solutions that concerns us here, the invariants C_R and C_I are necessarily finite.

Similarly, from the definitions (6), (7) we obtain the following formulas

$$\alpha(\tau, 0) = \sum_m (x_m^+ + x_m^-), \quad \nu(\tau, 0) = \sum_m (y_m^+ + y_m^-). \quad (17)$$

They represent the sum of the changes in the averaged position of each pair of corresponding vortices which we term a cell. From (16) and (17), we derive the following physical interpretations for the linear growth of $\alpha(\tau, 0)$ and $\nu(\tau, 0)$:

An initial (localized and finite) increment or decrement in the **vertical** distance between the vortex rows is a necessary and sufficient cause for **horizontal** motion of vortex pairs (cell).

Similarly, an initial **horizontal** shift of some vortices in the upper row relative to the lower row (that adds up to a nonzero sum) is a necessary and sufficient cause for **vertical** motion of vortex pairs.

In order to discuss the action of (13) at wavenumbers beside $\phi = 0, 2\pi$, it is convenient to introduce the ℓ_2 -norm,

$$\|z\|^2 \equiv \sum_{m=-\infty}^{\infty} (|x_m^+|^2 + |x_m^-|^2 + |y_m^+|^2 + |y_m^-|^2). \quad (18)$$

Solutions of Eq. (5) which decay spatially along the lattice and evolve from initial data that have compact support can be treated as elements of the Hilbert space ℓ_2

with the above norm. We note that normal modes (periodic solutions) are excluded in this analysis.

The generating functions, (6), are discrete Fourier Transforms of the physical variables, (4). They can be regarded as elements of the space $L_2[0, 2\pi]$ with the norm

$$\|Z\|^2 \equiv \int_0^{2\pi} (|X^+|^2 + |X^-|^2 + |Y^+|^2 + |Y^-|^2) d\phi. \quad (19)$$

It follows that

$$\|Z\| = \|z\|. \quad (20)$$

The L_2 -norm, $\|Z\|$, is directly related to the quantities

$$\begin{aligned} \|(\alpha, \beta)\|^2 &\equiv \int_0^{2\pi} (|\alpha|^2 + |\beta|^2) d\phi \\ \|(\nu, \mu)\|^2 &\equiv \int_0^{2\pi} (|\nu|^2 + |\mu|^2) d\phi \end{aligned} \quad (21)$$

through (7). Since solutions of Eq. (5) are conveniently expressed in terms of the generating functions, (7), we compute the evolution of the $\mathcal{L}_2$ -norm, (18) in terms of (21).

First, we compute lower and upper bounds for $\|(\alpha, \beta)\|$. Our earlier calculations show that growth of the functions $\alpha(\tau, \phi)$ and $\nu(\tau, \phi)$ occur only at $\phi = 0, 2\pi$. This suggests the following partition

$$\|(\alpha, \beta)\|^2(\tau) = \int_0^\delta (|\alpha|^2 + |\beta|^2) d\phi + \int_\delta^{2\pi-\delta} (|\alpha|^2 + |\beta|^2) d\phi + \int_{2\pi-\delta}^{2\pi} (|\alpha|^2 + |\beta|^2) d\phi \quad (22)$$

where δ is some sufficiently small positive number to be determined later. Thus we obtain

$$\int_0^\delta (|\alpha|^2 + |\beta|^2) d\phi \leq \|(\alpha, \beta)\|^2. \quad (23)$$

From the solution operator, (13) we find that the integrand in (23) evolves as follows

$$\begin{aligned} (|\alpha|^2 + |\beta|^2)(\tau) &= \cos^2 K\tau (|\alpha_0|^2 + |\beta_0|^2) \\ &\quad + \frac{\sin^2 K\tau}{K^2} [(a-c)^2 \beta_0^2 + (a+c)^2 \alpha_0^2] + \frac{2 \sin 2K\tau}{K} a\alpha_0\beta_0. \end{aligned} \quad (24)$$

This immediately implies the bounds

$$\begin{aligned} &\frac{\sin^2 K\tau}{K^2} (a-c)^2 \beta_0^2 - 4|a| |\alpha_0| |\beta_0| \tau \\ &\leq (|\alpha|^2 + |\beta|^2)(\tau) \\ &\leq (|\alpha_0|^2 + |\beta_0|^2) + [(a+c)^2 \alpha_0^2 + (a-c)^2 \beta_0^2] \tau^2 + 4|a| |\alpha_0| |\beta_0| \tau. \end{aligned} \quad (25)$$

When we substitute this into the integral, we find

$$\begin{aligned}
 & \tau^2 \int_0^\delta \left(\frac{\sin K\tau}{K\tau} \right)^2 (a-c)^2 \beta_0^2 d\phi - 4\tau \int_0^\delta |\alpha_0 \beta_0 a| d\phi \\
 & \leq \|(\alpha, \beta)\|^2 \\
 & \leq \int_0^{2\pi} (|\alpha_0|^2 + |\beta_0|^2) d\phi + \tau^2 \int_0^{2\pi} [(a+c)^2 \alpha_0^2 + (a-c)^2 \beta_0^2] d\phi \\
 & \quad + 4\tau \int_0^{2\pi} |a\alpha_0 \beta_0| d\phi.
 \end{aligned} \tag{26}$$

In the appendix, we estimate the integrals in the lower and upper bounds in (26). As a result of these calculations, we obtain the bounds

$$\begin{aligned}
 & \frac{\pi^4}{72} \beta_0(0)^2 \tau D(\tau) - 4D(\tau) \pi^2 |\alpha_0(0)| |\beta_0(0)| \leq \|(\alpha, \beta)\|(\tau)^2 \\
 & \|(\alpha, \beta)\|^2 \leq (1 + 2M\tau + M\tau^2) \int_0^{2\pi} (|\alpha_0|^2 + |\beta_0|^2) d\phi
 \end{aligned} \tag{27}$$

where

$$M = \max_{\phi} \{(|a| + |c|)^2\}. \tag{28}$$

The quantity $D(\tau)$, computed in the appendix, is linear in τ for small τ but takes the constant value $\frac{\pi}{4}$ for large τ . Therefore, the lower bound in (27) becomes unbounded with time if $\beta_0(\phi = 0) \neq 0$ which proves that the vanishing of $\beta(\phi)$ at $\phi = 0$ is a necessary condition for boundedness of $\|(\alpha, \beta)\|$. That it is also sufficient is established in the appendix.

Similar arguments for $\|(\nu, \mu)\|^2(\tau)$ prove that $\mu(\phi = 0) = 0$ is a necessary and sufficient condition for boundedness of $\|(\nu, \mu)\|$. Putting these together with (20) and using the formulas (16), we derive one of the main results: $C_R = C_I = 0$ are necessary and sufficient conditions for the solutions of (5) to be bounded in ℓ_2 .

Furthermore, the upper bound in (27) and a similar one for (ν, μ) imply the well-posedness of the initial value problem (5) in the space ℓ_2 .

The unbounded solutions of (5) for the von Karman trail at special aspect ratio suggests that the equilibrium is unstable. It is easy to verify that no matter how small the initial ℓ_2 -norm, $\|z(0)\|$, we can pick initial data such that $C_R \neq 0$ or $C_I \neq 0$ which lead to unbounded solutions in ℓ_2 . Therefore, the von Karman trail at the special aspect ratio is *linearly Lagrange-unstable* with respect to the ℓ_2 -norm. This is a comparatively weak notion of instability in the sense that the growth rate is algebraic and not exponential.

IV. Bounded and unbounded solutions, Part II. We now know that initial data which violates the conditions

$$C_R = C_I = 0 \tag{29}$$

leads to unbounded solutions in ℓ_2 . *A priori*, it is not obvious that such solutions are also unbounded in the space ℓ_∞ with the norm

$$\|z\|_\infty^2 \equiv \max_m (|x_m^+|^2 + |x_m^-|^2 + |y_m^+|^2 + |y_m^-|^2). \tag{30}$$

From the previous section, we know that the generating functions grow only at wavenumber $\phi = 0$ (and 2π by symmetry). This fact combined with the nonlinearity of the dispersion relation, (11) (cf. Fig. 2) suggest that the growing disturbance travel at two well-defined group speeds, associated with the slopes of the dispersion curves at $\phi = 0$. The growing component of the disturbance is not dispersing; we therefore expect that such disturbances increase pointwise in the asymptotic limit, $\tau \rightarrow \infty$.

Now we establish the above claim that growing disturbances in ℓ_2 are also unbounded in ℓ_∞ . The solutions (13) of the spectral problem (8) are inverted to give the physical solution (14). We recall from the appendix that $K = \sqrt{c^2 - a^2} \sim O(\phi)$ and $(a+c) \sim O(\phi)$ while $(a-c)|_{\phi=0} = \pi^2$. From this and the properties of the spectral solutions (13), it is clear that the most significant term of

$$x_m^+(\tau) = \frac{1}{2\pi} \int_0^{2\pi} (\alpha + \mu) \exp(-im\phi) d\phi$$

is

$$I = \int_0^{2\pi} \beta_0(\phi) \frac{(a-c)}{K} \sin K\tau \exp(ib\tau - im\phi) d\phi. \quad (31)$$

Subscript zero denotes initial values as usual.

We consider the cases $\beta_0(\phi = 0) = 0$ and $\beta_0(\phi = 0) \neq 0$ separately. In the first case, a direct application of Kelvin's method [13] yields waves that decay as $\tau \rightarrow \infty$. In the latter case, the largest contribution to the integral, (31) can be expected to come from the neighborhood of the origin where the integral grows linearly with τ .

If the stationary point ϕ_0 is not the origin, the standard procedure yields waves travelling at the group speeds $[d\omega^\pm(\phi)/d\phi](\phi_0)$ and decaying as $O(1/\sqrt{\tau})$. For stationary point, $\phi_0 = 0$, the integral I can be effectively replaced by

$$\begin{aligned} J &= 2i \int_0^{2\pi} \frac{\sin K\tau}{\phi} \exp(ib\tau - im\phi) d\phi \\ &= \int_0^{2\pi} \left[\frac{\exp(i\omega^+\tau) - \exp(i\omega^-\tau)}{\phi} \right] \exp(-im\phi) d\phi \end{aligned} \quad (32)$$

where $\omega^\pm(\phi)$ are given by the dispersion relations, (11). Each of the exponential integrals of (32) is divergent and the standard asymptotic method does not apply. To treat this case, we consider

$$J_\varepsilon = \int_0^{2\pi} d\phi \frac{\exp(i\omega^+\tau - im\phi)}{\phi^{1-\varepsilon}} - \int_0^{2\pi} d\phi \frac{\exp(i\omega^-\tau - im\phi)}{\phi^{1-\varepsilon}} \quad (33)$$

under the limit $\varepsilon \rightarrow 0$. By expanding $\omega^\pm$ in powers of ϕ we can write each of the above integrals in the following way

$$J_\varepsilon^\pm = \int_0^{2\pi} d\phi \frac{\exp(-im\phi)}{\phi^{1-\varepsilon}} \exp[i(b_1\phi + b_2\phi^2)\tau \pm i(K_1\phi + K_2\phi^2)\tau] \quad (34)$$

where

$$\begin{aligned} b_1 &= \frac{\pi}{\sqrt{2}} + \frac{h\pi^2}{2\ell}, & b_2 &= \pi \frac{h}{\ell}, \\ K_1 &= \left[\pi^2 \left(\frac{\pi^2 h^2}{4\ell^2} + \frac{\pi h}{\ell\sqrt{2}} - \frac{1}{2} \right) \right]^{1/2}, \\ K_2 &= - \left(\frac{\pi^2 h^2 / 2\ell^2 + \pi h / (\ell\sqrt{2}) - \frac{1}{2}}{\pi^2 h^2 / 4\ell^2 + \pi h / (\ell\sqrt{2}) - \frac{1}{2}} \right) \frac{K_1}{\pi}. \end{aligned} \quad (35)$$

The stationary point at the origin corresponds to two values of the group velocity namely,

$$b_1 \pm K_1 = \frac{m}{\tau}. \quad (36)$$

Since the analysis for both cases are similar, we only discuss one of them, viz. $b_1 + K_1 = m/\tau$. Application of a standard procedure [13] then yields

$$\begin{aligned} J_\varepsilon^+ &\approx \int_0 \frac{d\phi}{\phi^{1-\varepsilon}} \exp[i\tau(b_2 + K_2)\phi^2] \approx \frac{\exp(i\pi\varepsilon/4)}{\varepsilon} [\tau(b_2 + K_2)]^{-\varepsilon/2} \\ J_\varepsilon^- &\approx \int_0 \frac{d\phi}{\phi^{1-\varepsilon}} \exp[-2i\tau K_1\phi] \approx \frac{\exp(i\pi\varepsilon/2)}{\varepsilon} (2K_1\tau)^{-\varepsilon}. \end{aligned} \quad (37)$$

In the limit $\varepsilon \downarrow 0$ we obtain

$$J = \lim_{\varepsilon \rightarrow 0} (J_\varepsilon^+ - J_\varepsilon^-) \approx \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left\{ \exp \left[- \left(\frac{\varepsilon}{2} \right) (\ln \tau + \ln C_1) \right] - \exp[-\varepsilon(\ln \tau + \ln(2))] \right\}, \quad (38)$$

where the constants C_1, C_2 are derived from the coefficients in (37). Expansion of the exponential yields.

$$I = O(\ln \tau). \quad (39)$$

Thus we have established the results:

- (i) condition (29) is necessary and sufficient for solutions of (5) to be bounded in ℓ_∞ ;
- (ii) if condition (29) holds, the ℓ_∞ -norm (30) of the solution decays as $\tau \rightarrow \infty$;
- (iii) on the other hand, if condition (29) is violated, the ℓ_∞ -norm grows as $O(\ln \tau)$.

The pair of group velocities given by (36) was compared with Tritton's measurements [14] of wave speeds in the vortex street behind a circular cylinder and found to agree with experimental error bounds [6, 7]. Next, we interpret the above result on unbounded solutions as an instability of the von Karman vortex trail at special aspect ratio. No matter how small the initial bound on $\|z\|_\infty$, we can choose data such that condition (29) fails. Thus the von Karman trail is *linearly Lagrange-unstable* with respect to the ℓ_∞ -norm. Again, this is a rather weak notion of instability in view of the above observation that the growth rate is **logarithmic** in time.

V. Focussing. We turn our attention now to the class of bounded solutions characterized by the condition (29). We recall from Sec. IV that the $\|z\|_\infty$ of bounded solutions decay as $\tau \rightarrow \infty$. Combined with the invariance of Eqs.(2) and also

(5) under the transformation (3), this dispersive decay implies the following result:

Suppose that an arbitrarily large time T and a positive number B are given, then no matter how small we pick $\varepsilon > 0$, there exists initial data for (5) such that

$$\|z^*(0)\|_\infty < \varepsilon \quad \text{but} \quad \|z^*(\tau)\|_\infty > B \quad \text{for some } \tau > T.$$

We establish this result by constructing an example of such initial data. First, two adjacent vortices in the upper row (say) are perturbed in the vertical direction by equal but opposite amounts of magnitude B ; this disturbance is denoted by (**),

$$y_0^+(\tau=0) = B, \quad y_1^+(\tau=0) = -B, \quad \text{all other vortices are unperturbed.} \quad (**)$$

This data clearly satisfies the condition (29). After a sufficiently long time, in particular $\tau > T$, the disturbance (**) has dispersed and its norm $\|z^{**}(\tau)\|_\infty < \varepsilon$. An immediate corollary of invariance under the transformation (3) implies that if at this time τ , we stop the clock and reverse the signs of circulation, i.e., $\kappa \rightarrow -\kappa$, then the final state reached at time 2τ is exactly the original state (**). Thus by picking the initial state $z^*(0) = z^{**}(\tau)$ we have an example of the type required to establish the above result.

In statistical mechanics, this mathematical construction has been named a *Loschmidt demon* in analogy with Maxwell's demon but has different powers. While Maxwell's demon acts at the microscopic level, i.e., on individual particles, to bring the system to a particular state, Loschmidt's demon, by *macroscopic actions* only, returns the system to a previous dynamical state. For our example, the macroscopic action consists of reversing the sign of the circulation, κ . The fact that Loschmidt's demon can be realized in practice has been demonstrated in experiments on the dipole coupled nuclear spins in solids [15], [16]. The **well-posedness** of the initial value problem (5) in ℓ_2 (cf. Sec. III) now implies that a small open neighborhood of the initial data constructed above, e.g., $z^*(0)$, is mapped into a neighborhood of the waveform (**) at time τ . This means that the example constructed above is not an isolated case but is robust under small perturbations.

In conclusion, we note that although each bounded solution of the Eqs. (5) decay as $\tau \rightarrow \infty$, dispersion can focus an arbitrarily small waveform into a relatively large signal at an arbitrarily large but finite time τ before the asymptotic decay sets in.

Appendix.

I. The elements of matrix $F(\phi)$:

$$\begin{aligned} a &= [\phi(2\pi - \phi) - \pi^2]/2 \\ b &= \left[\sqrt{2}\pi\phi \sinh \frac{h}{\ell}(\pi - \phi) + \pi^2 \sinh \frac{h}{\ell}\phi \right] / 2 \\ c &= \left[\pi^2 \cosh \frac{h}{\ell}\phi - \pi\phi\sqrt{2} \cosh \frac{h}{\ell}(\pi - \phi) \right] / 2. \end{aligned} \quad (\text{A1})$$

II. Upper and Lower Bounds for $\|(\alpha, \beta)\|_{L_2}$. We estimate the integrals in the bounds (26) to obtain expressions that contain the invariant $\beta_0(0)$ explicitly. The integrals in the lower bound in (26) are evaluated on a small interval of $\phi = 0$, i.e.,

$\delta \ll 1$. Thus estimates for the integrands near $\phi = 0$ are required. First we want to estimate the quantity $(\sin K\tau/K\tau)$ in the first integral. We use the fact

$$\left| \frac{\sin x}{x} \right| > \frac{1}{3} \quad \text{for } |x| \leq \frac{\pi}{2} \quad (\text{A2})$$

and given $T > 0$, choose $\delta(T) \ll 1$ so that

$$K(\phi)\tau \leq \frac{\pi}{2} \quad \text{for } \phi \leq \delta(T), \tau \leq T. \quad (\text{A3})$$

From (A1) it is easily seen that $K(\phi)$ is an increasing function for ϕ sufficiently small and, furthermore, $K(\phi) \sim O(\phi)$. Thus by choosing $\delta(T)$ to satisfy this criterion for all $\phi \leq \delta(T)$ and also

$$\delta(T)T \leq \frac{\pi}{4} \quad (\text{A4})$$

it is clear that we have met the condition (A3). Since it follows from (A1) that

$$\begin{aligned} a(\phi = 0) &= \frac{-\pi^2}{2} \\ c(\phi = 0) &= \frac{\pi^2}{2} \end{aligned} \quad (\text{A5})$$

we also want $\delta(T)$ to be small enough so that

$$\begin{aligned} (a - c)^2(\phi) &\geq \frac{\pi^4}{2} \\ |\beta_0(\phi)| &\geq \frac{|\beta_0(0)|}{2} \\ |\alpha_0(\phi)| &\geq \frac{|\alpha_0(0)|}{2} \end{aligned} \quad (\text{A6})$$

for $\phi \leq \delta(T)$. Thus we obtain the lower bound

$$\|(\alpha, \beta)\|^2(\tau) \geq \frac{\pi^4}{72} \beta_0^2(0) \tau D(\tau) - 4\pi^2 |\alpha_0(0)| |\beta_0(0)| D(\tau) \quad (\text{A7})$$

where

$$D(\tau) \equiv \tau \delta(\tau). \quad (\text{A8})$$

For $T \ll 1$, $\delta(\tau)$ is some sufficiently small constant but, for τ large enough, we choose $\delta(\tau)$ to meet the equality in (A4). Therefore $D(\tau)$ is linear in τ for $\tau \ll 1$ but achieves the constant value of $\frac{\pi}{4}$ for τ large enough.

To obtain the upper bound, it is sufficient to note that the second integral in the upper bound in (26) can be estimated in following way

$$\begin{aligned} \tau^2 \int_0^{2\pi} [(a+c)^2 \alpha_0^2 + (a-c)^2 \beta_0^2] d\phi &\leq \tau^2 \int_0^{2\pi} (|a| + |c|)^2 (\alpha_0^2 + \beta_0^2) d\phi \\ &\leq \tau^2 \max_{\phi} \{(|a| + |c|)^2\} \int_0^{2\pi} (\alpha_0^2 + \beta_0^2) d\phi \end{aligned} \quad (\text{A9})$$

while the third integral

$$\begin{aligned} 4\tau \int_0^{2\pi} |a\alpha_0\beta_0| d\phi &\leq 2\tau \int_0^{2\pi} a(\alpha_0^2 + \beta_0^2) d\phi \\ &\leq 2\tau \max_{\phi} \{(|a| + |c|)^2\} \int_0^{2\pi} (\alpha_0^2 + \beta_0^2) d\phi. \end{aligned} \quad (\text{A10})$$

Putting the three integrals in the upper bound together we obtain

$$\|(\alpha, \beta)\|^2 \leq (1 + 2M\tau + M\tau^2) \int_0^{2\pi} (|\alpha_0|^2 + |\beta_0|^2) d\phi. \quad (\text{A11})$$

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