UNIFORM L^1 BEHAVIOR IN A CLASS OF LINEAR VOLTERRA EQUATIONS*

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Abstract. We find sufficient conditions for the solution of the equation $u'(t) + \int_0^t \sum_{i=1}^n \lambda_i a_i(t-s)u(s) ds = 0$, $u(0) = 1$, to satisfy $\int_0^\infty \sup_{\lambda_1, \dots, \lambda_n \geq 1} (\lambda_1 + \dots + \lambda_n)^{-1/2} \times u'(t, \lambda_1, \dots, \lambda_n) dt < \infty$. Our results generalize the case $n = 1$. Applications to a related equation in Hilbert space are given.

1. Introduction. We study the solution $u = u(t) = u(t, \lambda_1, \dots, \lambda_n)$ of the equation

$$\begin{aligned} u'(t) + \int_0^t (\lambda_1 a_1(\tau) + \dots + \lambda_n a_n(\tau)) u(t - \tau) d\tau, \\ u(0) = 1, \quad t > 0, \quad \lambda_i \geq 1, \quad i = 1, \dots, n \quad \left(' = \frac{d}{dt} \right). \end{aligned} \quad (1.1)$$

The functions a_i , $i = 1, \dots, n$, satisfy

$$\int_0^1 a_i(t) dt < \infty, \quad a_i \text{ is nonconstant, nonnegative, nonincreasing,} \\ \text{and convex on } (0, \infty), \quad i = 1, \dots, n. \quad (1.2)$$

We study the question: When is $u'(t)$ integrable uniformly with respect to the parameters $\lambda_1, \dots, \lambda_n$? In particular when is it true that the function

$$v(t, \lambda_1, \dots, \lambda_n) = u'(t, \lambda_1, \dots, \lambda_n) (\lambda_1 + \dots + \lambda_n)^{-1/2} \quad (1.3)$$

satisfies

$$\begin{aligned} \text{(i)} \quad & \int_0^\infty \sup_{\lambda_1, \dots, \lambda_n \geq 1} v(t, \lambda_1, \dots, \lambda_n) dt < \infty, \\ \text{(ii)} \quad & \lim_{t \rightarrow \infty} \sup_{\lambda_1, \dots, \lambda_n \geq 1} v(t, \lambda_1, \dots, \lambda_n) = 0. \end{aligned} \quad (1.4)$$

For the example $a_i(t) = e^{-t}$, $i = 1, \dots, n$, (1.1) reduces to

$$u''(t) + u'(t) + (\lambda_1 + \dots + \lambda_n)u(t) = 0, \quad u(0) = 1, \quad u'(0) = 0,$$

which has the solution

$$u(t, \lambda_1, \dots, \lambda_n) = e^{-t/2} \left(\cos \mu t + \frac{1}{2\mu} \sin \mu t \right),$$

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where $\mu = (\lambda_1 + \cdots + \lambda_n - \frac{1}{4})^{1/2}$. This shows the need to scale $u'(t)$ as in (1.3) before taking the supremum in order to obtain a finite valued function of t .

For certain operator kernels of the form

$$L(t) = \sum_{k=1}^n a_k(t) L_k, \quad (1.5)$$

the results in (1.4) give information on the derivative $V(t) = \frac{d}{dt} U(t)$ of the resolvent

$$U(t) = \int_0^\infty u(t; \lambda) dE_\lambda \quad (1.6)$$

for the problem

$$y'(t) + \int_0^t L(t-s)y(s) ds = f(t), \quad y(0) = y_0, \quad (1.7)$$

in a Hilbert space H . Here $\{E_\lambda\}$ is a fixed resolution of the identity in H . The requirement that the L_k have spectral decompositions with respect to a common resolution of the identity $\{E_\lambda\}$ greatly restricts the applicability of the results for (1.7), but see [7] for applications. For example, results on the asymptotic behavior of solutions to the following problems may be obtained.

$$w_t(x, y, t) = \int_0^t [a_0(t-s)w_{xx}(x, y, s) + b_0(t-s)w_{yy}(x, y, s)] ds + f(x, y, t),$$

$$w(x, y, 0) = w_0(x, y), \quad \alpha < x < \beta, \quad \alpha < y < \delta, \quad t > 0,$$

with boundary conditions

$$w(\alpha, y, t) = w(\beta, y, t) = w(x, \alpha, t) = w(x, \delta, t) = 0.$$

This problem arises in a linear model for heat flow in a rectangular, orthotropic material with memory [3], [8] in which the axes of orthotropy are parallel to the edges of the rectangle. Another equation where our results apply is the equation

$$w_t(x, t) = \int_0^t [a_0(t-s)w_{xx}(x, s) - a_1(t, s)w_{xxx}(x, s)] ds + f(x, t),$$

with selfadjoint boundary conditions. See [7] for details.

Included in [11] is the result that

$$\int_0^\infty \sup_{\lambda_1, \dots, \lambda_n \geq 1} |u(t, \lambda_1, \dots, \lambda_n)| dt < \infty \quad (1.8)$$

holds for functions a_i satisfying (1.2),

$$-a'_i(t) \text{ is convex}, \quad i = 1, \dots, n, \quad (1.9)$$

the condition

$$\int_1^\infty \frac{du}{uA(t)} dt < \infty, \quad (1.10)$$

where $A(t) = \int_0^t a_1(s) + \cdots + a_n(s) ds$, a condition stated in terms of the Fourier transform of the functions a_i , $i = 1, \dots, n$, and a mild technical condition. In [7], (1.8) is

shown for functions a_i that are completely monotonic on $(0, \infty)$, $((-1)^j a_i^{(j)}(t) \geq 0, j = 0, 1, 2, 3, \dots)$ and satisfy conditions that are similar to (1.10) and the transform condition used in [11]. By contrast in Theorems 1.2 and 1.3 and in the corollary, no condition like (1.10) is used to prove (1.4). Condition (1.8) implies that

$$\int_0^\infty \|U(t)\| dt < \infty. \quad (1.11)$$

Our results (1.4) (i) and (ii) imply, respectively,

$$\begin{aligned} \text{(i)} \quad & \int_0^\infty \|V(t)(L_1 + \dots + L_n)^{-1/2}\| dt < \infty \\ \text{(ii)} \quad & V(t)(L_1 + \dots + L_n)^{-1/2} \rightarrow 0, \quad \text{as } t \rightarrow \infty. \end{aligned} \quad (1.12)$$

The function V also appears in the alternate resolvent formula

$$y(t) = F(t) + \int_0^t V(t-s)F(s) ds$$

for the integrated version of (1.7), that is,

$$y(t) + \int_0^t \sum_{k=1}^n A_{a_k}(t-s)L_k y(s) ds = F(t),$$

where $A_{a_k}(t) = \int_0^t a_k(s) ds$, $F(t) = \int_0^t f(s) ds + y_0$.

Assuming (1.2), the Fourier transform

$$\hat{a}_k(\tau) = \phi_k(\tau) - i\tau\theta_k(\tau) = \int_0^\infty e^{-i\tau t} a_k(t) dt$$

of a_k is continuous, and $\phi_k(\tau)$ and $\theta_k(\tau)$ are nonnegative for $\tau > 0$, $k = 1, \dots, n$ (see [4]). Formally, the Fourier transform of $(\lambda_1 + \dots + \lambda_n)^{1/2}v$ is

$$(\lambda_1 + \dots + \lambda_n)^{1/2}\hat{v}(\tau, \lambda_1, \dots, \lambda_n) = \frac{-(\lambda_1\hat{a}_1(\tau) + \dots + \lambda_n\hat{a}_n(\tau))}{\lambda_1\hat{a}_1(\tau) + \dots + \lambda_n\hat{a}_n(\tau) + i\tau}, \quad (1.13)$$

so $v(\cdot, \lambda_1, \dots, \lambda_n) \notin L^1(R^+)$ if the denominator equals zero for some τ . By [4], $\phi_i(\tau) \geq 0$; moreover, $\phi_i(\tau) > 0$ ($\tau > 0$) unless $a_i(t)$ is piecewise linear with changes of slope only at integral multiples of a fixed number t_0 (taken as large as possible) and τ is an integral multiple of $2\pi/t_0$ ($i = 1, \dots, n$). If $\phi_i(\tau) > 0$ for some i , then the denominator of (1.12) is nonzero, the hypotheses of [12, Theorem 2] hold, $v(\cdot, \lambda_1, \dots, \lambda_n) \in L^1(R^+)$ and (1.13) holds. Throughout this paper, we will restrict ourselves to this case by assuming that for some j ,

$$\phi_j(\tau) > 0, \quad \tau > 0. \quad (1.14)$$

As in [1], the equation

$$\lambda_1\theta_1(\omega) + \dots + \lambda_n\theta_n(\omega) = 1 \quad (1.15)$$

defines a continuous function $\omega = \omega(\lambda_1, \dots, \lambda_n)$ for $\lambda_1, \dots, \lambda_n \geq \lambda_0$ (some $\lambda_0 > 0$), that is strictly increasing as a function of each $\lambda_i \geq \lambda_0$, $i = 1, \dots, n$. We extend ω to $[1, \infty) \times \dots \times [1, \infty)$ if necessary by defining $\omega(\lambda_1, \dots, \lambda_n) = \omega(\lambda_0, \dots, \lambda_0) \equiv \rho$ ($1 \leq \lambda_i \leq \lambda_0$, $i = 1, \dots, n$).

For $\lambda_i \geq \lambda_0$, $i = 1, \dots, n$, we have

$$\begin{aligned} \int_0^\infty |v(t, \lambda_1, \dots, \lambda_n)| dt &\geq |\hat{v}(\omega, \lambda_1, \dots, \lambda_n)| \\ &= (\lambda_1 + \dots + \lambda_n)^{-1/2} \left| -1 + \frac{i\omega}{\lambda_1 \phi_1(\omega) + \dots + \lambda_n \phi_n(\omega)} \right| \\ &\geq \frac{\omega}{(\lambda_1 + \dots + \lambda_n)^{1/2} (\lambda_1 \phi_1(\omega) + \dots + \lambda_n \phi_n(\omega))}. \end{aligned}$$

This proves our first result.

THEOREM 1.1. Suppose that (1.2) and (1.14) hold. If (1.4)(i) holds, then

$$\sup_{\lambda_1, \dots, \lambda_n \geq 1} \frac{\omega}{(\lambda_1 + \dots + \lambda_n)^{1/2} (\lambda_1 \phi_1(\omega) + \dots + \lambda_n \phi_n(\omega))} < \infty. \quad (1.16)$$

Our next theorem uses a condition which is only slightly stronger than (1.16) to conclude that (1.4) (i) holds. For the proof, we employ the mild technical condition that

$$\int_0^\infty t \sum_i a_i(t) dt > 5. \quad (1.17)$$

THEOREM 1.2. (i) Suppose that the following conditions hold: (1.2), (1.14), (1.17), and

$$\sup_{\lambda_1, \dots, \lambda_n \geq 1} \frac{\omega}{(\lambda_1 + \dots + \lambda_n)^{1/2-\delta} (\lambda_1 \phi_1(\omega^*) + \dots + \lambda_n \phi_n(\omega^*))} < \infty, \quad (1.18)$$

for some $\delta > 0$, where $\omega^* = \min_{\omega/2 \leq \tau \leq 2\omega} (\lambda_1 \phi_1(\tau) + \dots + \lambda_n \phi_n(\tau))$. Then (1.4)(i) holds.

(ii) Suppose that (1.2), (1.14), and (1.16) are satisfied. Then (1.4)(ii) follows.

Sometimes the criteria in the next theorem are weaker.

THEOREM 1.3. If (1.2), (1.14), (1.17), and

$$\sup_{\lambda_1, \dots, \lambda_n \geq 1} \frac{\omega^2}{(\lambda_1 + \dots + \lambda_n) (\lambda_1 \phi_1(\omega^*) + \dots + \lambda_n \phi_n(\omega^*))} < \infty \quad (1.19)$$

are satisfied, then (1.4)(i) holds.

COROLLARY. If each a_i , $i = 1, \dots, n$, satisfies (1.2), (1.17), and (1.9), then (1.4)(i) and (ii) hold.

For results on the problem (1.1) $n = 1$, with applications, see [1,2,5,6,9,10]. We finish this section with the conjecture that (1.10) is not needed to conclude that (1.8) holds.

Conjecture. If each a_i , $i = 1, \dots, n$ satisfies (1.2) and (1.9) then (1.8) holds.

2. Proofs. Most of the proofs go through just as in [2], with obvious changes made because the current problem has n parameters instead of one. We will therefore only give outlines of our proofs and give full details where our proof differs from [2].

The proof of Theorem 1.2 (ii) is obtained by using the representation (see [2])

$$\pi u' = \operatorname{Re} \left\{ \frac{1}{t} \int_0^\infty e^{i\tau t} \left[\frac{\tau \sum_i \lambda_i \hat{a}'_i(\tau) - \sum_i \lambda_i \hat{a}_i(\tau)}{D^2} \right] \right\}, \quad t > 0, \quad (2.1)$$

where $D = D(\tau, \lambda_1, \dots, \lambda_n) = \sum_{i=1}^n \lambda_i \hat{a}_i(\tau) + i\tau$, and estimating the integrand to obtain (1.4) (ii).

Theorems 1.2 (i) and 1.3 rely on estimates on u' obtained by using (2.1) as well as an estimate on u' obtained directly from (1.1). These estimates are then combined to obtain (1.4) (i).

Now we will outline the proof of Theorem 1.2 (i). Following [2] we need to introduce two auxiliary functions. Define $\sigma = \sigma(\lambda_1, \dots, \lambda_n)$ by the formula $\sigma = \sum_{i=1}^n \lambda_i A_{a_i}(1/\sigma)$, and define $\omega = \omega(\lambda_1, \dots, \lambda_n)$ as follows. The equation $\sum_{i=1}^n \lambda_i \theta_i(\omega) = 1$ defines a C^1 function $\tilde{\omega} = \tilde{\omega}(\lambda_1, \dots, \lambda_n)$ on a subdomain $[\lambda_0^1, \infty) \times \dots \times [\lambda_0^n, \infty)$ of $(0, \infty) \times \dots \times (0, \infty)$. Fix $t_1 > 0$, such that $\min\{a_i(t_1), i = 1, \dots, n\} > 0$ and let $\rho = 6/t_1$. Set $\omega(\lambda_1, \dots, \lambda_n) = \max\{\rho, \tilde{\omega}(\lambda_1, \dots, \lambda_n)\}$ if $\tilde{\omega}$ is defined, $\omega(\lambda_1, \dots, \lambda_n) = \rho$ otherwise. We have inequalities for ω and σ similar to those in [2]:

$$|u'(t, \lambda_1, \dots, \lambda_n)| \leq 8\sigma, \quad \lambda_i \geq 1, \quad i = 1, \dots, n, \quad (2.2)$$

$$|u'(t, \lambda_1, \dots, \lambda_n)| \geq \frac{\sigma}{64}, \quad \frac{1}{32\sigma} \leq t \leq \frac{1}{16\sigma}, \quad (2.3)$$

and

$$\frac{1}{5} \sum_{i=1}^n \lambda_i A_{1a_i} \left(\frac{1}{\omega} \right) \leq \sum_{i=1}^n \lambda_i \theta_i(\omega) \leq 12 \sum_{i=1}^n \lambda_i A_{1a_i} \left(\frac{1}{\omega} \right), \quad (2.4)$$

where $A_{1a_i}(x) = \int_0^x r a_i(r) dr$. The proofs of these are exactly like those in [2] with obvious modifications.

These functions are used to estimate the integrand in the representation (2.1), as was done in [2, pp. 470, 471] to obtain

$$|u'(t, \lambda_1, \dots, \lambda_n)| \leq \frac{M}{t} \left[1 + \frac{\omega}{\sum_i \lambda_i \phi_i(\omega^*)} \right], \quad t < \frac{1}{\varepsilon} \quad (2.5)$$

where ε is a constant (see (2.7) below) and $\omega^* = \omega^*(\lambda_1, \dots, \lambda_n)$ is defined so that

$$\frac{\omega}{2} \leq \tau \leq 2\omega \quad \text{and} \quad \sum_{i=1}^n \lambda_i \phi_i(\omega^*) = \min_{\omega/2 \leq \tau \leq 2\omega} \sum_{i=1}^n \lambda_i \phi_i(\tau) :$$

for instance, ω^* could be the smallest such number.

Estimates (2.2) and (2.5) are then combined, using (1.18), as in [2, p. 469] to obtain

$$|v(t, \lambda_1, \dots, \lambda_n)| \leq \frac{M}{t^p}, \quad t < \frac{1}{\varepsilon}, \quad (2.6)$$

where M is a constant and $0 < p < 1$. This shows that

$$\int_0^{1/\varepsilon} \sup_{\lambda_1, \dots, \lambda_n \geq 1} |v(t, \lambda_1, \dots, \lambda_n)| dt < \infty.$$

To estimate v for $t \geq 1/\varepsilon$, we employ the technical condition (1.17) to obtain the estimate (proved in [11]),

$$|D| \geq \frac{1}{M} \int_0^{1/\tau} \sum_i \lambda_i a_i(s) ds, \quad 0 < \tau < \varepsilon, \quad (2.7)$$

where ε is a positive constant. This inequality, and its use in the estimates below, is where the proof differs from that in [2].

We write the integral in (2.1) as $\int_0^{1/t} + \int_{1/t}^\infty$. Then we have

$$\pi u' = \operatorname{Re} \left\{ \frac{1}{t} \left(\int_0^{1/t} + \int_{1/t}^\infty \right) I dt \right\}, \quad (2.8)$$

where I is the integrand in (2.1). We need the following inequalities (see [1, Lemma 4.1]):

$$\frac{1}{2\sqrt{2}} A_{a_i}(1/\tau) \leq |\hat{a}_i(\tau)| \leq 4A_{a_i}(1/\tau), \quad \tau > 0, \quad (2.9)$$

$$|\hat{a}'_i(\tau)| \leq 40 \int_0^{1/\tau} r a_i(r) dr, \quad \tau > 0. \quad (2.10)$$

Now we may estimate the first term in (2.8). For the rest of the paper we let M be a constant whose exact value may change each time that it appears. For $t > 1/\varepsilon$ we use (2.7), (2.9), and (2.10) to obtain

$$\begin{aligned} \left| \frac{1}{t} \int_0^{1/t} I dt \right| &\leq \frac{M}{t} \int_0^{1/t} \frac{\sum_i \lambda_i A_{a_i}(\tau^{-1})}{(\sum_i \lambda_i A_{a_i}(\tau^{-1}))^2} dt \\ &= \frac{M}{t} \int_t^\infty \frac{du}{u^2 \sum_i \lambda_i A_{a_i}(u)} \leq \frac{M}{t} \int_t^\infty \frac{du}{u^2 A(u)} \\ &\leq \frac{M}{t} \int_t^\infty \frac{du}{u^2} \leq \frac{M}{t^2} \end{aligned} \quad (2.11)$$

(see the definition of A on the line below (1.10)), and clearly $\int_{1/\varepsilon}^\infty 1/t^2 dt < \infty$.

To handle the second term in (2.8) we integrate by parts to obtain

$$\operatorname{Re} \left(\frac{1}{t} \int_{1/t}^\infty I d\tau \right) = \operatorname{Re} \frac{1}{it^2} \left(B - \int_{1/t}^\infty (J + K) d\tau \right), \quad (2.12)$$

where

$$B = \frac{(1/t) \sum_i \lambda_i \hat{a}'_i(1/t) - \sum_i \lambda_i \hat{a}_i(1/t)}{(\sum_i \lambda_i \hat{a}_i(1/t) + i/t)^2} e^i, \quad J = \frac{\tau \sum_i \lambda_i \hat{a}''_i(\tau)}{(\sum_i \lambda_i \hat{a}_i(\tau) + i\tau)^2}, \quad (2.13)$$

and

$$K = \frac{2(\sum_i \lambda_i \hat{a}'_i(\tau)\tau - \sum_i \lambda_i \hat{a}_i(\tau))(\sum_i \lambda_i \hat{a}_i(\tau) + i)}{(\sum_i \lambda_i \hat{a}_i(\tau) + i\tau)^3}.$$

By (2.7), (2.9), and (2.10) we have

$$\left| \frac{1}{t^2} B \right| \leq \frac{M}{t^2 \sum_i A_{a_i}(t)} \leq \frac{M}{t^2}. \quad (2.14)$$

To estimate the integral term we first estimate $1/t \int_{1/t}^\varepsilon J + K d\tau$. We will need the inequality (see [1, Lemma 5.1])

$$|\hat{a}''_i(\tau)| \leq 6000 \int_0^{1/\tau} r^2 a_i(r) dr \leq 6000 \tau^{-2} A_{a_i}(\tau^{-1}), \quad \tau > 0.$$

This estimate depends on assumption (1.9). By (2.7), (2.9), and (2.10) we now have

$$\begin{aligned} 1/t^2 \int_{1/t}^{\varepsilon} |J + K| dt &\leq M/t^2 \int_{1/t}^{\varepsilon} \frac{\tau^{-1}}{\sum_i A_{a_i}(\tau^{-1})} + \frac{1}{(\sum_i A_{a_i}(\tau^{-1}))^2} d\tau \\ &= M/t^2 \int_{1/\varepsilon}^t \left(\frac{u}{\sum_i A_{a_i}(u)} + \frac{1}{(\sum_i A_{a_i}(u))^2} \right) \left(\frac{du}{u^2} \right) \\ &\leq M/t^2 \int_{1/\varepsilon}^t 1/u + 1/u^2 du \leq M/t^2. \end{aligned}$$

If $\varepsilon \leq \rho$ then we have

$$\begin{aligned} \int_{\varepsilon}^{\rho} |J + K| d\tau &\leq M \int_{\varepsilon}^{\rho} \left(\frac{\sum_i \lambda_i A_{a_i}(\tau^{-1}) \tau^{-1}}{(\sum_i \lambda_i \phi_i(\tau))^2} + \frac{\sum_i \lambda_i A_{a_i}(\tau^{-1}) (\sum_i \lambda_i A_{a_i}(\tau^{-1}) \tau^{-1} + 1)}{(\sum_i \lambda_i \phi_i(\tau))^3} \right) d\tau \\ &\leq M, \end{aligned}$$

because the integrand is bounded by

$$M \left(\frac{\sum_i A_{a_i}(\tau^{-1}) \tau^{-1}}{\sum_i \phi_i(\tau)} + \frac{\sum_i A_{a_i}(\tau^{-1}) (\sum_i A_{a_i}(\tau^{-1}) \tau^{-1} + 1)}{\sum_i \phi_i(\tau)} \right),$$

a continuous function on $\varepsilon \leq \tau \leq \rho$.

The estimates for $(1/t^2) \int_{\rho}^{\infty} J + K d\tau$ are done exactly as in [2]. Thus, for $t \geq 1/\varepsilon$, we have $u'(t, \lambda_1, \dots, \lambda_n) \leq G(t)$, where $G(t)$ is some function in $L^1(1/\varepsilon, \infty)$. Combined with the result (2.6), (1.4) (i) follows. This completes the proof of Theorem 2.1 (i).

The proofs of Theorem 1.3 and the corollary are the same as the analogous theorems in [2], with obvious changes, and will be omitted.

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