THE JEFFCOTT EQUATIONS IN NONLINEAR ROTORDYNAMICS*

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Abstract. The Jeffcott equations are a system of coupled differential equations that represent the behavior of a rotating shaft. This is a simple model that allows investigation of the basic dynamic behavior of rotating machinery. Nonlinearities can be introduced by taking into consideration deadband, side force, and rubbing, among others.

In this paper we study the properties of the solutions of the Jeffcott equations with deadband. In particular, we show how bounds for the solutions of these equations can be obtained from bounds for the solutions of the linearized equations. By studying the behavior of the Fourier transforms of the solutions, we are also able to predict the onset of destructive vibrations. These conclusions are verified by means of numerical solutions of the equations, and of power spectrum density (PSD) plots.

This study offers insight into a possible detection method to determine pump stability margins during flight and hot fire tests, and was motivated by the need to explain a phenomenon observed in the development phase of the cryogenic pumps of the space shuttle, during hot fire ground testing: namely, the appearance of vibrations at frequencies that could not be accounted for by means of linear models.

1. Introduction. Jeffcott [10] was one of the first to study the vibration characteristics of an unbalanced, uniform, flexible shaft supported by bearings. He did so by considering a linear system of differential equations of the form

$$\begin{aligned} y'' + Cy' + Ay &= F \cos wt, \\ z'' + Cz' + Az &= F \sin wt, \end{aligned} \tag{1}$$

where the differentiation is with respect to the parameter t , and the shaft is assumed to rotate along the x -axis with angular velocity w , y and z describe the displacement of the center of the shaft, and the coefficients have the following physical interpretation: $C = C_s/m$, $A = K_s/m$, $K = K_b/m$, and $F = uw^2$, where m is the mass of the shaft, C_s is the seal damping, K_s and K_b are the seal and bearing stiffnesses, and u is the displacement of the shaft's center of mass from the geometric center.

In [12] Yamamoto studied the effect of the nonlinearities induced by deadbands (i.e., the clearing between housing and bearing), but his treatment was not rigorous.

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FIG. 1

Other works dealing with nonlinearities in this context include [3, 4, 8, 11]. In [7], Day used the method of multiple scales to gain new insight into the properties of the solutions. He discovered a frequency, which he termed "nonlinear natural frequency," that appears in the PSD plots of solutions of the nonlinear model and is absent from the PSD plots of solutions of the linear model. The nonlinear natural frequency seems to have been observed during early ground testing of both LOX and fuel pumps of the second stage main engine of the space shuttle but, until now, there has been no explanation of its origin. If $r = (y^2 + z^2)^{1/2}$, δ is the deadband, $B = Q_s/m$, with Q_s denoting the cross-coupling stiffness of the seal, and

$$h(t) = \begin{cases} 1 & \text{if } r \leq \delta \\ \delta/r & \text{if } r > \delta, \end{cases} \quad (2)$$

then the model studied by Day can be described by the system

$$\begin{aligned} y'' + Cy' + [A + K(1 - h(t))]y + Bz &= f_1(t), \\ z'' + Cz' - By + [A + K(1 - h(t))]z &= f_2(t), \end{aligned} \quad (3)$$

where $f_1(t) = F \cos wt$, $f_2(t) = F \sin wt$, and $K(1 - h(t))$ is the nonlinearity associated with the deadband (see Fig. 1). Note that (1) is a particular case of (3).

In this paper we shall explain the nature of the nonlinear natural frequency and apply our conclusions to the signature analysis of the nonlinear Jeffcott model described by (3), where $f_1(t)$ and $f_2(t)$ are arbitrary bounded and continuous functions, B , C , K , and δ are positive, and A and t are nonnegative.

2. Properties of the solutions of Jeffcott's equations.

2.1. *Existence, uniqueness, and a representation formula.* If $x_1 = y$, $x_2 = y'$, $x_3 = z$, $x_4 = z'$, $r = (x_1^2 + x_3^2)^{1/2}$ and $h(t)$ is given by (2), then (3) is equivalent to

$$\begin{aligned}x_1' &= x_2, \\x_2' &= -[A + K(1 - h)]x_1 - Cx_2 - Bx_3 + f_1(t), \\x_3' &= x_4, \\x_4' &= Bx_1 - [A + K(1 - h)]x_3 - Cx_4 + f_2(t),\end{aligned}\tag{4}$$

or, more concisely,

$$\mathbf{x}' = \mathbf{g}(\mathbf{x}, t).$$

Since $\mathbf{g}(\mathbf{x}, t)$ is continuous, and satisfies a Lipschitz condition on $\mathbf{x}$, from standard existence and uniqueness theorems (cf., e.g. [5]), we know that every initial value problem for (4) has a unique solution. Thus, we also infer that every initial value problem for (3) has a unique solution. Let $v = y + iz$, $f(t) = f_1(t) + if_2(t)$, and $M = A + K - iB$; then, as remarked by Day, (3) is also equivalent to

$$v'' + Cv' + Mv - Kh(t)v = f(t).\tag{5}$$

Before studying (5), let us first consider a linear system of the form

$$v'' + Cv' + Mv = g(t).\tag{6}$$

Then

$$v = c_1 \exp(\lambda_1 t) + c_2 \exp(\lambda_2 t) + v_p,$$

where

$$c_1 \exp(\lambda_1 t) + c_2 \exp(\lambda_2 t)$$

is a solution of

$$v'' + Cv' + Mv = 0,\tag{7}$$

and therefore $\lambda_{1,2} = (1/2)[-C \pm (C^2 - 4M)^{1/2}]$. If $Q = C^2 - 4(A + K)$, a straightforward computation shows that $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha' - i\beta$, where

$$\beta = 8^{-1/2}[-Q + (Q^2 + 16B^2)^{1/2}]^{1/2},\tag{8}$$

$$\alpha = [\beta^{-1}B - C]/2, \quad \alpha' = -[\beta^{-1}B + C]/2,\tag{9}$$

and therefore

$$\alpha' = -\beta^{-1}B + \alpha.\tag{10}$$

Applying, e.g., [5, Theorem 6.4] we readily deduce that the Green's function of the differential operator $(d^2/dt^2) + C(d/dt) + M$ is $G(t - s)$, where

$$G(t) = (\lambda_1 - \lambda_2)^{-1}[\exp(\lambda_1 t) - \exp(\lambda_2 t)],$$

i.e., $\int_0^t G(t - s)g(s)ds$ is a particular solution of (6), and therefore

$$v = c_1 \exp(\lambda_1 t) + c_2 \exp(\lambda_2 t) + \int_0^t G(t - s)g(s)ds.$$

If, in particular, $g(s) = Kh(s)v(s) + f(s)$, then (6) reduces to (5), and we have

$$v = c_1 \exp(\lambda_1 t) + c_2 \exp(\lambda_2 t) + \int_0^t G(t - s)f(s)ds + K \int_0^t G(t - s)h(s)v(s)ds.$$

In other words,

$$v(t) = u(t) + P(t), \quad (11)$$

where

$$u(t) = c_1 \exp(\lambda_1 t) + c_2 \exp(\lambda_2 t) + \int_0^t G(t-s)f(s) ds \quad (12)$$

is a solution of the linear differential equation $v'' + Cv' + Mv = f(t)$, (henceforth called the *linear part* of (5)), and

$$P(t) = K \int_0^t G(t-s)h(s)v(s) ds. \quad (13)$$

Note that we have a closed form formula for $u(t)$, whereas $P(t)$ is expressed in terms of the unknown function $v(t)$. Our analyses will be based on a study of the properties of the perturbation term $P(t)$.

2.2. *Bounds.* The definition (2) of $h(s)$ implies that $|h(t)v(t)| \leq \delta$; thus,

$$\begin{aligned} |P(t)| &\leq K\delta \int_0^t |G(t-s)| ds \\ &\leq K\delta(\beta^{-2}B^2 + 4\beta^2)^{-1/2} \int_0^t [\exp(\alpha(t-s)) + \exp(\alpha'(t-s))] ds. \end{aligned}$$

Since (9) and (10) imply that if $\alpha = 0$, then $\beta = B/C$, and when $\alpha = \beta^{-1}B$ then $\alpha' = 0$, we have

$$|P(t)| \leq \begin{cases} K\delta(\beta^{-2}B^2 + 4\beta^2)^{-1/2}[(1/\alpha)(\exp(\alpha t) - 1) \\ \quad + (1/\alpha')(\exp(\alpha' t) - 1)] & \text{if } \alpha \neq 0, \beta^{-1}B, \\ K\delta(C^2 + 4B^2/C^2)^{1/2}[t + (2/C)(1 - \exp[(-C/2)t])] & \text{if } \alpha = 0, \\ K\delta(\beta^{-2}B^2 + 4\beta^2)^{-1/2}[(\beta/B)(\exp[(B/\beta)t] - 1) + t] & \text{if } \alpha = \beta^{-1}B. \end{cases} \quad (14)$$

From (11), (12), (13), and (14) we derive the following conclusions:

1. If $\alpha < 0$ and $|\int_0^t G(t-s)f(s) ds| \leq M$, then the steady state solution v_∞ of (5) satisfies the following inequality:

$$|v_\infty| \leq M + K\delta(\beta^{-2}B^2 + 4\beta^2)^{-1/2}|1/\alpha + 1/\alpha'|.$$

2. If $\alpha = 0$, the perturbation term $P(t)$ can grow at most linearly.

3. If $\alpha > 0$, the order of magnitude of $P(t)$ cannot exceed $\exp(\alpha t)$; note that the order of magnitude of all nonzero solutions of (7) cannot exceed $\exp(\alpha t)$.

Thus, since we have assumed that $f(t)$ is bounded, the study of the boundedness of the solutions of (5) reduces to the study of the boundedness of the solutions of its homogeneous part. If $\alpha < 0$ we shall say that the system (5) is stable, if $\alpha > 0$ that (5) is unstable, and if $\alpha = 0$ that (5) has reached the stability boundary. This nomenclature is consistent with that used for linear systems (cf. [9], pp. 83, 84).

2.3. *Estimates for β .* In this section we will prove that β is between B/C and $(A + K)^{1/2}$. The importance of this observation will become clear in the sequel. In what follows, let $\gamma = A + K$ and $\xi = (B/C)^2$; thus $Q = C^2 - 4\gamma$.

Assume that $\alpha < 0$; then from (9) we see that $B/C < \beta$. Squaring and applying (8), we have

$$\xi < (1/8)[-Q + (Q^2 + 16B^2)^{1/2}].$$

Thus, ξ satisfies the inequality $4\xi^2 + Q\xi - B^2 < 0$, which can be written as

$$4\xi^2 + [C^2 - 4\gamma]\xi - B^2 < 0,$$

or

$$C^2 - 4\gamma < (B^2 - 4\xi^2)\xi^{-1} = B^2\xi^{-1} - 4\xi.$$

Since $\xi = (B/C)^2$, we have that $C^2 - 4\gamma < C^2 - 4\xi$, and therefore $\gamma > \xi$, i.e., $B^2 < C^2\gamma$. Thus,

$$16\gamma^2 + C^4 - 8C^2\gamma + 16B^2 < 16\gamma^2 + C^4 + 8C^2\gamma,$$

i.e.,

$$Q^2 + 16B^2 < (4\gamma + C^2)^2,$$

and therefore

$$[Q^2 + 16B^2]^{1/2} < 4\gamma + C^2.$$

Subtracting Q from both sides of this inequality we see that

$$8\beta^2 = -Q + [Q^2 + 16B^2]^{1/2} < 8\gamma,$$

and we conclude that $\beta < (A + K)^{1/2}$. The cases $\alpha = 0$ and $\alpha > 0$ are treated similarly, and we shall omit the details. The conclusions are the following.

1. If $\alpha < 0$, then $B/C < \beta < (A + K)^{1/2}$, and $\alpha' < 0$.
2. If $\alpha = 0$, then $B/C = \beta = (A + K)^{1/2}$, and $\alpha' < 0$.
3. If $\alpha > 0$, then $(A + K)^{1/2} < \beta < B/C$.

From these conclusions we also infer that if $f(t)$ is bounded, then (5) is stable if and only if $B/C < (A + K)^{1/2}$.

2.4. Resonance. From the results of 2.3 it is clear that (5) is in resonance if and only if its linear part is in resonance. If for example $f(t) = F_0 \exp(i\omega t)$, then we readily see that the linear part of (5) has a particular solution of the form $A_0 \exp(i\omega t)$, where

$$A_0 = F_0 / [(A + K - \omega^2) + i(\omega C - B)].$$

Since the denominator in the preceding formula vanishes if and only if $\omega = B/C$ and $\omega = (A + K)^{1/2}$, we deduce that $B/C = (A + K)^{1/2}$. The inequalities we obtained in Sec. 2.3 imply that $\alpha = 0$. Thus (5) can be in resonance only on the stability boundary.

3. Harmonic analysis of the solutions.

3.1. Introduction. In practice, the coefficients of (5) and, in general, the equations that describe the movement of rotating machinery, are imperfectly known. The approach taken is to sample the system response over a time interval (in our case, that would mean measuring $y(t_i)$ and $z(t_i)$, $i = 0, \dots, N$, where the t_i are equally spaced points), and to compute the discrete Fourier transform of the sequence $\{y(t_i) + iz(t_i)\}$. (See, e.g., [2].) The absolute value of the discrete Fourier transform is then divided by a normalizing constant and plotted on a graph called power spectrum density (PSD)

plot, which represents the response of the mechanical system at different frequencies. One then tries to determine the condition of the mechanical system by an examination of these plots. This is known as "signature analysis." (See, e.g., Collacott [6].) In this section we examine the properties of the Fourier transforms of the solutions of (5), whereas in Sec. 4 we show, by means of examples, how to apply these conclusions to the signature analysis of the system. From now on, we shall assume that $\alpha < 0$.

3.2. *Properties of the continuous Fourier transform.* Let $G(t)$, $v(t)$, $u(t)$, and $P(t)$ be defined to equal 0 for $t < 0$. Then (11) is valid on $(-\infty, \infty)$. If

$$q(t) = h(t)v(t),$$

it is clear that $q(t)$ vanishes for $t < 0$. Thus, since $G(t-x) = 0$ for $t < x$, from (13) we readily see that

$$P(t) = K \int_{-\infty}^{\infty} G(t-x)q(x)dx = K(G * q)(t), \quad (15)$$

where $*$ denotes the convolution product. If $p_1(t) = \exp(\lambda_1 t)$, $p_2(t) = \exp(\lambda_2 t)$ for $t \geq 0$, and equal zero for $t < 0$, and $u_p(t)$ denotes a particular solution of the linear part of (5), then (11) can be written in the form

$$v(t) = c_1 p_1(t) + c_2 p_2(t) + u_p(t) + P(t). \quad (16)$$

Note, moreover, that

$$G(t) = (\lambda_1 - \lambda_2)^{-1} [p_1(t) - p_2(t)]$$

and therefore

$$P(t) = K(\lambda_1 - \lambda_2)^{-1} [(p_1 - p_2) * q](t). \quad (17)$$

Let F denote the Fourier transform operator; thus, if $g(t)$ is integrable on $(-\infty, \infty)$, then $F[g](s) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} g(t) \exp(-ist) dt$. In particular, when $\alpha < 0$, we have

$$F[p_1](s) = (2\pi)^{-1/2} / [i(s - \beta) - \alpha] \quad (18)$$

and

$$F[p_2](s) = (2\pi)^{-1/2} / [i(s + \beta) - \alpha + \beta^{-1}B]. \quad (19)$$

If $u_p(t)$ and $q(t)$ are integrable on $(-\infty, \infty)$, from (16) and (17) we see that

$$F[v] = c_1 F[p_1] + c_2 F[p_2] + F[u_p] + K(\lambda_1 - \lambda_2)^{-1} (F[p_1] - F[p_2])F[q]. \quad (20)$$

Since $F[p_1](\beta)$ diverges as $\alpha \rightarrow 0^-$, we therefore conclude that if α approaches 0 from the left, then the graph of the absolute value of $F[v]$ may exhibit increasingly larger spikes near β , where α and β are linked by (9). At first glance, this does not appear to be very useful, since in most applications u_p will not be integrable on $(-\infty, \infty)$ (as for example when $f(t) = F \exp(i\omega t)$). We shall now show that the range of validity of our conclusions can be greatly extended if we consider almost periodic functions.

3.3. *Almost periodic functions, windowing, and discretization.*

3.3.1. *Almost periodic functions.* We shall call a function $f(t)$ almost periodic (a.p.) on $[0, \infty)$ if $f(|t|)$ is almost periodic on $(-\infty, \infty)$. (For the definition and properties of almost periodic functions, see Besicovitch [1], where they are called "uniform almost periodic.") Moreover, if $f(t) = f_1(t) + f_2(t)$, where $f_1(t)$ is continuous and tends to

zero as $t \rightarrow +\infty$, and $f_2(t)$ is a.p. on $[0, \infty)$, then $f(t)$ is called "asymptotically almost periodic" (on $[0, \infty)$). The "mean value" of a function f , denoted by $M(f)$ is defined by

$$M(f) = \lim_{b \rightarrow +\infty} b^{-1} \int_0^b f(t) dt.$$

If $f(t)$ is a.p., this limit is known to exist (cf. [1]). Moreover, if $f(t)$ is a continuous function that vanishes at $+\infty$, it is readily seen that $M(f) = 0$. Indeed, let $\varepsilon > 0$ be given, and assume that $|f(t)| < \varepsilon/2$ if $t > b_0$; let L be an upper bound of $|f(t)|$, and let $b_1 = \max\{b_0, 2b_0L\varepsilon^{-1}\}$. If $b > b_1$ we have

$$\begin{aligned} \left| b^{-1} \int_0^b f(t) dt \right| &\leq b^{-1} \int_0^{b_0} |f(t)| dt + b^{-1} \int_{b_0}^b |f(t)| dt \\ &\leq b^{-1} [b_0L + (b - b_0)\varepsilon/2] \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Thus, $M(f)$ is defined for any asymptotically a.p. function.

In the sequel we shall assume that the solutions $u(t)$ (given by (12)) of the linearized equation, are asymptotically a.p. on $[0, \infty)$. This will be the case if, for instance, $\alpha \geq 0$ and $f(t) = A_0 \exp(i\omega t)$. We shall also assume that the solutions $v(t)$ of (5) are asymptotically almost periodic, since numerical evidence suggests that this is often the case. From, e.g., (11) we thus see that $P(t)$ will then be asymptotically a.p. Since $h(t) = 1/h_1(t)$, where $h_1(t) = \max\{1, |v(t)|/\delta\}$, it is also clear that $q(t)$ is asymptotically a.p. Setting $a(v, s) = M(v(x) \exp(-isx))$ and applying (16) we conclude that

$$a(v, s) = a(v_p, s) + a(P, s). \quad (21)$$

Let $P_k(x) = \int_0^x q(t) \exp[\lambda_k(x-t)] dt$, $k = 1, 2$. Integrating by parts we have

$$\begin{aligned} \int_0^b \exp(-isx) P_1(x) dx &= \int_0^b \int_0^x q(t) \exp(-\lambda_1 t) dt \exp[(\lambda_1 - is)x] dx \\ &= (is - \lambda_1)^{-1} \left[-\exp[(\lambda_1 - is)b] \int_0^b q(x) \exp(-\lambda_1 x) dx \right. \\ &\quad \left. + \int_0^b q(x) \exp(-isx) dx \right]. \end{aligned}$$

Since

$$\begin{aligned} \left| \exp[(\lambda_1 - is)b] \int_0^b q(x) \exp(-\lambda_1 x) dx \right| &\leq \delta \int_0^b \exp[\alpha(b-x)] dx \\ &= \delta \alpha^{-1} [\exp(\alpha b) - 1], \end{aligned}$$

dividing by b and passing to the limit we see that for $\alpha < 0$

$$a(P_1, s) = (is - \lambda_1)^{-1} a(q, s).$$

Similarly,

$$a(P_2, s) = (is - \lambda_2)^{-1} a(q, s),$$

whence, since λ_1 and λ_2 are solutions of $\lambda^2 + C\lambda + M = 0$ from (17) and (21) we obtain

$$a(v, s) = a(v_p, s) + K[(A + K - s^2) + i(Cs - B)]^{-1} a(q, s), \quad (22)$$

an analog of (20).

Thus, if the coefficients of (11) vary continuously with α , and, moreover, $a(q, \beta)$ remains bounded away from zero, and the solutions $v(t)$ are asymptotically a.p. as $\alpha \rightarrow 0^-$, then $\lim_{\alpha \rightarrow 0^-} |a(v, \beta)| = \infty$. (Note that β may vary with α .) We should however point out that if $v(t)$ is asymptotically a.p. and $\alpha = 0$, $a(q, \beta)$ may vanish. Indeed, integrating by parts, we readily see that $a(v', s) = isa(v, s)$, and therefore $a(v'', s) = -s^2 a(v, s)$, whence from (5) we deduce that

$$[(A + K - \beta^2) + i(\beta C - B)]a(v, \beta) - Ka(q, \beta) = a(f, \beta).$$

If $\alpha = 0$, from the estimates for β obtained in Sec. 2.3, it is clear that the coefficient of $a(v, \beta)$ vanishes, whence $-Ka(q, \beta) = a(f, \beta)$. If now $f(t) = A_0 \exp(i\omega t)$ and $\omega \neq \beta$, we see that $a(f, \beta) = 0$, and therefore $a(q, \beta) = 0$. We also note that, since $|q(t)| \leq \delta$, it is readily seen that $|m(q, s)| \leq \delta$. In view of (22) we thus conclude that $a(v, s)$ can become unbounded only if either $a(v_p, s)$ or $(is - \lambda_1)^{-1}$ become unbounded.

We identify the nonlinear natural frequency with those frequencies at which the plot of $|a(v, s)|$ has relative maxima that become unbounded as $\alpha \rightarrow 0^-$, but such that $a(v_p, s)$ remains bounded, with the caveat that there is no obvious reason why these frequencies should be unique. For α sufficiently small this observation is immaterial, since all these frequencies will coalesce with $\beta/2\pi$.

All that remains to be done, is to find a method to compute $a(v, s)$ and $a(q, s)$.

3.3.2. Windowing and discretization. As we mentioned in Sec. 3.1, in order to monitor a mechanical system, we study the behavior of its discrete Fourier transform, computed on sets Λ_n of the form $\{a_{0,n}, a_{1,n}, \dots, a_{m-1,n}\}$, $n = 0, 1, 2, \dots$, where $a_{k,n} = nd + kh$, $k = 0, \dots, m-1$, $d = mh$ and both d and h are constant. (Note that $a_{m,n} = a_{0,n+1}$.) Assuming that $v(t)$ is asymptotically a.p., we now show the connection between its discrete Fourier transforms, continuous Fourier transforms and $a(v, s)$. Let $D_n v(s)$ denote the discrete Fourier transform of $v(t)$ on Λ_n . Thus

$$D_n v(s) = m^{-1} \sum_{k=0}^{m-1} v(a_{k,n}) \exp(-isa_{k,n}).$$

Let $b_n = a_{m,n}$. If $V_n(s)$ denotes the Fourier transform of v over the interval (b_n, b_{n+1}) , i.e., $V_n(s) = \int_{b_n}^{b_{n+1}} v(t) \exp(-sti) dt$, applying the composite trapezoid rule we have

$$\begin{aligned} V_n(s) &= (h/2)[v(a_{0,n}) \exp(-isa_{0,n}) + v(a_{m,n}) \exp(-isa_{m,n})] \\ &\quad + h \sum_{k=1}^{m-1} v(a_{k,n}) \exp(-isa_{k,n}) + hdE_m(n, s) \end{aligned}$$

where (since the integrand is complex valued) $|E_m(n, s)| \leq (h/6)M$, M being the maximum of the second derivative of $v(t) \exp(ist)$ on $[0, \infty)$. This can be written as

$$V_n(s) = h \sum_{k=0}^{m-1} v(a_{k,n}) \exp(-isa_{k,n}) + hdF_m(n, s),$$

where $F_m(n, s) = (1/2d)[v(b_{n+1}) \exp(-isb_{n+1}) - v(b_n) \exp(-isb_n)] + E_m(n, s)$. Thus, $F_m(n, s)$ is bounded, uniformly on m, n, h ($h < 1$), and s , and

$$V_n(s) = dD_nv(s) + hdF_m(n, s). \quad (23)$$

It is known that if $g(t)$ is an a.p. function, then $\lim_{m \rightarrow \infty} m^{-1} \sum_{k=0}^{m-1} g(x + kh) = g_h(x)$, where $g_h(x)$ is a periodic function of period h , and the convergence is uniform in x (cf. [1, p. 44]). Since $v(t) \exp(ist)$ is asymptotically a.p., we conclude that

$$D_nv(s) = p(b_n, s) + G_m(n, s), \quad (24)$$

where, for each s , $p(x, s)$ is h -periodic, and $G_m(n, s)$ approaches zero as $m \rightarrow \infty$, uniformly on n . Since $b_n = nmh$, we see that

$$p(b_n, s) = p(0, s). \quad (25)$$

In other words, we have shown that for given s and h , if m (or d) is sufficiently large, then $D_nv(s)$ will be approximately constant. Combining (23), (24), and (25) we also have

$$V_n(s) = dp(0, s) + dH(m, n, s) \quad (26)$$

where $H(m, n, s) = G_m(n, s) + hF_m(n, s)$. At this point, we should remark that only a finite set of values of s is considered, say $\{s_j, j = 0, \dots, N\}$. Let $\varepsilon > 0$ be given. Since $F_m(n, s)$ is bounded uniformly on m, n, h ($h < 1$), and d , we can choose an h small enough so that $h|F_m(n, s_j)| < \varepsilon/4$. Once this h has been chosen, we can find an m (and therefore a value of d) such that

$$|G_m(n, s_j)| < \varepsilon/4, \quad j = 0, \dots, N; \quad n = 1, 2, 3, \dots \quad (27)$$

Thus

$$|H(m, n, s_j)| < \varepsilon/2; \quad j = 0, \dots, N; \quad n = 1, 2, 3, \dots, \quad (28)$$

and from (26) and (28) we deduce that

$$\frac{1}{nd} \int_0^{nd} \exp(s_j ti) v(t) dt = \frac{1}{nd} \sum_{k=0}^{n-1} V_k(s_j) = p(0, s) + R(n, s_j),$$

where $R(n, s_j) = \frac{1}{n} \sum_{k=0}^{n-1} H(m, k, s_j)$, and therefore $|R(n, s_j)| < \varepsilon/2$. Making $n \rightarrow \infty$, we have

$$a(v, s_j) = p(0, s_j) + R(s_j), \quad j = 0, \dots, N;$$

with $|R(s_j)| < \varepsilon/2$. Finally, from (24), (25), (27), and the preceding identity, we conclude that

$$a(v, s_j) = D_nv(s_j) + T_n(s_j), \quad j = 0, \dots, N; \quad n = 1, 2, 3, \dots,$$

and $|T_n(s_j)| < \varepsilon$.

3.3.3. The transient frequency. The quantity $\beta/(2\pi)$ will be called the "transient frequency." To explain the reason for this name, let $g^{(a,b)}(t) = g(t)$ if $a < t < b$, and let $g^{(a,b)}(t)$ equal zero otherwise. Clearly

$$F[p_1^{(a,b)}](s) = (2\pi)^{-1/2} [\exp(\lambda_1 - si)b - \exp(\lambda_1 - si)a]/(\lambda_1 - si),$$

and

$$F[p_2^{(a,b)}](s) = (2\pi)^{-1/2} [\exp(\lambda_2 - si)b - \exp(\lambda_2 - si)a] / (\lambda_2 - si).$$

By elementary calculus we see that $|F[p_1^{(a,b)}](s)|$ has its absolute maximum at β . Thus, the graph of the absolute value of $F[v^{(a,b)}]$ may have spikes near β . However, since $\lim_{a \rightarrow \infty} F[p_1^{(a,b)}](s) = \lim_{a \rightarrow \infty} F[p_2^{(a,b)}](s) = 0$, we conclude that for fixed α and sufficiently large a , these spikes may be detected only if α is extremely close to 0. (Whether they will be detected at all depends on the numerical stability of the computations.) This is consistent with $a(p_1, s)$ being identically zero.

3.3.4. Conclusions. In summary, we have shown that if

$$a(v, s) = \lim_{b \rightarrow +\infty} b^{-1} \int_0^b v(t) \exp(-ist) dt,$$

and $v(t)$ is assumed to be asymptotically almost periodic, then

1. If the coefficients of (5) vary continuously with α , and β_0 is the value of β that corresponds to $\alpha = 0$, then the ratio $a(v, s)/a(q, s)$ will diverge if and only if $s \rightarrow \beta_0$ as $\alpha \rightarrow 0^-$.

2. For each s and α there are sufficiently small h and sufficiently large m such that the discrete Fourier transforms $D_n v(s)$ and $D_n q(s)$, $n = 0, 1, 2, \dots$, taken over sets of the form $\{h(mn + k); k = 0, \dots, m - 1\}$ are approximately constant with respect to n , and are good approximations to $a(v, s)$ and $a(q, s)$, respectively. Therefore $D_n v(s)$ may have a spike near β or, in the frequency domain, near the transient frequency $\beta/(2\pi)$.

In [7, p. 784], Day equates the nonlinear natural frequency with the ratio Q_s/C_s (which, in our notation, equals B/C). Later on (on p. 786), he notes that the nonlinear natural frequency is actually not B/C , but a number close to it. In this paper we have gone one step further and shown that for all practical purposes, if α is sufficiently small then $\beta/(2\pi)$ (i.e., the transient frequency), and the nonlinear natural frequency are one and the same. This is a surprising result.

On the practical side, our conclusions suggest that, all other things being equal, the introduction of nonlinearities (as induced, e.g., by deadbands) in a mechanical system, may give an earlier warning of the approach to the instability boundary.

4. Examples. We now study the behavior of the solutions of (5) for $C = 240$, $A = 0$, $\delta = 0.0000285$, $K = 1,305,000$, $f(t) = uw^2 \exp(iwt)$, $u = 0.000060915$, and $w = 1,000\pi s$, where s will vary. We assume throughout that the various systems considered satisfy the initial conditions $v(0) = \delta$, $v'(0) = 1$. We also make the realistic assumption—based on empirical data—that the bearing stiffness changes with the forcing frequency w , by setting $B = 60w$. Let f_c (the “critical frequency”) be defined to equal $(A + K)^{1/2}/(2\pi) = K^{1/2}/(2\pi)$. Since, as we have already shown, when $\alpha = 0$ we have $B/C = (A + K)^{1/2}$, we readily see that $f_c = 181.8$ Hz, and that the value of s that corresponds to f_c is $s_c \doteq 1.4545$. Our example is related to Example 1 of [7]. Let $f_1 = w/(2\pi)$ denote the forcing frequency and $f_2 = \beta/(2\pi)$ the transient frequency; clearly $f_1 = 500$ s. We know that (5) will be stable for $s < s_c$, and unstable for $s > s_c$. In Figs. 2–5 we show, for various values of s between 0.5 and 1.8, plots of the numerical solutions of (5) (obtained by solving (4) using a

FIG. 2

fourth-order Runge-Kutta algorithm), and of PSD plots for v . From (8) we readily deduce that f_2 will vary between 180.92 Hz and 182.34 Hz. The solution plots are for $0.100 < t < 0.256$, and the PSD plots for the window $[0, 0.256]$. Note that all the PSD plots have two distinct spikes: one corresponding to the forcing frequency, and one corresponding to the nonlinear natural frequency.

For $s = 0.5$ (i.e., far from the stability boundary), the nonlinear natural frequency and f_2 are far apart. Moreover, the forcing function $f(t)$ dominates the perturbation term $P(t)$; thus, the solution is nearly circular (we see a thick circular curve; the thickness is caused by $P(t)$), and the PSD plots exhibit larger spikes for the forcing frequency than for the nonlinear natural frequency. As s increases, the nonlinear natural frequency gets closer to f_2 , the solution becomes annular, and the nonlinear natural frequency begins to dominate. Finally, for $s > s_c$ the solutions begin to diverge. Note that there is no obvious qualitative difference between PSD plots for values of s close to s_c , even if the values of s are on opposite sides of s_c .

FIG. 3

Our plots have been obtained using 256 points connected by linear interpolation, and the frequency range considered is from 0 to 1,000 Hz; thus, when α is near zero (i.e., when s is near 1.4545), our plots should exhibit spikes at about 180 Hz. If we agree that the nonlinear natural frequency f corresponds to the largest spike (other than that corresponding to the forcing frequency), then f equals approximately 168 Hz for $s = 0.5, 1.1$, and 1.2 , 160 Hz for $s = 1$, 172 Hz for $s = 1.3$, 176 Hz for $s = 1.4$, 180 Hz for $s = 1.5$, and 184 Hz for $s = 1.8$. In fact, the transition from 176 Hz to 180 Hz occurs somewhere between $s = 1.44$ and $s = 1.45$, i.e., very close to the stability boundary. This suggests that the location of the nonlinear natural frequency in PSD plots might not be a good predictor of stability margins.

Fig. 6 shows PSD plots for $s = 1.2$ and $s = 1.3$ for the window $2.560 < t < 2.816$. For $s = 1.2$, the nonlinear natural frequency has disappeared. For $s = 1.3$ the nonlinear natural frequency has shifted to the left (from 176 Hz to 164 Hz), and slightly decreased in magnitude. This can be explained by saying that the influence of the transient terms, $\exp(\lambda_1 t)$ and $\exp(\lambda_2 t)$, in the perturbation term $P(t)$ has subsided.

FIG. 4

We have obtained PSD plots for several intervals of the form $0.256n < t < 0.256(n+1)$, with n ranging from 10 to 500. In every case, these plots are practically identical with those depicted in Fig. 6.

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FIG. 5

FIG. 6

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