SEMICOERCIVE HEMIVARIATIONAL INEQUALITIES. ON THE DELAMINATION OF COMPOSITE PLATES*

By

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Abstract. In this paper semicoercive hemivariational inequalities are studied in the framework of a concrete mechanical problem: the delamination effect of laminated plates. The interlaminar bonding forces are described by a nonmonotone multivalued law which may be written as the generalized gradient of a nonconvex superpotential in the sense of F. H. Clarke. Then necessary conditions are proved for the existence of the solution, as well as sufficient conditions using compactness and average value arguments.

1. Introduction. In the debonding problem for laminated plates we do not know a priori the region where the delamination takes place. The interaction between the laminae is described, due to the adhesive, by a nonmonotone possibly multivalued law between the interlaminar bonding forces and the corresponding relative displacements. This law permits the formulation of the problem as a hemivariational inequality (cf. [1]) if classical boundary conditions hold. Here we assume that the boundary conditions permit rigid-body displacements of the plates and therefore a semicoercive hemivariational inequality arises. The aim of the present paper is to formulate and study this semicoercive hemivariational inequality. The mathematical method presented is general and may be applied to most classes of hemivariational inequalities arising in mechanics ([2]–[4]) when they are semicoercive.

The mechanical problem providing the mathematical framework is a unilateral one because its variational formulation, which expresses the principle of virtual work, is an inequality. Coercive and semicoercive unilateral BVP's in the theory of plates were first studied in [5]–[7] in the context of convexity, i.e., for monotone, possibly multivalued boundary conditions which are derived from a convex superpotential [8] through subdifferentiation, both for the Kirchhoff and the von Kármán plate theory (i.e., the small and the large deflection theory, respectively). Due to the convexity they give rise to variational inequalities which are studied by means of monotonicity arguments. Here the nonmonotone interlaminar law is expressed in terms of a nonconvex superpotential [3], [9] by forming its generalized gradient [10]. Hemivariational inequalities for a single Kirchhoff and von Kármán plate have been studied in

*Received September 1, 1988.

[11], [12]. In [1] the coercive delamination problem for laminated Kirchhoff plates is studied, when the plates are subjected to monotone multivalued boundary conditions. In this case we obtain a coercive variational-hemivariational inequality. In the present paper the delamination problem for laminated von Kármán plates (or any type of layered plates or sandwich plates) is formulated and the resulting semicoercive hemivariational inequality is studied.

FIG. 1 The geometry of the von Kármán plates.

2. Classical and variational formulation of the problem. Consider a laminated plate consisting of r laminae and the binding material between them (Fig. 1). In the undeformed state the middle surface of lamina j ($j = 1, \dots, r$) occupies an open, bounded, and connected subset Ω_j of $\mathbb{R}^2$, referred to a fixed right-handed Cartesian coordinate system $0x_1x_2x_3$. Let Γ_j be the boundary of the j th lamina; Γ_j is assumed to be appropriately regular (in general, a Lipschitz boundary $C^{0,1}$ is sufficient). We assume also that (cf. [1]) the interlaminar binding material occupies a subset Ω' such that $\Omega' \subset \Omega_j \cap \Omega_{j+1}$ and $\Omega' \cap \Omega_j = \emptyset$, $\Omega' \cap \Omega_{j+1} = \emptyset$. Let us denote by

$\zeta^{(j)}(x)$, $x = \{x_1, x_2\}$, the vertical deflexion of the point $x \in \Omega_j$ on the j th lamina, and by $f^{(j)} = \{0, 0, f_3^{(j)}(x)\}$ the distributed vertical load acting on the j th lamina per unit area of Ω_j . Further, let $u^{(j)} = \{u_1^{(j)}, u_2^{(j)}\}$ be the in-plane displacements of the j th lamina. We assume that the j th lamina has constant thickness h_j , while the interlaminar binding layer has constant thickness h considered as small and is assumed to exhibit negligible shearing resistance in the interface level. Moreover, for each lamina the von Kármán plate theory is assumed to hold (cf. [13] p. 57). Thus the following system of differential equations is formulated.

$$K_j \Delta \Delta \zeta^{(j)} - h_j (\sigma_{\alpha\beta}^{(j)} \zeta_{,\beta}^{(j)})_{,\alpha} = f^{(j)} \quad \text{in } \Omega_j, \quad (2.1)$$

$$\sigma_{\alpha\beta}^{(j)} = 0 \quad \text{in } \Omega_j, \quad (2.2)$$

and

$$\sigma_{\alpha\beta}^{(j)} = C_{\alpha\beta\gamma\delta}^{(j)} (\varepsilon_{\gamma\delta}^{(j)}(u^{(j)}) + \frac{1}{2} \zeta_{,\gamma}^{(j)} \zeta_{,\delta}^{(j)}) \quad \text{in } \Omega_j. \quad (2.3)$$

Here the subscripts $\alpha, \beta, \gamma, \delta = 1, 2$ refer to the coordinate directions; the subscript $j = 1, \dots, r$ refers to the j th lamina; $\{\sigma_{\alpha\beta}^{(j)}\}$, $\{\varepsilon_{\alpha\beta}^{(j)}\}$, and $\{C_{\alpha\beta\gamma\delta}^{(j)}\}$ are the stress, strain, and elasticity tensors in the plane of the plate. The components of $C^{(j)}$ are assumed to be elements of $L^\infty(\Omega_j)$ and to satisfy the usual symmetry and ellipticity properties. Moreover, $K_j = Eh_j^3/12(1 - \nu^2)$ is the bending rigidity of the j th plate with E Young's modulus, and ν Poisson's ratio. For the sake of the simplicity, we consider here isotropic homogeneous plates of constant thicknesses. However, in place of (2.1–2.3), the appropriate equations for orthotropic or anisotropic, homogeneous or inhomogeneous von Kármán plates could have been considered. Then the bending term $K_j \Delta \Delta \zeta^{(j)}$ should be replaced by the corresponding more complicated terms describing orthotropy, etc.

In laminated or layered plates, the interlaminar normal stresses σ_{33} are mainly responsible for delamination effects in the normal direction to the interfaces [14]. In order to model this phenomenon, we use the method of [1], i.e., $f^{(j)}$ is split into $\bar{f}^{(j)}$, which describes the normal interaction of the two plates, and $\bar{\bar{f}}^{(j)} \in L^2(\Omega_j)$, which represents the external loading applied on the j th plate:

$$f^{(j)} = \bar{f}^{(j)} + \bar{\bar{f}}^{(j)} \quad \text{in } \Omega_j. \quad (2.4)$$

Let f denote the force in the interlaminar binding layer, say of the first two plates, i.e., (Fig. 1b)

$$f = \bar{f}^{(1)} = -\bar{f}^{(2)} \quad \text{in } \Omega'. \quad (2.5)$$

Then a phenomenological possibly multivalued and nonmonotone law connecting f with the corresponding relative deflection of the plates (cf. in [1] Fig. 2a)

$$[\zeta] = \zeta^{(1)} - \zeta^{(2)} \quad (2.6)$$

is introduced, i.e., that

$$-f \in \hat{\beta}([\zeta]) \quad \text{in } \Omega'. \quad (2.7)$$

To complete the definition of $\bar{f}^{(j)}$ we assume that

$$\bar{f}^{(j)} = 0 \quad \text{in } \Omega_j - \Omega', \quad j = 1, 2. \quad (2.8)$$

As in [1] relation (2.7) is put in a superpotential form. We assume that $\hat{\beta}: \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is defined in the following way [15]: Let $\beta \in L_{\text{loc}}^{\infty}(\mathbb{R})$ and let us consider the functions

$$\bar{\beta}_{\varepsilon}(\xi) = \text{ess sup}_{|\xi_1 - \xi| \leq \varepsilon} \beta(\xi_1) \quad \text{and} \quad \underline{\beta}_{\varepsilon}(\xi) = \text{ess inf}_{|\xi_1 - \xi| \leq \varepsilon} \beta(\xi_1). \quad (2.9)$$

They are increasing and decreasing functions of ε , respectively. Thus, the limits for $\varepsilon \rightarrow 0_+$ exist; let these limits be denoted by $\bar{\beta}(\xi)$ and $\underline{\beta}(\xi)$, respectively. The multivalued function $\hat{\beta}: \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is now defined as the interval

$$\hat{\beta}(\xi) = [\underline{\beta}(\xi), \bar{\beta}(\xi)]. \quad (2.10)$$

If $\beta(\xi_{\pm 0})$ exists for every $\xi \in \mathbb{R}$, then a locally Lipschitz (nonconvex) function $J: \mathbb{R} \rightarrow \mathbb{R}$ can be determined up to an additive constant [16], such that

$$\hat{\beta}(\xi) = \bar{\partial} J(\xi). \quad (2.11)$$

$\bar{\partial}$ is here the generalized gradient of F. H. Clarke [10]. Relation (2.7) can now be written in the form

$$-f \in \hat{\beta}([\zeta]) = \bar{\partial} J([\zeta]), \quad (2.12)$$

which by definition, is equivalent to the hemivariational inequality

$$J^0([\zeta], \xi - [\zeta]) \geq -f(\xi - [\zeta]), \quad \forall \xi \in \mathbb{R}. \quad (2.13)$$

In (2.13) $J^0(\cdot, \cdot)$ denotes the directional derivative in the sense of Clarke and J is the nonconvex superpotential of the adhesive contact between the plates. We refer to [1] for the definition of $J^0(\cdot, \cdot)$ and for the mechanical problems expressed by means of (2.12); in [10] the complete theory of the generalized gradient can be found.

From (2.1), assuming sufficiently regular functions, multiplying by $z^{(j)} - \zeta^{(j)}$, integrating and applying the Green–Gauss theorem, we get the expressions:

$$\begin{aligned} & \alpha(\zeta^{(j)}, z^{(j)} - \zeta^{(j)}) + \int_{\Omega_j} h_j \sigma_{\alpha\beta}^{(j)} \zeta_{,\alpha}^{(j)} (z^{(j)} - \zeta^{(j)})_{,\beta} d\Omega \\ &= \int_{\Gamma_j} h_j \sigma_{\alpha\beta}^{(j)} \zeta_{,\beta}^{(j)} n_{\alpha}^{(j)} (z^{(j)} - \zeta^{(j)}) d\Gamma \\ &+ \int_{\Omega_j} \bar{f}^{(j)} (z^{(j)} - \zeta^{(j)}) d\Omega \\ &+ \int_{\Gamma_j} \bar{Q}_j(\zeta^{(j)}) (z^{(j)} - \zeta^{(j)}) d\Gamma \\ &- \int_{\Gamma_j} M_j(\zeta^{(j)}) \frac{\partial}{\partial n^{(j)}} (z^{(j)} - \zeta^{(j)}) \quad \alpha, \beta = 1, 2. \end{aligned} \quad (2.14)$$

Here $n^{(j)}$ denotes the outward normal unit vector to Γ_j and (we omit the index j)

$$\alpha(\zeta, z) = K \int_{\Omega} [(1 - \nu) \zeta_{,\alpha\beta} z_{,\alpha\beta} + \nu \Delta \zeta \Delta z] d\Omega, \quad \alpha, \beta = 1, 2, \quad (2.15)$$

$$M(\zeta) = -K [\nu \Delta \zeta + (1 - \nu) (2n_1 n_2 \zeta_{,12} + n_1^2 \zeta_{,11} + n_2^2 \zeta_{,22})], \quad (2.16)$$

$$\bar{Q}(\zeta) = -K \left[\frac{\partial \Delta \zeta}{\partial n} + (1 - \nu) \frac{\partial}{\partial \tau} [n_1 n_2 (\zeta_{,22} - \zeta_{,11}) + (n_1^2 - n_2^2) \zeta_{,12}] \right]. \quad (2.17)$$

Here ν is the Poisson ratio, τ is the unit vector tangential to Γ , such that n , τ and the $0x_3$ -axis form a right-handed system for each plate, and M and $\bar{Q}$ are the bending moment and the total shearing force ($\bar{Q} = Q - \frac{\partial M}{\partial \tau}$, e.g. [17]) on the boundary Γ , respectively. Applying the same technique we get from (2.2) the expression

$$\int_{\Omega_j} \sigma_{\alpha\beta}^{(j)} \varepsilon_{\alpha\beta}^{(j)} (v^{(j)} - u^{(j)}) d\Omega = \int_{\Gamma_j} \sigma_{\alpha\beta}^{(j)} n_{\beta}^{(j)} (v_{\alpha}^{(j)} - u_{\alpha}^{(j)}) d\Gamma, \quad \alpha, \beta = 1, 2. \quad (2.18)$$

The following notations are used:

$$R(m, k) = \int_{\Omega} C_{\alpha\beta\gamma\delta} m_{\alpha\beta} k_{\gamma\delta} d\Omega, \quad \alpha, \beta, \gamma, \delta = 1, 2 \quad (2.19)$$

and

$$P(\zeta, z) = \{\zeta, z\}, \quad P(\zeta, \zeta) = P(\zeta), \quad (2.20)$$

where $m = \{m_{\alpha\beta}\}$ and $k = \{k_{\alpha\beta}\}$, $\alpha, \beta = 1, 2$ are 2×2 tensors. Let us also introduce the functional framework for the B.V.P. We assume that $u^{(j)}, v^{(j)} \in [H^1(\Omega_j)]^2$ and that $\zeta^{(j)}, z^{(j)} \in H^2(\Omega_j)$. For the in-plane stresses let the boundary condition

$$\sigma_{\alpha\beta}^{(j)} n_{\alpha}^{(j)} = 0 \quad \alpha, \beta = 1, 2 \quad (2.21)$$

hold. Moreover we assume for the deflexions $\zeta^{(j)}$ $j = 1, \dots, r$ such boundary conditions which guarantee the semicoerciveness of the bilinear forms $a(\zeta^{(j)}, \zeta^{(j)})$ [7], [4], i.e., they allow each plate to have a rigid-plate displacement, if it is considered independently of the other plates. We shall denote by Z_j the closed linear subspace of $H^2(\Omega_j)$ to which $\zeta^{(j)}, z^{(j)}$ belong. Obviously if the plate Ω_j is free, $Z_j = H^2(\Omega_j)$. Because of the semicoerciveness of the bilinear forms $\alpha(\cdot, \cdot)$, $\text{Ker } \alpha^{(j)} = \{q^{(j)} | q^{(j)} \in Z^{(j)} \alpha(q^{(j)}, q^{(j)}) = 0\} \neq \{0\}$. Then the bilinear form $\alpha(\zeta^{(j)}, \zeta^{(j)})$ is coercive on the quotient space $H^2(\Omega_j) / \text{Ker } \alpha^{(j)}$. Note that $\text{Ker } \alpha^{(j)}$ is the space of polynomials of degree ≤ 1 . It is well known from the plate theory (cf., e.g., [5]) that the norm $\|\zeta^{(j)}\|$ on $H^2(\Omega_j)$ is equivalent to the norm $\|\zeta^{(j)}\| = p(\tilde{\zeta}^{(j)}) + |q^{(j)}|_2$, where $|\cdot|_2$ denotes the $L^2(\Omega_j)$ -norm, $p(\zeta^{(j)}) = (\alpha(\zeta^{(j)}, \zeta^{(j)}))^{1/2}$ and $\zeta^{(j)} = \tilde{\zeta}^{(j)} + q^{(j)}$, $q^{(j)} \in \text{Ker } \alpha^{(j)}$, $\tilde{\zeta}^{(j)} \in \text{Ker } \alpha^{(j)\perp}$.

Now we take into account the notation (2.19), (2.20), the variational expressions (2.14) and (2.18), the boundary condition (2.21) as well as the interface conditions (2.5), (2.8), (2.12), and the inequality (2.13) which defines the generalized gradient $\bar{\partial}$. Thus we obtain from (2.14), if we formulate it for $j = 1, \dots, r$ and we add (in order to get the relative deflexions $[\zeta^{(m)}] = \zeta^{(m)} - \zeta^{(m+1)}$, $m = 1, \dots, m'$, where $m' = r - 1$) and from (2.18) the following Problem P_0 .

Problem P_0 . Find $\zeta^{(j)} \in Z_j$ and $u^{(j)} \in [H^1(\Omega_j)]^2$, $j = 1, \dots, r$, such as to satisfy the hemivariational inequality

$$\begin{aligned} & \sum_{j=1}^r \alpha_j(\zeta^{(j)}, z^{(j)} - \zeta^{(j)}) + \sum_{j=1}^r h_j R(\varepsilon(u^{(j)})) + \frac{1}{2} P(\zeta^{(j)}), P(\zeta^{(j)}, z^{(j)} - \zeta^{(j)}) \\ & + \sum_{m=1}^{m'=r-1} \int_{\Omega'_m} J_m^0([\zeta]^{(m)}, [z]^{(m)} - [\zeta]^{(m)}) d\Omega \geq \sum_{j=1}^r \int_{\Omega_j} \bar{f}^{(j)}(z^{(j)} - \zeta^{(j)}) d\Omega, \end{aligned} \quad (2.22)$$

$$\forall z^{(j)} \in Z_j, \quad j = 1, \dots, r$$

and the variational equalities

$$R(\varepsilon(u^{(j)}) + \frac{1}{2}P(\zeta^{(j)}), \varepsilon(v^{(j)} - u^{(j)})) = 0, \quad \forall v^{(j)} \in [H^1(\Omega_j)]^2, \quad j = 1, \dots, r. \quad (2.23)$$

First we note that $R(\cdot, \cdot)$ is a continuous, symmetric, coercive bilinear form on $[L^2(\Omega)]^4$, and that $P: [H^2(\Omega)]^2 \rightarrow [L^2(\Omega)]^4$ is a completely continuous operator (cf., e.g., [18], [6], [7]). Thus (2.23) and the Lax–Milgram theorem imply that to every deflexion $\zeta^{(j)} \in Z_j$, $j = 1, 2, \dots, r$ there corresponds a plane displacement $u^{(j)}(\zeta^{(j)}) \in [H^2(\Omega_j)]^2$. Indeed, due to Korn's inequality [19], $R(\varepsilon(u), \varepsilon(v))$ is a bilinear coercive form on the quotient space $[H^1(\Omega)]^2/\bar{R}$, where $\bar{R}$ is the space of in-plane rigid-plate displacements defined by:

$$\bar{R} = \{\bar{r} \mid \bar{r} \in [H^1(\Omega)]^2, \bar{r}_1 = a_1 + bx_2, \bar{r}_2 = a_2 - bx_2, a_1, a_2, b \in \mathbb{R}\}.$$

From (2.23) we also obtain that

$$\varepsilon^{(j)}(u^{(j)}(\zeta^{(j)})): Z_j \rightarrow [L^2(\Omega_j)]^4$$

is uniquely determined and is a completely continuous quadratic function of $\zeta^{(j)}$, $j = 1, 2, \dots, r$, since $\varepsilon^{(j)}(u^{(j)}(\zeta^{(j)}))$ is a linear continuous function of $P(\zeta^{(j)})$. Now the completely continuous, quadratic functions $G_j: Z_j \rightarrow [L^2(\Omega_j)]^4$ defined by

$$\zeta^{(j)} \rightarrow G_j(\zeta^{(j)}) = \varepsilon^{(j)}(u^{(j)}(\zeta^{(j)})) + \frac{1}{2}P(\zeta^{(j)}) \quad (2.24)$$

are introduced [7]. Let us now define the operators $A_j: Z_j \rightarrow Z'_j$ and $C_j: Z_j \rightarrow Z'_j$ such that

$$\alpha(\zeta^{(j)}, z^{(j)}) = \langle A_j \zeta^{(j)}, z^{(j)} \rangle \quad (2.25)$$

and

$$h_j R(G_j(\zeta^{(j)}), P(\zeta^{(j)}, z^{(j)})) = \langle C_j(\zeta^{(j)}), z^{(j)} \rangle. \quad (2.26)$$

The A_j are linear continuous monotone operators, C_j are completely continuous operators, and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between Z_j and Z'_j . Moreover due to (2.23), (2.26) implies that

$$\langle C_j(\zeta^{(j)}), \zeta^{(j)} \rangle = h_j R(G_j(\zeta^{(j)}), 2G_j(\zeta^{(j)})) \geq 0, \quad \forall \zeta^{(j)} \in Z_j, \quad j = 1, 2, \dots, r. \quad (2.27)$$

Since the $\alpha_j(\zeta^{(j)}, \zeta^{(j)})$ are semicoercive on Z_j we obtain

$$\begin{aligned} \langle (A_j + C_j)(\zeta^{(j)}), \zeta^{(j)} \rangle &= \langle T_j(\zeta^{(j)}), \zeta^{(j)} \rangle \geq c[p(\zeta^{(j)})]^2 \\ \forall \zeta^{(j)} \in Z_j, \quad j &= 1, 2, \dots, r, \quad c \text{ const} > 0. \end{aligned} \quad (2.28)$$

Moreover it can be verified [7] that

$$\langle T_j(\zeta^{(j)}), q^{(j)} \rangle = 0 \quad \forall q^{(j)} \in \text{Ker } \alpha^{(j)}. \quad (2.29)$$

After the elimination of the in-plane displacements Problem P_0 takes the following

form:

Problem P. Find $\zeta^{(j)} \in Z_j$, $j = 1, 2, \dots, r$ so as to satisfy the hemivariational inequality:

$$\begin{aligned} & \sum_{j=1}^r \langle T_j(\zeta^{(j)}), z^{(j)} - \zeta^{(j)} \rangle + \sum_{m=1}^{m'} \int_{\Omega'_m} J_m^0([\zeta]^{(m)}, [z]^{(m)} - [\zeta]^{(m)}) d\Omega \\ & \geq \sum_{j=1}^r \left(\bar{f}^{(j)}, z^{(j)} - \zeta^{(j)} \right), \quad \forall z^{(j)} \in Z_j. \end{aligned} \quad (2.30)$$

Here $(\cdot, \cdot)$ denotes the $L^2(\Omega_j)$ -product. Further a necessary and sufficient condition for the existence of the solution is proved.

3. Existence and approximation of the solution. The following proposition gives the necessary condition for the existence of the solution. We introduce the notations

$$\beta^{(m)}(-\infty) = \limsup_{\xi \rightarrow -\infty} \beta^{(m)}(\xi) \quad \text{and} \quad \beta^{(m)}(\infty) = \liminf_{\xi \rightarrow \infty} \beta^{(m)}(\xi) \quad (3.1)$$

for $m = 1, 2, \dots, m' = r - 1$. Moreover $[q]^m = q^{(m)} - q^{(m+1)}$ $m = 1, \dots, r - 1$ and $[q]_+$ (resp. $[q]_-$) denotes the positive (resp. negative) part of $[q]$, i.e., $[q]_+ = \frac{[q] + |[q]|}{2}$, $[q]_- = \frac{[q] - |[q]|}{2}$.

PROPOSITION 3.1. Let

$$\beta^{(m)}(-\infty) \leq \beta^{(m)}(\xi) \leq \beta^{(m)}(\infty) \quad \forall \xi \in \mathbb{R}; m = 1, \dots, r - 1. \quad (3.2)$$

Then a necessary condition for the existence of a solution $\zeta^{(j)} \in Z_j$ $j = 1, \dots, r$ of Problem P is the inequality

$$\begin{aligned} & \sum_{m=1}^{m'} \left(\int_{\Omega'_m} (\beta^{(m)}(-\infty)[q]_+^{(m)} - \beta^{(m)}(\infty)[q]_-^{(m)}) d\Omega \right) \\ & \leq \sum_{j=1}^r \left(\bar{f}^{(j)}, q^{(j)} \right) \\ & \leq \sum_{m=1}^{m'} \left(\int_{\Omega'_m} (\beta^{(m)}(\infty)[q]_+^{(m)} - \beta^{(m)}(-\infty)[q]_-^{(m)}) d\Omega \right) \\ & \quad \forall q^{(j)} \in \text{Ker } \alpha^{(j)} \quad j = 1, \dots, r. \end{aligned} \quad (3.3)$$

If at least one inequality in (3.2) holds strictly (with $<$ instead of $\leq$) the same happens in (3.3).

Proof. Let us set in (2.30) $z^{(j)} - \zeta^{(j)} = \pm q^{(j)} \in \text{Ker } \alpha^{(j)}$. We obtain

$$\begin{aligned} & \sum_{m=1}^{m'} \int_{\Omega'_m} J_m^0([\zeta]^{(m)}, \pm [q]^{(m)}) d\Omega \geq \sum_{j=1}^r \left(\bar{f}^{(j)}, \pm q^{(j)} \right) \\ & \quad \forall q^{(j)} \in \text{Ker } \alpha^{(j)}, \quad q^{(j)} \neq 0, \quad j = 1, \dots, r \end{aligned} \quad (3.4)$$

which implies due to the fact that $q \rightarrow J^0(\xi, q)$ is positively homogeneous that

$$\begin{aligned} \sum_{m=1}^{m'} \int_{\Omega'_m} J_m^0([\zeta]^{(m)}, [q]^{(m)}) d\Omega &\geq \sum_{j=1}^r \left(\bar{f}^{(j)}, q^{(j)} \right) \\ &\geq - \sum_{m=1}^{m'} \int_{\Omega'_m} J_m^0([\zeta]^{(m)}, [q]^{(m)}) d\Omega \\ &\quad \forall q^{(j)} \in \text{Ker } \alpha^{(j)}, q^{(j)} \neq 0, j = 1, \dots, r. \end{aligned} \quad (3.5)$$

From the definition of J_m and J_m^0 for $m = 1, \dots, r-1$ and from (3.2) we obtain that

$$\begin{aligned} \int_{\Omega'_m} J_m^0([\zeta]^{(m)}, [q]^{(m)}) d\Omega &= \int_{\Omega'_m} \limsup_{\substack{\lambda \rightarrow 0+ \\ h \rightarrow 0}} \frac{1}{\lambda} \int_{[\zeta]^{(m)}+h}^{[\zeta]^{(m)}+h+\lambda[q]^{(m)}} \beta^{(m)}(t) dt \\ &= \int_{[q]^{(m)}>0} \dots d\Omega + \int_{[q]^{(m)}<0} \dots d\Omega \\ &\leq \int_{[q]^{(m)}>0} \beta^{(m)}(\infty)[q]^{(m)} d\Omega \\ &\quad + \int_{[q]^{(m)}<0} \beta^{(m)}(-\infty)[q]^{(m)} d\Omega \\ &= \int_{\Omega'_m} (\beta^{(m)}(\infty)[q]_+^{(m)} - \beta^{(m)}(-\infty)[q]_-^{(m)}) d\Omega. \end{aligned} \quad (3.6)$$

Accordingly

$$\begin{aligned} \sum_{j=1}^r \left(\bar{f}^{(j)}, q^{(j)} \right) &\leq \sum_{m=1}^{m'} \left(\int_{\Omega'_m} (\beta^{(m)}(\infty)[q]_+^{(m)} - \beta^{(m)}(-\infty)[q]_-^{(m)}) d\Omega \right) \\ &\quad \forall q^{(j)} \in \text{Ker } \alpha^{(j)}, q^{(j)} \neq 0, j = 1, \dots, r \end{aligned} \quad (3.7)$$

and analogously for $-\int J_m^0([\zeta]^{(m)}, [q]) d\Omega$. Thus (3.3) is shown. The rest of the proposition is obvious. q.e.d.

Further a sufficient condition will be given. To this end we need to formulate the regularised Problem P_ε : Let p be a mollifier ($p \in C_c^\infty(-1, +1)$, $p \geq 0$ with $\int_{-\infty}^{+\infty} p(\xi) d\xi = 1$), and let

$$\beta_\varepsilon^{(m)} = p_\varepsilon * \beta^{(m)}, \quad \varepsilon > 0, m = 1, \dots, m' \quad (3.8)$$

where $p_\varepsilon(\xi) = (1/\varepsilon)p(\xi/\varepsilon)$ and $*$ denotes the convolution product. The regularized Problem P_ε reads:

Problem P_ε . Find $\zeta_\varepsilon^{(j)} \in Z_j$, $j = 1, \dots, r$, such as to satisfy the variational equality

$$\sum_{j=1}^r \langle T_j(\zeta_\varepsilon^{(j)}), z^{(j)} \rangle + \sum_{m=1}^{m'} \int_{\Omega'_m} \beta_\varepsilon^{(m)}([\zeta_\varepsilon]^{(m)})[z]^{(m)} d\Omega = \sum_{j=1}^r \left(\bar{f}^{(j)}, z^{(j)} \right), \quad \forall z^{(j)} \in Z_j. \quad (3.9)$$

Further we introduce a Galerkin basis for each one of the spaces Z_j , $j = 1, \dots, r$; let Z_{jn} denote the corresponding n -dimensional subspace of Z_j . Then from Problem P_ε the finite dimensional Problem $P_{\varepsilon n}$ results.

Problem $P_{\varepsilon n}$. Find $\zeta_{\varepsilon n}^{(j)} \in Z_{jn}$, $j = 1, 2, \dots, r$, such that

$$\sum_{j=1}^r \langle T_j(\zeta_{\varepsilon n}^{(j)}), z^{(j)} \rangle + \sum_{m=1}^{m'} \int_{\Omega'_m} \beta_\varepsilon^{(m)}([\zeta_{\varepsilon n}]^{(m)})[z]^{(m)} d\Omega = \sum_{j=1}^r (\bar{f}^{(j)}, z^{(j)}),$$

$$\forall z^{(j)} \in Z_{jn}, \quad j = 1, \dots, r. \quad (3.10)$$

PROPOSITION 3.2. Let

$$\beta^{(m)}(-\infty) < \beta^{(m)}(\infty) \quad m = 1, \dots, m'. \quad (3.11)$$

Then if

$$\begin{aligned} & \sum_{m=1}^{m'} \left(\int_{\Omega'_m} (\beta^{(m)}(-\infty)[q]_+^{(m)} - \beta^{(m)}(\infty)[q]_-^{(m)}) d\Omega \right) \\ & < \sum_{j=1}^r (\bar{f}^{(j)}, q^{(j)}) \\ & < \sum_{m=1}^{m'} \left(\int_{\Omega'_m} (\beta^{(m)}(\infty)[q]_+^{(m)} - \beta^{(m)}(-\infty)[q]_-^{(m)}) d\Omega \right) \end{aligned} \quad (3.12)$$

$$\forall q^{(j)} \in \text{Ker } \alpha^{(j)}, q^{(j)} \neq 0, \quad j = 1, \dots, r$$

Problem P has a solution.

Proof. From (3.11) we have that for some $\xi \in \mathbb{R}$

$$\sup_{(-\infty, -\xi)} \beta_\varepsilon^{(m)}(\xi) \leq \inf_{(\xi, \infty)} \beta_\varepsilon^{(m)}(\xi). \quad (3.13)$$

Thus we may determine for every m , $\rho_1^{(m)} > 0$ and $\rho_2^{(m)} > 0$ such that $\beta_\varepsilon^{(m)}(\xi) \geq 0$ if $\xi > \rho_1^{(m)}$, $\beta_\varepsilon^{(m)}(\xi) \leq 0$ if $\xi < -\rho_1^{(m)}$ and $|\beta_\varepsilon^{(m)}(\xi)| \leq \rho_2^{(m)}$ if $|\xi| \leq \rho_1^{(m)}$ and thus

$$\begin{aligned} \int_{\Omega'_m} \beta_\varepsilon^{(m)}([\zeta_{\varepsilon n}]^{(m)})[\zeta_{\varepsilon n}]^{(m)} d\Omega &= \int_{|[\zeta_{\varepsilon n}]^{(m)}| > \rho_1^{(m)}} \dots d\Omega + \int_{|[\zeta_{\varepsilon n}]^{(m)}| \leq \rho_1^{(m)}} \dots d\Omega \\ &\geq 0 - \rho_1^{(m)} \rho_2^{(m)} \text{mes } \Omega'_m. \end{aligned} \quad (3.14)$$

This estimate which was applied to the coercive problem [1] will now be ameliorated for the semicoercive problem. First we write (3.10) in the form

$$\langle \Lambda(\bar{\zeta}'_{\varepsilon n}), \bar{z}' \rangle = 0, \quad \text{for } \bar{z}' = \{z^{(j)}\}, \quad \forall z^{(j)} \in Z_{jn}, \quad j = 1, 2, \dots, r. \quad (3.15)$$

From (3.15), (3.14), and (2.27) we find that

$$\begin{aligned} \langle \Lambda(\bar{\zeta}'_{\varepsilon n}), \bar{\zeta}'_{\varepsilon n} \rangle &\geq c \sum_{j=1}^r [p(\bar{\zeta}_{\varepsilon n}^{(j)})]^2 - c \sum_{j=1}^r \|\bar{\zeta}_{\varepsilon n}^{(j)}\| \\ &\quad - \sum_{m=1}^{m'} \rho_1^{(m)} \rho_2^{(m)} \text{mes } \Omega'_m, \quad c \text{ const} > 0. \end{aligned} \quad (3.16)$$

Now we will apply Brouwer's theorem to show that (3.15) has at least one solution $\bar{\zeta}'_{\varepsilon n}$ and that $\{\|\bar{\zeta}_{\varepsilon n}^{(j)}\|\}$ is bounded for every j . According to this theorem (cf. [20] p. 53) it suffices to show that a number $M > 0$ can be determined such that

$$\|\bar{\zeta}_{\varepsilon n}^{(j)}\| > M \quad j = 1, \dots, r \quad \text{implies } \langle \Lambda(\bar{\zeta}'_{\varepsilon n}), \bar{\zeta}'_{\varepsilon n} \rangle > 0. \quad (3.17)$$

For the proof of (3.17) it is sufficient to prove that

$$(\Lambda(\bar{\zeta}'_{\varepsilon n}), \bar{\zeta}'_{\varepsilon n}) \leq 0 \quad \text{implies} \quad \|\zeta_{\varepsilon n}^{(j)}\| \leq c \quad j = 1, \dots, r. \quad (3.18)$$

Due to (3.16) we deduce that, if $(\Lambda(\bar{\zeta}'_{\varepsilon n}), \bar{\zeta}'_{\varepsilon n}) \leq 0$, then a constant $c > 0$ exists such that

$$\left(\sum_{j=1}^r p(\bar{\zeta}_{\varepsilon n}^{(j)}) \right)^2 \leq c \sum_{j=1}^r |q_{\varepsilon n}^{(j)}|_2 + c \quad (3.19)$$

where $\zeta_{\varepsilon n}^{(j)} = \bar{\zeta}_{\varepsilon n}^{(j)} + q_{\varepsilon n}^{(j)}$. Accordingly it is sufficient to prove that $(\Lambda(\bar{\zeta}'_{\varepsilon n}), \bar{\zeta}'_{\varepsilon n}) \leq 0$ implies $|q_{\varepsilon n}^{(j)}|_2 \leq c$, $j = 1, \dots, r$ or equivalently, that a number $R > 0$ can be determined such that

$$|q_{\varepsilon n}^{(j)}|_2 > R \quad j = 1, \dots, r \quad \text{and} \quad (3.19) \Rightarrow (\Lambda(\bar{\zeta}'_{\varepsilon n}), \bar{\zeta}'_{\varepsilon n}) > 0. \quad (3.20)$$

This last relation will be shown now. From the definition of $\beta_\varepsilon^{(m)}$ we prove that

$$\begin{aligned} \beta_\varepsilon^{(m)}(\infty) &= \lim_{\xi \rightarrow \infty} \beta_\varepsilon^{(m)}(\xi) = \lim_{\xi \rightarrow \infty} \int_{-\varepsilon}^{+\varepsilon} \beta^{(m)}(\xi - t) p_\varepsilon(t) dt \geq \lim_{\xi \rightarrow \infty} \operatorname{ess\,inf}_{|x - \xi| \leq \varepsilon} \beta^{(m)}(x) \\ &\geq \lim_{\xi \rightarrow \infty} \operatorname{ess\,inf}_{\xi - \varepsilon \leq x < \infty} \beta^{(m)}(x) = \lim_{x \rightarrow \infty} \inf \beta^{(m)}(x) = \beta^{(m)}(\infty). \end{aligned} \quad (3.21)$$

Similarly we show that $\beta_\varepsilon^{(m)}(-\infty) \leq \beta^{(m)}(-\infty)$. Thus (3.12) implies that

$$\begin{aligned} &\sum_{m=1}^{m'} \left(\int_{\Omega'_m} (\beta_\varepsilon^{(m)}(-\infty)[q]_+^{(m)} - \beta_\varepsilon^{(m)}(\infty)[q]_-^{(m)}) d\Omega \right) \\ &< \sum_{j=1}^r \left(\bar{f}^{(j)}, q^{(j)} \right) \\ &< \sum_{m=1}^{m'} \left(\int_{\Omega'_m} (\beta_\varepsilon^{(m)}(\infty)[q]_+^{(m)} - \beta_\varepsilon^{(m)}(-\infty)[q]_-^{(m)}) d\Omega \right), \\ &\quad \forall q^{(j)} \in \operatorname{Ker} \alpha^{(j)}, \quad q^{(j)} \neq 0, \quad j = 1, \dots, r. \end{aligned} \quad (3.22)$$

Due to (3.11) numbers $\tilde{M}^{(m)} > \rho_1^{(m)}$ (cf. (3.14)) can be chosen such that for functions $\bar{\zeta}^{(j)} \in Z_j$ with $[\bar{\zeta}(x)]^{(m)} > \tilde{M}^{(m)}$ and $\operatorname{sign}[\bar{\zeta}(x)]^{(m)} = \operatorname{sign}[q(x)]^{(m)}$ for almost every $x \in \Omega'_m$, we have from (3.22) that for $[q]^{(m)} > 0$, $[q]^{(m)} = [q]_+^{(m)}$, $[\bar{\zeta}(x)]^{(m)} > \tilde{M}^{(m)}$, $m = 1, \dots, m' = r - 1$,

$$\sum_{m=1}^{m'} \int_{\{x \mid [q(x)]^{(m)} > 0\}} \beta_\varepsilon^{(m)}([\bar{\zeta}(x)]^{(m)})[q(x)]^{(m)} d\Omega - \sum_{j=1}^r \left(\bar{f}^{(j)}, q^{(j)} \right) > 0, \quad (3.23)$$

$$\sum_{m=1}^{m'} \int_{\{x \mid [q(x)]^{(m)} > 0\}} -\beta_\varepsilon^{(m)}(-[\bar{\zeta}(x)]^{(m)})[q(x)]^{(m)} d\Omega + \sum_{j=1}^r \left(\bar{f}^{(j)}, q^{(j)} \right) > 0. \quad (3.24)$$

For $[q]^{(m)} < 0$ we have $[q]^{(m)} = -[q]_-^{(m)}$, $[\bar{\zeta}(x)]^{(m)} < -\tilde{M}^{(m)}$ and thus we obtain from (3.22)

$$\sum_{m=1}^{m'} \int_{\{x \mid [q(x)]^{(m)} < 0\}} \beta_\varepsilon^{(m)}([\bar{\zeta}(x)]^{(m)})[q(x)]^{(m)} d\Omega - \sum_{j=1}^r \left(\bar{f}^{(j)}, q^{(j)} \right) > 0, \quad (3.25)$$

$$\sum_{m=1}^{m'} \int_{\{x \mid |q(x)|^{(m)} < 0\}} -\beta_\varepsilon^{(m)}(-[\bar{\zeta}(x)]^{(m)})[q(x)]^{(m)} d\Omega + \sum_{j=1}^r (\bar{f}^{(j)}, q^{(j)}) > 0. \quad (3.26)$$

By an appropriate choice of the numbers $\delta^{(m)} \in (0, 1]$, $N^{(m)} > 1$, $\eta > 0$, and $\alpha^{(m)} > 0$, and by taking into account that $\beta_\varepsilon^{(m)}([\bar{\zeta}(x)]^{(m)})[\bar{\zeta}(x)]^{(m)} \geq 0$ and that $\text{sign}[\bar{\zeta}(x)]^{(m)} = \text{sign}[q(x)]^{(m)}$, $m = 1, \dots, m'$, these inequalities imply for every $[\bar{\zeta}]^{(m)}$ as above, the relations

$$\begin{aligned} & \sum_{m=1}^{m'} \left(1 - \frac{1}{N^{(m)}}\right) \int_{\{x \mid |q(x)|^{(m)} > \delta^{(m)} \alpha^{(m)}\}} \beta_\varepsilon^{(m)}([\bar{\zeta}(x)]^{(m)})[q(x)]^{(m)} d\Omega - \sum_{j=1}^r (f^{(j)}, q^{(j)}) \\ & > \sum_{j=1}^r \eta |q^{(j)}|_2 \end{aligned} \quad (3.27)$$

$$\begin{aligned} & \sum_{m=1}^{m'} - \left(1 - \frac{1}{N^{(m)}}\right) \int_{\{x \mid |q(x)|^{(m)} > \delta^{(m)} \alpha^{(m)}\}} \beta_\varepsilon^{(m)}(-[\bar{\zeta}(x)]^{(m)})[q(x)]^{(m)} d\Omega + \sum_{j=1}^r (f^{(j)}, q^{(j)}) \\ & > \sum_{j=1}^r \eta |q^{(j)}|_2 \end{aligned} \quad (3.28)$$

as is obvious by taking $\delta^{(m)} \rightarrow 0$. We take now $\zeta_{\varepsilon n}^{(j)} = \bar{\zeta}_{\varepsilon n}^{(j)} + q_{\varepsilon n}^{(j)}$. Then for $N^{(m)}$ as in (3.27), (3.28) and for $\alpha^{(m)} > \alpha_0^{(m)} = \tilde{M}^{(m)} \delta^{-1(m)} (1 - \frac{1}{N^{(m)}})^{-1}$ for $m = 1, \dots, m'$ we have (we omit the index (m) in (3.29)–(3.32) for the sake of simplicity)

$$\begin{aligned} \int_{\Omega'_m} \beta_\varepsilon([\zeta_{\varepsilon n}])[\zeta_{\varepsilon n}] d\Omega &= \int \dots + \int \dots + \int \dots \\ & \quad \begin{array}{l} |[\zeta_{\varepsilon n}(x)]| < \frac{\delta\alpha}{N} \quad |[\zeta_{\varepsilon n}(x)]| \geq \frac{\delta\alpha}{N} \quad |[\zeta_{\varepsilon n}(x)]| \geq \delta\alpha \\ |[\zeta_{\varepsilon n}(x)]| > \delta\alpha \quad |[\zeta_{\varepsilon n}(x)]| > \delta\alpha \end{array} \\ & \geq \int_{\substack{|[\zeta_{\varepsilon n}(x)]| < \frac{\delta\alpha}{N} \\ |[\zeta_{\varepsilon n}(x)]| > \delta\alpha}} \beta_\varepsilon([\zeta_{\varepsilon n}])[\zeta_{\varepsilon n}] d\Omega - \rho_1 \rho_2 \text{mes } \Omega'_m \\ & \geq \left(1 - \frac{1}{N}\right) \int_{\substack{|[\zeta_{\varepsilon n}(x)]| < \frac{\delta\alpha}{N} \\ |[\zeta_{\varepsilon n}(x)]| > \delta\alpha}} [\zeta_{\varepsilon n}] \beta_\varepsilon([\zeta_{\varepsilon n}]) d\Omega - \rho_1 \rho_2 \text{mes } \Omega'_m. \end{aligned} \quad (3.29)$$

Indeed for $|[\zeta_{\varepsilon n}(x)]| < \delta\alpha/N$ and $|[\zeta_{\varepsilon n}(x)]| > \delta\alpha$ we have that for $\alpha > \alpha_0$

$$|[\zeta_{\varepsilon n}]| = |[\bar{\zeta}_{\varepsilon n} + q_{\varepsilon n}]| > \left(1 - \frac{1}{N}\right) \delta\alpha > \tilde{M} \quad (3.30)$$

and thus $\beta_\varepsilon([\zeta_{\varepsilon n}])[\zeta_{\varepsilon n}] \geq 0$, and $\beta_\varepsilon([\zeta_{\varepsilon n}])[q_{\varepsilon n}] \geq 0$. Further it can be verified that for $[q_{\varepsilon n}(x)] > \delta\alpha$

$$[\zeta_{\varepsilon n}(x)] = [\bar{\zeta}_{\varepsilon n}(x)] + [q_{\varepsilon n}(x)] > -\frac{\delta\alpha}{N} + [q_{\varepsilon n}(x)] > [q_{\varepsilon n}(x)] \left(1 - \frac{1}{N}\right) \quad (3.31)$$

and therefore

$$\beta_\varepsilon([\zeta_{\varepsilon n}])[\zeta_{\varepsilon n}] \geq \left(1 - \frac{1}{N}\right) [q_{\varepsilon n}] \beta_\varepsilon([\zeta_{\varepsilon n}]). \quad (3.32)$$

Similarly for $[q_{\varepsilon n}^{(x)}] < -\delta\alpha$. Accordingly we obtain from (3.10) the inequality

$$\begin{aligned} \langle \Lambda(\bar{\zeta}'_{\varepsilon n}), \bar{\zeta}'_{\varepsilon n} \rangle &\geq \sum_{m=1}^{m'} \left(1 - \frac{1}{N^{(m)}} \right) \\ &\quad \times \int_{\substack{||[\zeta_{\varepsilon n}(x)]^{(m)}| < \frac{\delta^{(m)}\alpha^{(m)}}{N^{(m)}} [q_{\varepsilon n}]^{(m)} \beta_{\varepsilon}^{(m)} ([\zeta_{\varepsilon n}]^{(m)}) d\Omega \\ ||[q_{\varepsilon n}(x)]^{(m)}| > \delta^{(m)}\alpha^{(m)}}} \\ &\quad - \sum_{m=1}^{m'} \rho_1^{(m)} \rho_2^{(m)} \text{mes } \Omega'_m - \sum_{j=1}^r \left(\bar{f}^{(j)}, q_{\varepsilon n}^{(j)} \right) - \sum_{j=1}^r \left(\bar{f}^{(j)}, \tilde{\zeta}_{\varepsilon n}^{(j)} \right). \end{aligned} \quad (3.33)$$

By taking into account (3.27) and (3.28) we obtain for $\alpha^{(m)} > \alpha_0^{(m)}$ sufficiently large from (3.33), that

$$\begin{aligned} \langle \Lambda(\bar{\zeta}'_{\varepsilon n}), \bar{\zeta}'_{\varepsilon n} \rangle &> \sum_{j=1}^r \eta |q_{\varepsilon n}^{(j)}|_2 - \sum_{m=1}^{m'} \rho_1^{(m)} \rho_2^{(m)} \text{mes } \Omega'_m - \sum_{j=1}^r \left(\bar{f}^{(j)}, \tilde{\zeta}_{\varepsilon n}^{(j)} \right) \\ &\geq \sum_{j=1}^r \eta |q_{\varepsilon n}^{(j)}|_2 - c_1 - c_2 \sum_{j=1}^r \|\tilde{\zeta}_{\varepsilon n}^{(j)}\| \\ &\geq \sum_{j=1}^r \eta |q_{\varepsilon n}^{(j)}|_2 - c_1 - c'_2 \sum_{j=1}^r \|\tilde{\zeta}_{\varepsilon n}^{(j)}\| \\ &\geq \sum_{j=1}^r \eta |q_{\varepsilon n}^{(j)}|_2 - c_1 - c'_2 \sum_{j=1}^r \|\zeta_{\varepsilon n}^{(j)}\| \\ &= \sum_{j=1}^r \eta |q_{\varepsilon n}^{(j)}|_2 - c_1 - c'_2 \sum_{j=1}^r p(\zeta_{\varepsilon n}^{(j)}) \quad c_1, c_2, c'_2 \text{ const} > 0. \end{aligned} \quad (3.34)$$

From (3.34) we obtain due to (3.19) the final estimate

$$\langle \Lambda(\bar{\zeta}'_{\varepsilon n}), \bar{\zeta}'_{\varepsilon n} \rangle > \sum_{j=1}^r \eta |q_{\varepsilon n}^{(j)}|_2 - c_1 - \sqrt{c \sum_{j=1}^r |q_{\varepsilon n}^{(j)}|_2 + c}, \quad c, c_1 \text{ const} > 0. \quad (3.35)$$

The right-hand side of (3.35) is positive for $|q_{\varepsilon n}^{(j)}|_2 > R$ $j = 1, \dots, r$ where R is an appropriately large number. Thus (3.20) has been proved. Thus by Brouwer's fixed point theorem, Problem $P_{\varepsilon n}$ has a solution $\zeta_{\varepsilon n}^{(j)}$ with $\|\zeta_{\varepsilon n}^{(j)}\| < c$ $j = 1, \dots, r$. Now we may determine subsequences, again denoted by $\{\zeta_{\varepsilon n}^{(j)}\}$, $j = 1, \dots, r$, such that for $\varepsilon \rightarrow 0$, $n \rightarrow \infty$

$$\zeta_{\varepsilon n}^{(j)} \rightarrow \zeta^{(j)} \quad \text{weakly in } Z_j, \quad j = 1, \dots, r, \quad (3.36)$$

and because of the compact imbeddings $H^2(\Omega_j) \hookrightarrow \text{CL}^2(\Omega_j)$, $j = 1, \dots, r$

$$\zeta_{\varepsilon n}^{(j)} \rightarrow \zeta^{(j)} \quad \text{strongly in } L^2(\Omega_j), \quad j = 1, \dots, r. \quad (3.37)$$

Accordingly

$$\zeta_{\varepsilon n}^{(j)} \rightarrow \zeta^{(j)} \quad \text{a.e. in } \Omega_j, \quad j = 1, \dots, r. \quad (3.38)$$

Further we may omit the indices (j) and (m) if no ambiguity occurs. Now we prove that $\beta_\varepsilon^{(m)}([\zeta_{\varepsilon n}]^{(m)})$ is weakly precompact in $L^1(\Omega'_m)$, $m = 1, \dots, m'$: By the Dunford–Pettis theorem (e.g., [21] p. 239) it will be shown that, for each $\mu_m > 0$, a $\delta_m(\mu_m) > 0$ can be determined such that for $\omega_m \subset \Omega'_m$ with $\text{mes } \omega_m < \delta_m$.

$$\int_{\omega_m} |\beta_\varepsilon^{(m)}([\zeta_{\varepsilon n}]^{(m)})| d\Omega_m < \mu_m. \quad (3.39)$$

The inequality

$$\xi_0 |\beta_\varepsilon(\xi)| \leq |\beta_\varepsilon(\xi)\xi| + \xi_0 \sup_{|\xi| \leq \xi_0} |\beta_\varepsilon(\xi)| \quad (3.40)$$

implies that

$$\begin{aligned} \int_{\omega} |\beta_\varepsilon([\zeta_{\varepsilon n}])| d\Omega &\leq \frac{1}{\xi_0} \int_{\omega} |\beta_\varepsilon([\zeta_{\varepsilon n}])[\zeta_{\varepsilon n}]| d\Omega \\ &\quad + \int_{\omega} \sup_{|[\zeta_{\varepsilon n}(x)]| \leq \xi_0} |\beta_\varepsilon([\zeta_{\varepsilon n}])| d\Omega. \end{aligned} \quad (3.41)$$

But

$$\begin{aligned} &\int_{\Omega'} |\beta_\varepsilon([\zeta_{\varepsilon n}])[\zeta_{\varepsilon n}]| d\Omega \\ &= \int_{|[\zeta_{\varepsilon n}(x)]| > \rho_1} |\dots| d\Omega + \int_{|[\zeta_{\varepsilon n}(x)]| \leq \rho_1} |\dots| d\Omega \\ &= \int_{|[\zeta_{\varepsilon n}(x)]| > \rho_1} |\dots| d\Omega - \int_{|[\zeta_{\varepsilon n}(x)]| \leq \rho_1} |\dots| d\Omega + 2 \int_{|[\zeta_{\varepsilon n}(x)]| \leq \rho_1} |\dots| d\Omega \quad (3.42) \\ &\leq \int_{|[\zeta_{\varepsilon n}(x)]| > \rho_1} |\dots| d\Omega + \int_{|[\zeta_{\varepsilon n}(x)]| \leq \rho_1} |\dots| d\Omega + 2 \int_{|[\zeta_{\varepsilon n}(x)]| \leq \rho_1} |\dots| d\Omega \\ &= \int_{\Omega'} \beta_\varepsilon([\zeta_{\varepsilon n}])[\zeta_{\varepsilon n}] d\Omega + 2 \int_{|[\zeta_{\varepsilon n}(x)]| \leq \rho_1} |\beta_\varepsilon([\zeta_{\varepsilon n}])[\zeta_{\varepsilon n}]| d\Omega. \end{aligned}$$

Now let us set $z^{(1)} = \tilde{z}^{(1)} \in Z_{1n}$ in (3.10) such that $\tilde{z}^{(1)} = z^{(2)}$ on Ω'_1 , $z^{(2)} = \tilde{z}^{(2)} \in Z_{2n}$ such that $\tilde{z}^{(2)} = z^{(3)}$ on $\Omega'_2, \dots, z^{(j-1)} = \tilde{z}^{(j-1)} \in Z_{(j-1)n}$ such that $\tilde{z}^{(j-1)} = \zeta_{\varepsilon n}^{(j)}$ on $\Omega'_{(j-1)}$, $z^{(j)} = \zeta_{\varepsilon n}^{(j)}$ on Ω_j , $z^{(j+1)} = \zeta_{\varepsilon n}^{(j+1)}$ on Ω_{j+1} , $z^{(j+2)} = \tilde{z}^{(j+2)} \in Z_{(j+2)n}$ such that $\tilde{z}^{(j+2)} = \zeta_{\varepsilon n}^{(j+1)}$ on Ω'_{j+1} and finally $z^{(r)} = \tilde{z}^{(r)} \in Z_{rn}$ such that $\tilde{z}^{(r)} = z^{(r-1)}$ on Ω'_{r-1} . Thus, using (3.42) we obtain

$$\begin{aligned} &\int_{\Omega'_m} |\beta_\varepsilon^{(m)}([\zeta_{\varepsilon n}]^{(m)})[\zeta_{\varepsilon n}]^{(m)}| d\Omega \\ &\leq \sum_{j=1}^r (\bar{f}^{(j)}, \zeta_{\varepsilon n}^{(j)}) - \sum_{r=1}^r \langle T_j(\zeta_{\varepsilon n}^{(j)}), \zeta_{\varepsilon n}^{(j)} \rangle \\ &\quad + 2 \int_{|[\zeta_{\varepsilon n}(x)]^{(m)}| \leq \rho_1^{(m)}} |\beta_\varepsilon^{(m)}([\zeta_{\varepsilon n}]^{(m)})[\zeta_{\varepsilon n}]^{(m)}| d\Omega \\ &\leq c + 2\rho_1^{(m)} \rho_2^{(m)} \text{mes } \Omega'_m \\ &\leq c + 2\rho_1^{(m)} \rho_2^{(m)} \text{mes } \Omega'_m, \quad \forall m = 1, 2, \dots, m'. \end{aligned} \quad (3.43)$$

The definition of the mollifier and (3.8) easily imply the estimate

$$\sup_{|\xi| \leq \xi_0} |\beta_\varepsilon(\xi)| \leq \operatorname{ess\,sup}_{|\xi| \leq \xi_0+1} |\beta(\xi)|. \quad (3.44)$$

Choosing ξ_0 , such that for all ε and n

$$\frac{1}{\xi_0} \int_{\omega} |\beta_\varepsilon([\zeta_{\varepsilon n}])[\zeta_{\varepsilon n}]| d\Omega' \leq \frac{1}{\xi_0} (c + 2\rho_1\rho_2 \operatorname{mes} \Omega') \leq \frac{\mu}{2}, \quad (3.45)$$

and δ such that for $\operatorname{mes} \omega < \delta$

$$\operatorname{ess\,sup}_{|\xi| \leq \xi_0+1} |\beta(\xi)| \leq \frac{\mu}{2\delta}, \quad (3.46)$$

we obtain that

$$\int_{\omega} \sup_{|[\zeta_{\varepsilon n}(x)]| \leq \xi_0} |\beta_\varepsilon([\zeta_{\varepsilon n}])| d\Omega \leq \operatorname{ess\,sup}_{|[\zeta_{\varepsilon n}(x)]| \leq \xi_0+1} |\beta([\zeta_\varepsilon])| \operatorname{mes} \omega \leq \frac{\mu}{2\delta} \delta = \frac{\mu}{2}. \quad (3.47)$$

From (3.41), (3.43), (3.45), and (3.47) results (3.39), i.e., the weak precompactness of $\beta_\varepsilon^{(m)}([\zeta_{\varepsilon n}]^{(m)})$, $m = 1, 2, \dots, m'$ in $L^1(\Omega'_m)$. Thus, as $\varepsilon \rightarrow 0$ and $n \rightarrow \infty$, a subsequence again denoted by $\{\beta_\varepsilon^{(m)}([\zeta_{\varepsilon n}]^{(m)})\}$ can be determined such that

$$\beta_\varepsilon^{(m)}([\zeta_{\varepsilon n}]^{(m)}) \rightarrow \chi^{(m)} \text{ weakly in } L^1(\Omega'_m), \quad m = 1, 2, \dots, m'. \quad (3.48)$$

From (3.36) and (3.48) passing to the limit $n \rightarrow \infty$, $\varepsilon \rightarrow 0$ we obtain that

$$\begin{aligned} \sum_{j=1}^r \langle T_j(\zeta^{(j)}), z^{(j)} \rangle + \sum_{m=1}^{m'} \int_{\Omega'_m} \chi^{(m)}[z]^{(m)} d\Omega = \sum_{j=1}^r (\bar{f}^{(j)}, z^{(j)}), \\ \forall z^{(j)} \in Z_j, \quad j = 1, 2, \dots, r, \end{aligned} \quad (3.49)$$

where we have used the fact that

$$[z]^{(m)} \in H^2(\Omega'_m) \subset L^\infty(\Omega'_m) \quad \text{for } \Omega'_m \subset \mathbb{R}^2, \quad \forall m = 1, \dots, m'. \quad (3.50)$$

To complete the proof we have to show that

$$\chi^{(m)} \in \hat{\beta}^{(m)}([\zeta]^{(m)}), \quad m = 1, 2, \dots, m'. \quad (3.51)$$

To show (3.51) we proceed as follows: (3.38) implies by Egoroff's theorem, that for every $\alpha > 0$, we can determine ω_j , $j = 1, \dots, r$ with the $\operatorname{mes} \omega_j > \alpha$ such that

$$\zeta_{\varepsilon n}^{(j)} \rightarrow \zeta^{(j)} \quad \text{uniformly in } \Omega_j - \omega_j, \quad j = 1, 2, \dots, r. \quad (3.52)$$

with $\zeta^{(j)} \in L^\infty(\Omega_j - \omega_j)$. Due to this uniform convergence, for every $\alpha > 0$ an ω with $\operatorname{mes} \omega < \alpha$ can be found such that for any $\mu > 0$ and for $\varepsilon < \varepsilon_0 < \mu/2$ and $n > n_0 > 2/\mu$

$$|[\zeta_{\varepsilon n}] - [\zeta]| < \frac{\mu}{2}, \quad \forall x \in \Omega' - \omega. \quad (3.53)$$

From (3.8) we obtain easily the inequality

$$\beta_\varepsilon(\xi) = (p_\varepsilon * \beta)(\xi) = \int_{-\varepsilon}^{+\varepsilon} \beta(\xi - t) p_\varepsilon(t) dt \leq \operatorname{ess\,sup}_{|t| \leq \varepsilon} \beta(\xi - t), \quad (3.54)$$

and analogously

$$\operatorname{ess\,inf}_{|t| \leq \varepsilon} \beta(\xi - t) \leq \beta_\varepsilon(\xi). \quad (3.55)$$

From (3.53) and (3.54), (3.55) we have that

$$\begin{aligned} \beta_\varepsilon([\zeta_{\varepsilon n}]) &\leq \operatorname{ess\,sup}_{|[\zeta_{\varepsilon n}] - \xi| \leq \varepsilon} \beta(\xi) \leq \operatorname{ess\,sup}_{|[\zeta_{\varepsilon n}] - \xi| < \mu/2} \beta(\xi) \\ &\leq \operatorname{ess\,sup}_{|[\zeta] - \xi| < \mu} \beta(\xi) = \bar{\beta}_\mu([\zeta]) \end{aligned} \quad (3.56)$$

and

$$\underline{\beta}_\mu([\zeta]) \leq \operatorname{ess\,inf}_{|[\zeta] - \xi| < \mu} \beta(\xi) \leq \beta_\varepsilon([\zeta_{\varepsilon n}]), \quad (3.57)$$

respectively. Due to (3.56), (3.57) we obtain by choosing $e \geq 0$ a.e. in $\Omega' - \omega$ with $e \in L^\infty(\Omega' - \omega)$ that

$$\int_{\Omega' - \omega} \underline{\beta}_\mu([\zeta]) e \, d\Omega \leq \int_{\Omega' - \omega} \beta_\varepsilon([\zeta_{\varepsilon n}]) e \, d\Omega \leq \int_{\Omega' - \omega} \bar{\beta}_\mu([\zeta]) e \, d\Omega \quad (3.58)$$

which as $\varepsilon \rightarrow 0$ and $n \rightarrow \infty$ becomes

$$\int_{\Omega' - \omega} \underline{\beta}_\mu([\zeta]) e \, d\Omega \leq \int_{\Omega' - \omega} \chi e \, d\Omega \leq \int_{\Omega' - \omega} \bar{\beta}_\mu([\zeta]) e \, d\Omega. \quad (3.59)$$

Now passing to the limit $\mu \rightarrow 0$ we have by Lebesgue's theorem

$$\int_{\Omega' - \omega} \underline{\beta}([\zeta]) e \, d\Omega \leq \int_{\Omega' - \omega} \chi e \, d\Omega \leq \int_{\Omega' - \omega} \bar{\beta}([\zeta]) e \, d\Omega, \quad (3.60)$$

and since $e \geq 0$ is arbitrary, (3.60) yields that

$$\chi \in \hat{\beta}([\zeta]) \quad \text{a.e. in } \Omega' - \omega.$$

Now taking α as small as possible implies (3.51). *q.e.d.*

4. The abstract problem and additional results.

(i) The proof of Proposition 3.2 as it has been carried out holds for a more abstract problem. Let us consider again Problem P. We assume that Z_j are given Hilbert spaces and $\alpha^j(\cdot, \cdot): Z_j \times Z_j \rightarrow R$ are symmetric continuous bilinear forms. Let Z'_j be the dual space of Z_j and assume that for Ω_j open and bounded subset of R^n

$$Z_j \subset L^2(\Omega_j) \subset Z'_j \quad j = 1, \dots, r \quad (4.1)$$

where the injections are continuous. We denote by $(\cdot, \cdot)$ the L^2 -product and duality pairing, the norm of Z_j by $\|\cdot\|$ and the norm of $L^2(\Omega_j)$ by $|\cdot|_2$. We recall here that the linear form $(\cdot, \cdot)$ extends uniquely from $Z_j \times L^2(\Omega_j)$ to $Z_j \times Z'_j$ [22]. Further let us assume that

$$Z_j \subset L^2(\Omega_j) \quad \text{is compact} \quad (4.2)$$

and that

$$Z_j \cap L^\infty(\Omega_j) \quad \text{is dense in } Z_j \text{ for the } \|\cdot\| \text{-norm.} \quad (4.3)$$

The bilinear form is assumed to have a nonzero kernel in Z_j , i.e., that $\operatorname{Ker} \alpha^{(j)} = \{q | \alpha^{(j)}(q, q) = 0\} \neq \{0\}$, and let

$$\operatorname{Ker} \alpha^{(j)} \quad \text{be finite dimensional.} \quad (4.4)$$

The norm $\|z\|$ on Z_j is assumed to be equivalent to $\|z\| = p^{(j)}(\tilde{z}) + |q|_2$, where $z = \tilde{z} + q$, $q \in \text{Ker } \alpha^{(j)}$, $\tilde{z} \in \text{Ker } \alpha^{(j)\perp}$ (i.e. $(\tilde{z}, q) = 0 \ \forall q \in \text{Ker } \alpha^{(j)}$), $p^{(j)}(\tilde{z})$ is a seminorm on Z_j such that $p^{(j)}(z) = p(z + q) \ \forall z \in Z_j, q \in \text{Ker } \alpha^{(j)}$, and let

$$\alpha^{(j)}(z, z) \geq c(p^{(j)}(z))^2 \quad \forall z \in Z_j, c \text{ const} > 0. \quad (4.5)$$

The operators C_j have the same properties as in (2.30) as well as the functions J_m . Moreover $\bar{f}^{(j)} \in L^2(\Omega_j)$. For this more abstract problem Propositions 3.1 and 3.2 hold. The proofs are the same with only one small modification: since in the present general case (3.50) does not hold (because Z_j is not compactly imbedded into $C^0(\bar{\Omega}_j)$) we have to consider for the formulation of Problem $P_{\varepsilon n}$ Galerkin bases in the spaces $Z_j \cap L^\infty(\Omega_j)$. Then we can pass to the limit $n \rightarrow \infty, \varepsilon \rightarrow 0$ and obtain (3.49) due to the assumption (4.3).

(ii) For the present specific plate problem the imbeddings $H^2(\Omega) \subset C^0(\bar{\Omega}) \subset L^\infty(\Omega)$ hold. Moreover β_ε is a C^∞ -function and thus (3.48) results immediately without using the Dunford–Pettis theorem. Also the proof of (3.51) is simplified (no need of Egoroff's theorem) but still follows the general pattern of (3.58–3.60) (for $\omega = \emptyset$). However, the procedure followed in the proof is necessary for any other type of functional setting. Indeed in any hemivariational inequality formulated in the H^1 -space the above procedure is necessary because $H^1(\Omega) \not\subset C^0(\bar{\Omega})$ [4]. The same holds for plate problems (not considered in the present paper) with nonmonotone conditions at the boundary, which give rise to hemivariational inequalities: For instance for a plate problem with a nonmonotone boundary condition $M \in \hat{\beta}(\frac{\partial \zeta}{\partial n})$ (cf., e.g., [9], [23]), $\frac{\partial \zeta}{\partial n} \in H^{1/2}(\Gamma)$ but $H^{1/2}(\Gamma) \not\subset L^\infty(\Gamma)$. On the contrary, for a boundary condition $-Q \in \hat{\beta}(\zeta)$ the imbeddings $H^2(\Omega) \subset C^0(\bar{\Omega}) \subset C^0(\Gamma) \subset L^\infty(\Gamma)$ hold and the general procedure of the Dunford–Pettis theorem can be omitted.

REMARK. The proof of Proposition 3.2 holds for $T_j \ j = 1, \dots, r$ pseudomonotone which satisfy the assumption $H1 \div H3$ of [7].

For the problem studied the following approximation result holds.

PROPOSITION 4.1. Let the assumptions of Proposition 3.2 hold. Moreover suppose that there exist constants $c^{(m)} > 0$ and $q^{(m)} \geq 1, m = 1, \dots, m' = r - 1$ such that

$$|\beta^{(m)}(\xi)| \leq c^{(m)}(1 + |\xi|^{q^{(m)}}) \quad \forall \xi \in R. \quad (4.6)$$

Then for $\varepsilon \rightarrow 0, n \rightarrow \infty$

$$\tilde{\zeta}_{\varepsilon n}^{(j)} \rightarrow \tilde{\zeta}^{(j)} \quad \text{strongly in } Z_j. \quad (4.7)$$

The proof is the same as that in [1] and is omitted here. The sufficient condition in Proposition 3.2 is similar to the sufficient conditions derived in the Landesman–Lazer theory. Concerning the number of solutions both for the coercive and the semicoercive problem most questions are still open. There are some results in the area of the contemporary development of the Landesman–Lazer theory, which might be extended or directly applied to the theory or hemivariational inequalities. However, the problem of the multiplicity of solutions should be considered as still open.

(iii) Until now we have assumed that the adhesive prevents negligibly the interface sliding. If this is not the case the mechanical model is much more complicated and the corresponding mathematical problem is still open.

FIG. 2. Tangential interface action.

Let us denote by $[w_\alpha^{(m)}]$ the relative slipping in the α -direction at the m -interface. We may write that (Fig. 2a)

$$[w_\alpha^{(m)}] = u_\alpha^{(m)} - u_\alpha^{(m+1)} + \frac{1}{2} h_{m+1} \zeta_{,\alpha}^{(m+1)} + \frac{1}{2} h_m \zeta_{,\alpha}^{(m)} \quad \alpha = 1, 2. \quad (4.8)$$

Moreover it is clear that $S_{(j)\alpha}^+ = S_{(j+1)\alpha}^-$, because the thickness h of the adhesive is assumed as small. We denote the common value of $S_{(m)\alpha}^+$ and $S_{(m+1)\alpha}^-$ by $S_{(m)\alpha}$, and let us assume that

$$-S_{(m)\alpha} \in \hat{b}_\alpha([w_\alpha^{(m)}]) = \bar{\partial} \varphi_\alpha([w_\alpha^{(m)}]) \quad \alpha = 1, 2. \quad (4.9)$$

Here $\hat{b}_\alpha$ and φ_α result from $b_\alpha \in L_{\text{loc}}^\infty(R)$ as before (cf. (2.9)–(2.13)). Note that instead of (4.9) we may consider a much more general relation of the form

$$-\{S_{(m),\alpha}\} \in \bar{\partial} \varphi(\{[w_\alpha^{(m)}]\}; f) \quad (4.10)$$

connecting the vector of the tangential interface forces with the relative slipping of the plates and the normal interface force f . By means of (4.9) we can describe several interface laws, e.g., the Coulomb's friction (Fig. 2b), where $\varphi_\alpha(\xi) = \mu f |\xi|$ and μ is the coefficient of friction, nonmonotone sawtooth interface laws depicting either a more realistic interface friction (Fig. 2b—dotted line—and Fig. 2c), or, in Figs. 2d and 2e the gradual failure of an adhesive interface material in tangential action. Here the dotted (resp. the full line) corresponds to semibrittle (resp. to brittle) failure. We refer also in this respect to [24].

The effect of the tangential forces acting on the upper and on the lower surface of each plate is equivalent to the effect of a stretching force

$$q_{\alpha}^{(j)} = S_{(j)\alpha}^{-} + S_{(j)\alpha}^{+} \quad (4.11)$$

acting on the middle surface of each plate (obviously $S_{1(\alpha)}^{-} = S_{r(\alpha)}^{+} = 0$) and of two distributed bending moments

$$m_{\alpha}^{(j)} = \frac{1}{2} h_j (S_{(j)\alpha}^{-} - S_{(j)\alpha}^{+}) \quad \alpha = 1, 2. \quad (4.12)$$

Assuming that (4.9) holds we have that for $j = 1, \dots, r$

$$q_{\alpha}^{(j)} \in \bar{\partial} \varphi_{\alpha}([w_{\alpha}^{(j-1)}]) + \bar{\partial} \varphi_{\alpha}([w_{\alpha}^{(j)}]) \quad \alpha = 1, 2 \quad (4.13)$$

and

$$m_{\alpha}^{(j)} \in \frac{1}{2} h_j \{ \bar{\partial} \varphi_{\alpha}([w_{\alpha}^{(j-1)}]) - \bar{\partial} \varphi_{\alpha}([w_{\alpha}^{(j)}]) \} \quad \alpha = 1, 2. \quad (4.14)$$

Note that φ_{α} may be different on each interface.

Due to $m_{\alpha}^{(j)}$ and $q_{\alpha}^{(j)}$ $j = 1, \dots, r$ Eqs. (2.1–2.3) need a slight modification. In (2.1) $f^{(j)}$ should be replaced by $f^{(j)} - m_{1,2}^{(j)} - m_{2,1}^{(j)}$ and in the right-hand side of (2.2) the in-place force term $q_{\alpha}^{(j)}$ should be added. The resulting multivalued B.V.P. is still open.

Acknowledgments. The present research work was supported by the Scientific Division of NATO (Scientific cooperation grant No. 86/0694).

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