PSEUDODISSIPATIVE SYSTEMS III: GLOBAL BEHAVIOR

By

J. A. WALKER

Northwestern University, Evanston, Illinois

Abstract. A result is presented which suggests the form of global Liapunov functions for many nonlinear mechanical systems in the “pseudodissipative” class. Consequently, this result often provides a means of “global analysis” of the overall behavior of such a system. In each of three examples, the general behavior of a mechanical system is ascertained by means of the result presented.

1. Introduction. Stability analysis of “generalized equilibria” [9] provides considerable information about the behavior of any motion originating sufficiently near to generalized equilibria, and this also is true for stability analysis of “reduced equilibria” [10] which often correspond to steady motions. Unfortunately, almost all systems are nonlinear and stability analysis of equilibria gives little or no indication of “how near” is “sufficiently near.” In fact, for a nonlinear system the behaviors of most motions are not described at all by pure stability analysis of equilibria.

Our present objective is to demonstrate that some global conclusions, concerning the behaviors of all motions, can be reached for many nonlinear pseudodissipative systems. The construction and use of a global Liapunov function is considered in Sec. 3 (see Theorem 3.1), and the examples of Sec. 4 are analyzed on the basis of theory presented in Sec. 3. The following Sec. 2 describes our concepts and definitions.

2. Notation and terminology. Consider a collection of particles observed by some inertial observer, and choose n “generalized coordinates” $(q_1, q_2, \dots, q_n) \equiv q \in \mathcal{O} \subset \mathcal{R}^n$, where $\mathcal{O}$ is some open subset of $\mathcal{R}^n$; the dimension n of the *generalized position* $q \in \mathcal{R}^n$ need not be minimal. We denote by $u \in \mathcal{R}^n$ the corresponding *generalized speed*; i.e., $u(t) = \dot{q}(t)$ along motions $q(\cdot): \mathcal{R} \rightarrow \mathcal{R}^n$. The resulting *generalized kinetic energy* $T: \mathcal{R} \times \mathcal{O} \times \mathcal{R}^n \rightarrow \mathcal{R}$ and *generalized force* $Q: \mathcal{R} \times \mathcal{O} \times \mathcal{R}^n \rightarrow \mathcal{R}^n$ depend upon the *generalized state* $(q, u) \in \mathcal{O} \times \mathcal{R}^n$ and possibly the current time $t \in \mathcal{R}$. Any and all kinematic constraints on (q, u) may be accounted for by defining a *kinematically possible set* $\mathcal{C}(t) \subset \mathcal{O} \times \mathcal{R}^n$, consisting of all generalized states (q, u) kinematically possible at time $t \in \mathcal{R}$. A C^1 -smooth function $q(\cdot): \mathcal{R} \rightarrow \mathcal{R}^n$ is kinematically possible on $[t_1, t_2]$ if $(q(t), \dot{q}(t)) \in \mathcal{C}(t)$ for all $t \in [t_1, t_2]$.

Received July 24, 1989.

©1990 Brown University

Apart from some of the foregoing notation, we depart from the classical Lagrange formulation only by assuming that Q is explicitly known, continuous, and “pseudodissipative” in the following sense:

DEFINITION 2.1. The generalized force Q will be called *pseudodissipative* if there exists a C^1 -smooth function $U: \mathcal{R} \times \mathcal{O} \times \mathcal{R}^n \rightarrow \mathcal{R}$, affine with respect to its third argument $u \in \mathcal{R}^n$, and another function $D: \mathcal{R} \times \mathcal{O} \times \mathcal{R}^n \rightarrow \mathcal{R}^n$ such that

$$\sum_{l=1}^n (u_l - v_l) [D_l(t, q, u) - D_l(t, q, v)] \leq 0 \quad (1)$$

for all $(t, q, u, v) \in \mathcal{R} \times \mathcal{O} \times \mathcal{R}^n \times \mathcal{R}^n$, and along every kinematically possible $q(\cdot)$,

$$\begin{aligned} & \sum_{l=1}^n \delta_l(t) Q_l(t, q(t), \dot{q}(t)) \\ &= \sum_{l=1}^n \delta_l(t) D_l(t, q(t), \dot{q}(t)) \\ &+ \sum_{l=1}^n \delta_l(t) \left[\frac{d}{dt} \frac{\partial}{\partial u_l} U(t, q(t), \dot{q}(t)) - \frac{\partial}{\partial q_l} U(t, q(t), \dot{q}(t)) \right] \end{aligned} \quad (2)$$

for all C^1 -smooth $\delta(\cdot): \mathcal{R} \rightarrow \mathcal{R}^n$ such that $(q(t), \dot{q}(t) + \delta(t)) \in \mathcal{C}(t)$ for all $t \in \mathcal{R}$.

U will be called *pseudopotential* and the function $L \equiv T - U$ will be called the *Lagrangian*. Due to property (1), D will be called the *dissipative part* of Q .

The generalized force Q will be called *strongly pseudodissipative* if equality occurs in condition (1) *only* when $u = v$. Q will be called *pseudoconservative* if $D \equiv 0$.

Definition 2.1 was made and discussed recently [9], as was the following definition [10].

DEFINITION 2.2. If Q is pseudodissipative, then the last $n - r$ generalized coordinates $(q_{r+1}, q_{r+2}, \dots, q_n) \equiv w \in \mathcal{R}^{n-r}$ are said to be *ignorable* if the following conditions are met with $q \equiv (\tilde{q}, w) \in \mathcal{R}^n \times \mathcal{R}^{n-r}$, $\tilde{q} \equiv (q_1, q_2, \dots, q_r)$:

- (a) There exist selections for U and D such that D and $L \equiv T - U$ are independent of $w \in \mathcal{R}^{n-r}$; in particular, there exist an open set $\tilde{\mathcal{O}} \subset \mathcal{R}^r$ and functions $\tilde{L}: \mathcal{R} \times \tilde{\mathcal{O}} \times \mathcal{R}^n \rightarrow \mathcal{R}$, $\tilde{D}: \mathcal{R} \times \tilde{\mathcal{O}} \times \mathcal{R}^n \rightarrow \mathcal{R}^n$, such that $\mathcal{O} \subset \tilde{\mathcal{O}} \times \mathcal{R}^{n-r}$ and

$$L(t, q, u) = \tilde{L}(t, \tilde{q}, u), \quad D(t, q, u) = \tilde{D}(t, \tilde{q}, u),$$

for all $(t, q, u) \in \mathcal{R} \times \mathcal{O} \times \mathcal{R}^n$.

- (b) There exist sets $\tilde{\mathcal{E}}(t) \subset \tilde{\mathcal{O}} \times \mathcal{R}^n$ and $\mathcal{I}(t, \tilde{q}) \subset \mathcal{R}^{n-r}$ such that

$$\mathcal{C}(t) = \{(q, u) \in \mathcal{O} \times \mathcal{R}^n | (\tilde{q}, u) \in \tilde{\mathcal{E}}(t), w \in \mathcal{I}(t, \tilde{q})\}, \quad t \in \mathcal{R}.$$

We say $w \in \mathcal{R}^{n-r}$ is an *ignorable part* of $q \in \mathcal{R}^n$, $\tilde{q} \in \mathcal{R}^r$ is the corresponding *reduced position*, $(\tilde{q}, u) \in \mathcal{R}^r \times \mathcal{R}^n$ is the corresponding *reduced state*, and $\tilde{\mathcal{E}}(t) \subset \mathcal{R}^r \times \mathcal{R}^n$ is the corresponding *reduced kinematically possible set*.

In [9] we discussed a “generalized state” $(q, u) \in \mathcal{R}^n \times \mathcal{R}^n$, while in [10] we discussed a “reduced state” $(\tilde{q}, u) \in \mathcal{R}^r \times \mathcal{R}^n$ on the assumption that at least one generalized coordinate was ignorable and treated as such (i.e., $n > r \geq 0$). In order to conserve space here, we shall combine these cases by omitting the adjectives “generalized” and “reduced”, utilizing the notation of Definition 2.2 throughout, and allowing the possibility that $r = n$. Note that if $r \equiv n$, which we can always choose*, we have $\tilde{q}(t) = q(t) \in \mathcal{R}^n$ and can make the substitutions

$$(r, \tilde{q}, \tilde{\mathcal{Q}}, \tilde{L}, \tilde{D}, \tilde{\mathcal{E}}(t)) \rightarrow (n, q, \mathcal{Q}, L, D, \mathcal{E}(t))$$

in all of the following.

If the generalized force is pseudodissipative and the last $n - r$ generalized coordinates are both ignorable and ignored ($n \geq r \geq 0$), Lagrange’s formulation produces the following consequences of Newton’s second law:

THEOREM 2.1. If a continuous function $(\tilde{q}(\cdot), u(\cdot)): \mathcal{R} \rightarrow \mathcal{R}^r \times \mathcal{R}^n$ is dynamically possible on $[t_0, t_f]$, then for all $t \in [t_0, t_f]$,

$$(\tilde{q}(t), u(t)) \in \tilde{\mathcal{E}}(t), \quad u_l(t) = \dot{q}_l(t) \quad \text{for } l \leq r, \quad (3)$$

and

$$0 = \sum_{l=1}^n \delta_l(t) \left[\frac{d^+}{dt} \frac{\partial}{\partial u_l} \tilde{L}(t, \tilde{q}(t), u(t)) - \frac{\partial}{\partial q_l} \tilde{L}(t, \tilde{q}(t), u(t)) - \tilde{D}_l(t, \tilde{q}(t), u(t)) \right] \quad (4)$$

for all C^1 -smooth $\delta(\cdot): \mathcal{R} \rightarrow \mathcal{R}^n$ such that $(\tilde{q}(t), u(t) + \delta(t)) \in \tilde{\mathcal{E}}(t)$, $t_0 \leq t < t_f$.

In [9] no coordinates were ignored ($r \equiv n$) and (3)–(4) were called the generalized motion equations. In [10] the last $n - r$ coordinates were both ignorable and ignored ($r < n$), so (3)–(4) were called the reduced motion equations. Here (3)–(4) are simply called the motion equations, whether $r = n$ or $r < n$.

The following presentation is based on four assumptions:

- (i) A particular initial instant t_0 has been chosen, and the time interval of interest is $[t_0, \infty)$.
- (ii) Q is known, continuous, and pseudodissipative on $[t_0, \infty)$.
- (iii) The last $n - r$ generalized coordinates are ignorable and ignored, $0 \leq r \leq n$, and $\tilde{L}$ is time-invariant.
- (iv) For each $(\tilde{q}^0, u^0) \in \tilde{\mathcal{E}}(t_0)$, there exists a continuous solution $(\tilde{q}(\cdot), u(\cdot)): [t_0, \infty) \rightarrow \mathcal{R}^r \times \mathcal{R}^n$ of (3)–(4) such that $\tilde{q}(t_0) = \tilde{q}^0$, $u(t_0) = u^0$. Moreover, at each $t > t_0$, $(\tilde{q}(t), u(t))$ depends continuously on $(\tilde{q}^0, u^0)$.

Assumptions (i)–(iv) are very mild and will be maintained henceforth without further comment.

Obviously, the simplest type of solution of the motion equations (3)–(4) would be a constant solution, $(\tilde{q}(t), u(t)) \equiv (\tilde{q}^e, u^e)$ for some fixed $(\tilde{q}^e, u^e) \in \mathcal{R}^r \times \mathcal{R}^n$;

*We may choose $r < n$ if the conditions of Definition 2.2 allow this, but we need not do so. Ignorable coordinates need not be ignored, and we can always choose $r = n$.

the corresponding initial state $(\tilde{q}^e, u^e) \in \tilde{\mathcal{E}}(t_0)$ is called an *equilibrium*. If $r < n$, each equilibrium corresponds to a family of steady motions $q(\cdot): [t_0, \infty) \rightarrow \mathcal{R}^n$ parameterized by the initial values of the $n - r$ ignored coordinates. If $r \equiv n$, then $u^e = 0 \in \mathcal{R}^n$ and $q(t) = \tilde{q}^e \in \mathcal{R}^n$ is constant. Whether $r = n$ or $r < n$, the following result [9, 10] is useful.

COROLLARY 2.1. $(\tilde{q}^e, u^e) \in \mathcal{R}^r \times \mathcal{R}^n$ is an equilibrium if and only if $u_l^e = 0$ for each $l \leq r$ and, for all $t \geq t_0$, $(\tilde{q}^e, u^e) \in \tilde{\mathcal{E}}(t)$ and

$$0 = \sum_{l=1}^n \delta_l(t) \left[\frac{\partial}{\partial q_l} \tilde{L}(t_0, \tilde{q}^e, u^e) + \tilde{D}_l(t, \tilde{q}^e, u^e) \right] \quad (5)$$

for all C^1 -smooth $\delta_l(\cdot): \mathcal{R} \rightarrow \mathcal{R}^n$ such that $(\tilde{q}^e, u^e + \delta(t)) \in \tilde{\mathcal{E}}(t)$, $t \geq t_0$.

In [10] $r < n$ and $(\tilde{q}^e, u^e) \in \mathcal{R}^r \times \mathcal{R}^n$ was called a *reduced equilibrium*. In [9] $r \equiv n$ and $(\tilde{q}^e, u^e) = (q^e, 0) \in \mathcal{R}^n \times \mathcal{R}^n$ was called a *generalized equilibrium*. Here we shall consider both cases, use the unadorned term “equilibrium,” and define $\mathcal{E} \subset \mathcal{R}^r \times \mathcal{R}^n$ to be the set of all equilibria; clearly $\mathcal{E} \subset \tilde{\mathcal{E}}(t)$ for all $t \geq t_0$.

Our principal interest lies in determining conditions sufficient to ensure that every motion approaches $\mathcal{E}$ as time increases. That is, we wish to ensure that for every $(\tilde{q}^0, u^0) \in \tilde{\mathcal{E}}(t_0)$,

$$\lim_{t \rightarrow \infty} \left(\inf_{(\tilde{q}, \hat{u}) \in \mathcal{E}} \|\tilde{q}(t) - \hat{q}\|_r + \|u(t) - \hat{u}\|_n \right) = 0$$

along the corresponding solution of (3)–(4). The results of the following section bear on this problem of showing that $\mathcal{E}$ is a “global attractor” [3].

3. General asymptotic behavior. In the following Theorem 3.1 we employ two somewhat arbitrary C^1 -smooth functions $h: \tilde{\mathcal{E}} \rightarrow \mathcal{R}^r$, $p: \tilde{\mathcal{E}} \rightarrow \mathcal{R}^n$, and three (h, p) -related functions defined as

$$\begin{aligned} G(\tilde{q}, u) &\equiv \sum_{l=1}^n (u_l - p_l(\tilde{q})) \frac{\partial}{\partial u_l} \tilde{L}(t_0, \tilde{q}, u) \\ &\quad - \tilde{L}(t_0, \tilde{q}, u) + \tilde{L}(t_0, h(\tilde{q}), p(\tilde{q})) \\ &\quad + \sum_{l=1}^r (\tilde{q}_l - h_l(\tilde{q})) \frac{\partial}{\partial q_l} \tilde{L}(t_0, h(\tilde{q}), p(\tilde{q})), \end{aligned} \quad (6)$$

$$\begin{aligned} S(\tilde{q}, u) &\equiv \sum_{i=1}^r \sum_{j=1}^n u_i \left(\frac{\partial}{\partial q_i} p_j(\tilde{q}) \right) \left[\frac{\partial}{\partial u_j} \tilde{L}(t_0, h(\tilde{q}), p(\tilde{q})) - \frac{\partial}{\partial u_j} \tilde{L}(t_0, \tilde{q}, u) \right] \\ &\quad + \sum_{l=1}^r p_l(\tilde{q}) \left[\frac{\partial}{\partial q_l} \tilde{L}(t_0, h(\tilde{q}), p(\tilde{q})) - \frac{\partial}{\partial q_l} \tilde{L}(t_0, \tilde{q}, u) \right], \end{aligned} \quad (7)$$

$$F(t, q, u) \equiv \sum_{l=1}^n (u_l - p_l(\tilde{q})) [D_l(t, \tilde{q}, u) - D_l(t, \tilde{q}, p(\tilde{q}))], \quad (8)$$

for all $(\tilde{q}, u) \in \tilde{\mathcal{E}} \times \mathcal{R}^n$ and $t \geq t_0$. Definition 2.1 implies that $F(t, \tilde{q}, u) \leq 0$.

The structure of G is of some interest. $\tilde{L}$ has the form

$$\tilde{L}(t_0, q, u) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n u_i u_j m_{ij}(\tilde{q}) + \sum_{l=1}^n u_l f_l(\tilde{q}) + f_0(\tilde{q})$$

for some C^1 -smooth $f_l: \tilde{\mathcal{O}} \rightarrow \mathcal{R}$ ($l = 0, 1, \dots, n$), where each

$$m_{ij}(\tilde{q}) \equiv \frac{\partial^2}{\partial u_i \partial u_j} \tilde{L}(t_0, \tilde{q}, u) = \frac{\partial^2}{\partial u_i \partial u_j} T(t, q, u) \quad (9)$$

is independent of $u \in \mathcal{R}^n$ and the $n \times n$ symmetric matrix $M(\tilde{q}) \equiv [m_{ij}(\tilde{q})]$ is positive semidefinite for each $\tilde{q} \in \tilde{\mathcal{O}}$. Hence, identifying $u \in \mathcal{R}^n$ with an $n \times 1$ matrix, we find that

$$G(\tilde{q}, u) = \frac{1}{2} (u - p(\tilde{q}))^T M(\tilde{q}) (u - p(\tilde{q})) + G(\tilde{q}, p(\tilde{q})) \quad (10)$$

and $G(\tilde{q}, u) \geq G(\tilde{q}, p(\tilde{q}))$ for all $(\tilde{q}, u) \in \tilde{\mathcal{O}} \times \mathcal{R}^n$. If $\tilde{L}$ is C^2 -smooth, then

$$|G(\tilde{q}, p(\tilde{q})) - (\tilde{q} - h(\tilde{q}))^T \tilde{K}(\tilde{q}) (\tilde{q} - h(\tilde{q})) / 2| \leq o(\|\tilde{q} - h(\tilde{q})\|_r^2) \quad (11)$$

where the $r \times r$ symmetric matrix $\tilde{K}(\tilde{q}) = [k_{ij}(\tilde{q})]$,

$$k_{ij}(\tilde{q}) \equiv -\frac{\partial^2}{\partial q_i \partial q_j} L(t_0, h(\tilde{q}), p(\tilde{q})), \quad i, j = 1, 2, \dots, r. \quad (12)$$

The local estimate (11) is of little value here, since we are interested in global results. However, such estimates were used in [9, 10] to prove a number of corollaries to stability theorems provided for an equilibrium $(\tilde{q}^e, u^e) \in \mathcal{R}^r \times \mathcal{R}^n$. Therein we chose $(h(\tilde{q}), p(\tilde{q})) \equiv (\tilde{q}^e, u^e)$ and found that $S(\tilde{q}, u) \equiv 0$.

Here the essence of our approach is to choose the functions h and p such that $S(\tilde{q}, u) \leq 0$ for all $(\tilde{q}, u) \in \tilde{\mathcal{E}}(t)$, $t \geq t_0$. Then, under certain additional assumptions, it will be possible to conclude that G is a global Liapunov function [8] and this will allow us to draw conclusions regarding the behavior of all solutions of the motion equations (3)–(4). See the following theorem.

THEOREM 3.1. Suppose that there exist C^1 -smooth functions $h: \tilde{\mathcal{O}} \rightarrow \mathcal{R}^r$, $p: \mathcal{O} \rightarrow \mathcal{R}^n$, such that the following conditions hold:

- (a) At every $t \geq t_0$, the statement $(\tilde{q}, u) \in \tilde{\mathcal{E}}(t)$ implies that $(\tilde{q}, p(\tilde{q})) \in \tilde{\mathcal{E}}(t)$.
- (b) $\frac{\partial}{\partial q_l} \tilde{L}(t_0, h(\tilde{q}), p(\tilde{q})) = \frac{\partial}{\partial q_l} \tilde{L}(t_0, h(\hat{q}), p(\hat{q}))$ for all $(\tilde{q}, \hat{q}) \in \tilde{\mathcal{O}} \times \tilde{\mathcal{O}}$, $l = 1, 2, \dots, r$.
- (c) For all $(\tilde{q}, u) \in \tilde{\mathcal{E}}(t)$ and $t \geq t_0$,

$$\sum_{l=1}^n u_l \left[\frac{\partial}{\partial q_l} L(t_0, h(\tilde{q}), p(\tilde{q})) + D_l(t, \tilde{q}, p(\tilde{q})) \right] = 0.$$

Then along every solution of (3)–(4)

$$\begin{aligned} \frac{d^+}{dt} G(\tilde{q}(t), u(t)) &= S(\tilde{q}(t), u(t)) + F(t, \tilde{q}(t), u(t)) \\ &\leq S(\tilde{q}(t), u(t)), \quad t \geq t_0. \end{aligned}$$

If in addition to (a)–(c),

(d) $S(\tilde{q}, u) \leq 0$ for all $(\tilde{q}, u) \in \tilde{\mathcal{E}}(t)$ and $t \geq t_0$,

then $G(\tilde{q}(\cdot), u(\cdot)): [t_0, \infty) \rightarrow \mathcal{R}$ is nonincreasing along every solution of (3)–(4).

If in addition to (a)–(d),

(e) there exists $g_0 \in \mathcal{R}$ such that $G(\tilde{q}, p(\tilde{q})) \geq g_0$ for all $\tilde{q} \in \tilde{\mathcal{O}}$,

then there exist limits

$$\lim_{\tau \rightarrow \infty} \int_{\tau}^{\tau+1} S(\tilde{q}(t), u(t)) dt = \lim_{\tau \rightarrow \infty} \int_{\tau}^{\tau-1} F(t, \tilde{q}(t), u(t)) dt = 0,$$

$$\lim_{t \rightarrow \infty} G(\tilde{q}(t), u(t)) \geq g_0,$$

along every solution of (3)–(4).

Sketch of Proof. Under assumptions (a)–(c), we consider some solution $(\tilde{q}(\cdot), u(\cdot)): [t_0, \infty) \rightarrow \mathcal{R}^r \times \mathcal{R}^n$ of (3)–(4). Choosing $\delta(t) \equiv p(\tilde{q}(t)) - u(t)$ in (4), we find under (a)–(c) that

$$\frac{d^+}{dt} G(\tilde{q}(t), u(t)) = S(\tilde{q}(t), u(t)) + F(t, \tilde{q}(t), u(t)) \leq S(\tilde{q}(t), u(t))$$

for all $t \geq t_0$. Under (d), $G(\tilde{q}(t), u(t))$ is nonincreasing on $[t_0, \infty)$. If condition (e) also holds, then (10) implies that $G(\tilde{q}(t), u(t))$ is bounded below by g_0 on $[t_0, \infty)$. Since F and S are nonpositive, the theory of nonincreasing functions [5] produces the remaining conclusions. This concludes our sketch of the proof.

Notice that conditions (a)–(d) imply that G is a Liapunov function on all of $\tilde{\mathcal{O}} \times \mathcal{R}^n$; i.e., G is a global Liapunov function. This means that for each $\mu \in \mathcal{R}$ the closed set

$$\{(q, u) \in \tilde{\mathcal{O}} \times \mathcal{R}^n \mid G(q, u) \leq \mu\}$$

and the open set

$$\{(q, u) \in \tilde{\mathcal{O}} \times \mathcal{R}^n \mid G(q, u) < \mu\}$$

are positive invariant [8] under the motion equations (3)–(4). So are each of their disjoint components if either set consists of a collection of disjoint subsets (components) of the same type (closed or open, respectively). This fact follows from continuity of the Liapunov function G .

If Q is strongly pseudodissipative and $\tilde{D}$ is $(t, \tilde{q})$ -invariant, things may become much simpler, as the following result indicates.

COROLLARY 3.1. Suppose that the following conditions hold for some particular equilibrium $(\tilde{q}^e, u^e) \in \mathcal{R}^r \times \mathcal{R}^n$ with $h(\tilde{q}) \equiv \tilde{q}^e$, $p(\tilde{q}) \equiv u^e$ in (6)–(7):

- $\tilde{\mathcal{E}}(t)$ is time-invariant and the statement $(\tilde{q}, u) \in \tilde{\mathcal{E}}(t_0)$ implies that both $(\tilde{q}, u^e) \in \tilde{\mathcal{E}}(t_0)$ and (5) holds with $\delta(t) \equiv u - u^e$.
- Q is strongly pseudodissipative, and $\tilde{D}(t, \tilde{q}, u) = \tilde{D}(t_0, \tilde{q}^e, u)$ for all $(\tilde{q}, u) \in \tilde{\mathcal{E}}(t_0)$, $t \geq t_0$.
- $\tilde{G}(\tilde{q}, u) \geq g_0$ for some $g_0 \in \mathcal{R}$ and all $(\tilde{q}, u) \in \tilde{\mathcal{E}}(t_0)$.
- There does *not* exist a solution of the motion equations (3)–(4) along which $G(\tilde{q}(t), u(t))$ is nonincreasing, $\int_{\tau}^{\tau+1} \|u(t) - u^e\|_n dt \rightarrow 0$ as $\tau \rightarrow \infty$, but $\|\tilde{q}(t)\|_r + \|u(t)\|_n$ is unbounded.

Then all conclusions of Theorem 3.1 hold and $u(t) \rightarrow u^e$, $(\tilde{q}(t), u(t)) \rightarrow \mathcal{E}$ as $t \rightarrow \infty$, along the solution of (3)–(4) for every $(\tilde{q}^0, u^0) \in \tilde{\mathcal{E}}(t_0)$. In fact, to each $(\tilde{q}^0, u^0) \in \mathcal{E}(t_0)$ there corresponds some $g^0 \leq G(\tilde{q}^0, u^0)$ such that $(\tilde{q}(t), u(t))$ approaches some bounded connected subset of $\mathcal{E}$ on which $u = u^e$ and $G(\tilde{q}, u) = g^0$.

Sketch of Proof. The assumptions (i)–(iv) of Sec. 2 and our current conditions (a)–(b) imply that $\tilde{L}$, $\tilde{D}$, $\tilde{\mathcal{E}}$ are time-invariant and the motion equations (3)–(4) generate a C^0 -semigroup on $\tilde{\mathcal{E}}(t_0)$ [8]. Our current choices for h and p , the equilibrium condition (5), and our current conditions (a)–(c) imply that the conditions (a)–(e) of Theorem 3.1 are met with $S(\tilde{q}, u) \equiv 0$.

Since $\mathcal{R}^n$ is a locally compact metric space, our current condition (b) and Definition 2.1 assure the existence of a strictly increasing function $f: \mathcal{R}^+ \rightarrow R$ such that $f(0) = 0$ and $F(t, \tilde{q}, u) \geq f(\|u - u^e\|_n)$ for all $u \neq u^e$. Hence, for each initial state $(\tilde{q}^0, u^0) \in \tilde{\mathcal{E}}(t_0)$ the conclusions of Theorem 3.1 imply that $G(\tilde{q}(t), u(t))$ is nonincreasing, $\int_t^{t+1} \|u(\tau) - u^e\|_n d\tau \rightarrow 0$, and $G(\tilde{q}(t), u(t)) \rightarrow g^0$ as $t \rightarrow \infty$.

Our current condition (d) now implies that $(\tilde{q}^0, u^0)$ has precompact positive orbit and $(\tilde{q}(t), u(t))$ approaches a bounded, connected, and invariant positive limit set $\Omega(\tilde{q}^0, u^0)$ [1, 4]. Since $u(t) \rightarrow u^e$, it follows that $u = u^e$ for all $(\tilde{q}, u) \in \Omega(\tilde{q}^0, u^0)$ and so, by invariance, $\Omega(\tilde{q}^0, u^0) \subset \mathcal{E}$. This concludes our sketch of the proof.

Condition (d) of Corollary 3.1 seems complicated but obviously is met if $G(\tilde{q}, u) \rightarrow \infty$ whenever $\|\tilde{q}\|_r + \|u\|_n \rightarrow \infty$; it is also met in many other circumstances. For example, if $r = 1$, $M(\tilde{q}) > 0$ for all $\tilde{q} \in \tilde{\mathcal{E}}$, and $G(\tilde{q}, u)$ is nontrivially periodic in $\tilde{q}$ (as in Examples 4.1–4.3), then condition (d) is met.

One implication of Corollary 3.1 is that u^e is the same for all equilibria $(\tilde{q}^e, u^e) \in \mathcal{E}$; it follows that the assumptions of Corollary 3.1 cannot be met unless $\mathcal{E}$ has this property, as in Examples 4.1–4.2. Another implication of Corollary 3.1 is that all motions are bounded and, if all $(\tilde{q}^e, u^e)$ are isolated in $\mathcal{R}^r \times \mathcal{R}^n$, each motion approaches some particular $(\tilde{q}^e, u^e)$ (which depends on $(\tilde{q}^0, u^0)$) as $t \rightarrow \infty$. See Examples 4.1–4.2, which are related to earlier examples of [10, 9].

In the following Examples 4.1–4.2 we find it possible to apply Corollary 3.1. In Example 4.3, Q is not strongly pseudodissipative and only the less restrictive Theorem 3.1 applies.

4. Examples.

Example 4.1. The housing γ of a motor with armature β is hinged to a clevis ξ which is driven about a vertical axis by another motor attached to the floor α , assumed to be inertial. See Fig. 1. Along motions, bearing friction at B and B' creates a torque $\tau_f(\dot{\theta}(t))$ on γ about the horizontal bearing axis Ax , where $\tau_f: \mathcal{R} \rightarrow \mathcal{R}$ is C^1 -smooth, $\tau_f(0) = 0$, and $\tau'_f(u_\theta) < 0$ for all $u_\theta \in \mathcal{R}$. Both motors have strictly declining delivered-torque/speed relations. In particular, the C^1 -smooth torques $\tau_1(u_\psi)$ and $\tau_2(u_\theta)$ (delivered to ξ and β by the lower and upper motors, respectively) have strictly negative derivatives, include the effects of motor-bearing friction, and become zero at known angular rates ω_1 and ω_2 , respectively. $Axyz$

is a principal coordinate system for both the armature β and the housing γ of the upper motor, and C is the mass center of this motor. I_A will denote the amount of inertia (about a vertical axis through A) of the clevis ξ (including the lower motor armature).

Choosing $\gamma + \beta + \xi$ to be our collection and defining $q \equiv (\theta, \psi, \phi) \in \mathcal{R}^3 \equiv \mathcal{Q}$, we see that $u = (u_\theta, u_\psi, u_\phi) \in \mathcal{R}^3$, $\mathcal{C}(t) = \mathcal{R}^3 \times \mathcal{R}^3$ for all t ,

$$T(t, q, u) = \frac{1}{2}[\bar{J}_1 u_\theta^2 + \bar{J}_2 u_\psi^2 \sin^2 \theta + J_3(u_\phi^2 + 2u_\phi u_\psi \cos \theta) + (I_A + \bar{J}_3 \cos^2 \theta)u_\psi^2] \quad (13)$$

FIG. 1. Mechanism of Examples 4.1-4.2.

with

$$\begin{aligned} \bar{J}_1 &= J_1 + I_1, & \bar{J}_2 &= J_1 + I_2, & \bar{J}_3 &= J_3 + I_3, \\ J_{Axyz}^{(\beta)} &= \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_1 & 0 \\ 0 & 0 & J_3 \end{bmatrix}, & J_{Axyz}^{(\gamma)} &= \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}, \end{aligned} \quad (14)$$

and Q is strongly pseudodissipative with

$$U(t, q, u) = mgl \cos \theta, \quad (15)$$

$$D(t, q, u) = [\tau_f(u_\theta), \tau_1(u_\psi), \tau_2(u_\phi)]. \quad (16)$$

Both ψ and ϕ are ignorable and we define $\hat{q} \equiv \theta \in \mathcal{R}^1 \equiv \tilde{\mathcal{Q}}$, $\tilde{\mathcal{E}}(t) \equiv \mathcal{R}^1 \times \mathcal{R}^3$, $\tilde{L} \equiv T - U$, and $\tilde{D} \equiv D$. For convenience we also define two new parameters

$$\sigma \equiv \omega_1^2(\bar{J}_3 - \bar{J}_2), \quad \zeta \equiv J_3\omega_1\omega_2 - mgl, \quad (17)$$

and note that

$$\frac{\partial}{\partial \theta} \tilde{L}(t, \theta, u) = [(\bar{J}_2 - \bar{J}_3)u_\psi^2 \cos \theta - J_3u_\psi u_\phi + mgl] \sin \theta.$$

Corollary 2.1 shows that the set of all equilibria is

$$\mathcal{E} = \{(\theta, u) \in \mathcal{R}^1 \times \mathcal{R}^3 \mid u = (0, \omega_1, \omega_2), (\sigma \cos \theta - \zeta) \sin \theta = 0\}.$$

We see that each equilibrium (θ^e, u^e) is isolated from all others. Some equilibria are unstable and the others are asymptotically stable; the stability properties of all equilibria were determined for almost all values of σ in ζ in [10]. See Table 1 where $(\theta^e, u^e)_i$ is any equilibrium in $\mathcal{E}_i$,

$$\begin{aligned} \mathcal{E}_1 &\equiv \{(\theta, u) \in \mathcal{E} \mid \cos \theta = 1\}, & \mathcal{E}_2 &\equiv \{(\theta, u) \in \mathcal{E} \mid \cos \theta = -1\}, \\ \mathcal{E}_3 &\equiv \{(\theta, u) \in \mathcal{E} \mid \sin \theta > 0\}, & \mathcal{E}_4 &\equiv \{(\theta, u) \in \mathcal{E} \mid \sin \theta < 0\}, \end{aligned}$$

and the claimed stability is asymptotic.

TABLE 1. Stability table for Example 4.1.

	$\sigma < - \zeta $	$\sigma > \zeta $	$\zeta < - \sigma $	$\zeta > \sigma $
$(\theta^e, u^e)_1$	UNSTABLE	STABLE	UNSTABLE	STABLE
$(\theta^e, u^e)_2$	UNSTABLE	STABLE	STABLE	UNSTABLE
$(\theta^e, u^e)_{3,4}$	STABLE	UNSTABLE	IMPOSSIBLE	

We choose $h(\theta) \equiv 0 \in \mathcal{R}^1$, $p(\theta) \equiv (0, \omega_1, \omega_2)$, and note that $(h(\theta), p(\theta)) \in \tilde{\mathcal{E}}$, $S(\theta, u) = 0$,

$$\begin{aligned} 2G(\theta, u) &= \bar{J}_1 u_\theta^2 + (\bar{J}_2 \sin^2 \theta + I_A + I_3 \cos^2 \theta)(u_\psi - \omega_1)^2 \\ &\quad + J_3[u_\phi - \omega_2 + (u_\psi - \omega_1) \cos \theta]^2 \\ &\quad + 2\zeta(1 - \cos \theta) + \sigma(1 - \cos^2 \theta). \end{aligned}$$

All conditions of Corollary 3.1 are met, so to each initial state $(\theta^0, u^0) \in \mathcal{R}^1 \times \mathcal{R}^3$ there corresponds some particular (and isolated) $(\theta^e, u^e) \in \mathcal{E}$ such that $(\theta(t), u(t)) \rightarrow (\theta^e, u^e)$ as $t \rightarrow \infty$. In fact, since $G(\theta(\cdot), u(\cdot)): [t_0, \infty) \rightarrow \mathcal{R}$ is nonincreasing, the approached (θ^e, u^e) belongs to the same (disjoint) component of the set

$$\mathcal{G}^0 \equiv \{(\theta, u) \in \mathcal{R}^1 \times \mathcal{R}^3 \mid G(\theta, u) \leq G(\theta^0, u^0)\} \quad (18)$$

in which (θ^0, u^0) lies, and $(\theta(t), u(t))$ belongs to this component for all $t \geq t_0$. For (θ^0, u^0) such that $G(\theta^0, u^0)$ is sufficiently small, the identified component will contain only one equilibrium which, therefore, is the unique equilibrium approached by $(\theta(t), u(t))$.

If (θ^0, u^0) is such that $G(\theta^0, u^0)$ is large and the identified component of $\mathcal{G}^0$ contains more than one equilibrium (θ^e, u^e) , it is impossible to tell which one of these equilibria is approached unless the “dissipative torques” τ_1 , τ_2 , and τ_f are completely specified. Here we have assumed only that these torques are strictly decreasing functions with known zeroes ω_1 , ω_2 , and 0, respectively.

Example 4.2. We modify Example 4.1 by replacing the motors with constant-speed motors (speed ω_1 for the lower, speed ω_2 for the upper). Defining $\psi(t) = \omega_1 t$ and $\phi(t) = \omega_2 t$, we see that we need only one generalized coordinate and define $q \equiv \theta \in \mathcal{O} \equiv \mathcal{R}^1$. Thus $u = u_\theta \in \mathcal{R}^1$, Q is strongly pseudodissipative (since $\tau'_f(u_\theta) < 0$ for all $u_\theta \in \mathcal{R}$), no coordinates are ignorable ($\tilde{q} = q \equiv \theta$, $r = n = 1$), $\mathcal{E}(t) \equiv \mathcal{R}^1 \times \mathcal{R}^1$, $D(t, \theta, u_\theta) = \tau_f(u_\theta) \in \mathcal{R}^1$, and

$$\begin{aligned} L(t, \theta, u_\theta) = & \frac{1}{2}[\bar{J}_1 u_\theta^2 + \bar{J}_2 \omega_1^2 \sin^2 \theta + J_3(\omega_2^2 + 2\omega_2 \omega_1 \cos \theta) \\ & + (I_A + \bar{J}_3 \cos^2 \theta) \omega_1^2] - mgl \cos \theta. \end{aligned} \quad (19)$$

We define parameters (σ, ζ) as in (17), and then determine the set of all equilibria

$$\mathcal{E} = \{(\theta, u_\theta) \in \mathcal{R}^1 \times \mathcal{R}^1 \mid u_\theta = 0, (\sigma \cos \theta - \zeta) \sin \theta = 0\}$$

by employing Corollary 2.1. We see that each equilibrium $(\theta^e, 0)$ is isolated from all others. Some equilibria are unstable and the others are asymptotically stable; the stability properties of all equilibria were determined for almost all values of σ and ζ in [9]. In [10] these stability properties were shown to parallel those for the equilibria of Example 4.1.

We choose $h(\theta) \equiv 0 \in \mathcal{R}^1$, $p(\theta) \equiv 0 \in \mathcal{R}^1$ and note that $(h(\theta), p(\theta)) \in \mathcal{E}$, $S(\theta, u_\theta) = 0$, and $2G(\theta, u_\theta) = J_1 u_\theta^2 + 2\zeta(1 - \cos \theta) + \sigma(1 - \cos^2 \theta)$.

All conditions of Corollary 3.1 are met, so to each initial state $(\theta^0, u_\theta^0) \in \mathcal{R}^1 \times \mathcal{R}^1$ there corresponds some particular (and isolated) $(\theta^e, u_\theta^e) \in \mathcal{E}$ such that $(\theta(t), u_\theta(t)) \rightarrow (\theta^e, u_\theta^e)$ as $t \rightarrow \infty$. All conclusions reached in Example 4.1 with respect to the set $\mathcal{G}^0$ of (18) remain valid here for the re-definition

$$\mathcal{G}^0 \equiv \{(\theta, u_\theta) \in \mathcal{R}^1 \times \mathcal{R}^1 \mid G(\theta, u_\theta) \leq G(\theta^0, u_\theta^0)\} \quad (20)$$

in terms of the current function G .

FIG. 2. Modified skateboard.

Example 4.3. Consider a skateboard (mass m , mass center C , $l > 0$) allowed to move upon a rough horizontal surface α (assumed inertial), but replace one wheel-assembly by a ball-bearing. See Fig. 2. We shall ignore both mass and bearing friction of the ball and the rear wheels, which are assumed to roll on α . Choosing the board β to be our collection and defining $q \equiv (\theta, x, y) \in \mathcal{R}^3 \equiv \mathcal{O}$, we see that $u = (u_\theta, u_x, u_y) \in \mathcal{R}^3$. T and $\mathcal{E}$ are time-invariant, $Q(t, q, u) \equiv 0 \in \mathcal{R}^3$ is pseudoconservative with $U(t, q, u) \equiv 0$, and coordinates (x, y) are ignorable. Hence, we may define $\tilde{q} \equiv \theta \in \tilde{\mathcal{O}} \equiv \mathcal{R}^1$ and

$$\begin{aligned}\tilde{L}(t, \theta, u) &= T(t, q, u) \\ &= \frac{m}{2}(u_x^2 + u_y^2) + \frac{I_B}{2}u_\theta^2 \\ &\quad + mlu_\theta(u_y \cos \theta - u_x \sin \theta), \\ \tilde{\mathcal{E}}(t) &= \{(\theta, u) \in \mathcal{R}^1 \times \mathcal{R}^3 \mid u_x \sin \theta = u_y \cos \theta\}, \\ \tilde{D}(t, \theta, u) &= D(t, q, u) = (0, 0, 0) \in \mathcal{R}^3.\end{aligned}$$

Constant solutions of (3)–(4) result from all initial states (θ^0, u^0) in the equilibrium set

$$\mathcal{E} = \{(\theta, u) \in \mathcal{R}^1 \times \mathcal{R}^3 \mid u_\theta = 0, u_x \sin \theta = u_y \cos \theta\}.$$

(See Corollary 2.1.) Note that no equilibrium is isolated in $\mathcal{R}^1 \times \mathcal{R}^3$, and $\mathcal{E}$ corresponds to all straight-line motions of the board.

Conditions (a) and (b) of Corollary 3.1 are violated, but the more general Theorem 3.1 may be applicable. We choose some $\mu \in \mathcal{R}$ and define $h_\mu(\theta) \equiv \theta \in \mathcal{R}^1$,

$$p_\mu(\theta) \equiv (0, \mu \cos \theta, \mu \sin \theta) \in \mathcal{R}^3,$$

for all $\theta \in \mathcal{R}^1$. Hence, $F_\mu(t, \theta, u) = 0$,

$$G_\mu(\theta, u) = \frac{m}{2}[(u_x - \mu \cos \theta)^2 + (u_y - \mu \sin \theta)^2] + \frac{I_B}{2}u_\theta^2,$$

$$S_\mu(\theta, u) = \mu m(u_x \sin \theta - u_y \cos \theta)u_\theta/2 - \mu m l u_\theta^2$$

in (6)–(8), and $S_\mu(\theta, u) = -uml u_\theta^2$ for all $(\theta, u) \in \tilde{\mathcal{E}}(t_0)$. For any $\mu \in \mathcal{R}$ conditions (a)–(c) and (e) of Theorem 3.1 are met. Condition (d) is met for every $\mu \geq 0$. Note that $G_0(\theta, u)$ is the total energy, which is entirely kinetic.

Consider any initial state $(\theta^0, u^0) \in \tilde{\mathcal{E}}(t_0)$ and note that, for any choice of $\mu > 0$, Theorem 3.1 implies $u_\theta(t) \rightarrow 0$ as $t \rightarrow \infty$. Let $\nu(\cdot): [t_0, \infty) \rightarrow \mathcal{R}$ be such that

$$u_x(t) = \nu(t) \cos \theta(t), \quad u_y(t) = \nu(t) \sin \theta(t),$$

along the solutions of (3)–(4). Theorem 3.1 implies that

$$\frac{d^+}{dt}[I_B u_\theta^2(t) + m|\nu(t) - \mu|^2] = -2\mu m l u_\theta^2(t) \quad (21)$$

for any $\mu \in \mathcal{R}$, and $\mu \equiv 0$ leads to the energy conservation result

$$\frac{d^+}{dt}[I_B u_\theta^2(t) + m\nu^2(t)] = 0. \quad (22)$$

Hence, $(\theta(t), u(t)) \rightarrow \mathcal{E}$ as $t \rightarrow \infty$, and (21)–(22) lead to

$$\frac{d^+}{dt}\nu(t) = l u_\theta^2(t) \geq 0, \quad (23)$$

$$\frac{d^+}{dt}u_\theta^2(t) = -\frac{2ml}{I_B}u_\theta^2(t)\nu(t), \quad (24)$$

for all $t \geq t_0$.

We see that the equilibrium set $\mathcal{E}$ is a global attractor and (21) implies that a subset of $\mathcal{E}$,

$$\mathcal{E}^s \equiv \{(\theta, u) \in \mathcal{R}^1 \times \mathcal{R}^3 \mid u = (0, \nu \cos \theta, \nu \sin \theta), \text{ and } \nu > 0\},$$

is a stable set (see Definition IV.3.1 of [8]), while the relative complement $\mathcal{E}^u \equiv \mathcal{E} \setminus \mathcal{E}^s$ is an unstable set. In fact, (23)–(24) imply that every $(\theta^e, u^e) \in \mathcal{E}^u$ is an unstable equilibrium, every $(\theta^e, u^e) \in \mathcal{E}^s$ is stable, and no (θ^e, u^e) is asymptotically stable. Finally, (23)–(24) imply that if $(\theta^0, u^0) \notin \mathcal{E}^u$, then $(\theta(t), u(t)) \rightarrow \mathcal{E}^s$ as $t \rightarrow \infty$. $\mathcal{E}^s$ is an attractor for all $(\theta^0, u^0) \notin \mathcal{E}^u$.

As every $(\theta^0, u^0) \in \mathcal{E} = \mathcal{E}^s \cup \mathcal{E}^u$ is an equilibrium, we have ascertained the general behavior of all solutions of (3)–(4). Note that the “almost-global attractor” $\mathcal{E}^s$ corresponds to all straight-line motions for which the wheels are behind the mass center.

5. Concluding remarks. In [9] the idea of a “pseudodissipative system” was presented and described as a means of accounting for the ubiquitous effects of friction and the properties of real motors. However, Examples 4.1–4.2 demonstrate that

these real-world phenomena often simplify greatly both the analysis of, and the conclusions on, general asymptotic behavior. If we were to replace the realistic motors of Example 4.1 by ideal zero-torque motors ($\tau_1(x) = 0 = \tau_2(x)$ for all $x \in \mathcal{R}$), the set $\mathcal{E}$ would be greatly enlarged (see Example 4.2 of [10]) and our conclusion on asymptotic behavior would be much less refined. If instead we were to assume $\tau_f(x) = 0$ for all $x \in \mathcal{R}$ in Example 4.2, we could not even conclude that $u_\theta(t) \rightarrow 0$ as $t \rightarrow \infty$.

In all of our examples, we have found it advantageous to choose $(h(\tilde{q}), p(\tilde{q})) \in \mathcal{E}$ for all $\tilde{q} \in \mathcal{Q}$, even though Theorem 3.1 does not require this (Corollary 3.1 does). The function G of (6) is related to a simpler function G of [9, 10] wherein $(h(\tilde{q}), p(\tilde{q})) \equiv (\tilde{q}^e, u^e) \in \mathcal{E}$ for some specific equilibrium under stability investigation. Corollary 3.1 requires such a choice but Theorem 3.1 does not. In Example 4.3, notice that we did not choose $(h(\theta), p(\theta))$ to be a specific equilibrium, even through our choice for $(h(\theta), p(\theta)) \in \mathcal{E}$ for all $\theta \in \tilde{\mathcal{Q}} = \mathcal{R}^1$. Such choices for $(h(\tilde{q}), p(\tilde{q}))$ seem to be the most useful in Theorem 3.1.

In Example 4.3 we decomposed $\mathcal{E}$ into two disjoint subsets $\mathcal{E}^s$ and $\mathcal{E}^u$. We found $\mathcal{E}^s$ to be a stable set while $\mathcal{E}^u$ was an unstable set; we also reached stability conclusions for each equilibrium. Such stability conclusions cannot be reached for Example 4.3 on the basis of the results presented in [10][†]. This current success at stability analysis has resulted from the special choice made for $(h(\theta), p(\theta))$ in Example 4.3.

Herein we have defined the set $\mathcal{E}$ of all equilibria of (3)–(4), and we have been concerned with showing that $(\tilde{q}(t), u(t)) \rightarrow \mathcal{E}$ as $t \rightarrow \infty$, for all initial states $(\tilde{q}^0, u^0) \in \tilde{\mathcal{E}}(t_0)$. We note that $\mathcal{E}$ always can be decomposed into two disjoint sets $\mathcal{E}^s$ and $\mathcal{E}^u$ such that $\mathcal{E}^u$ consists solely of equilibria $(\tilde{q}^e, u^e)$ which are unstable. In any physical system there always are random small disturbances which are not taken into account by a deterministic mathematical model, so a conclusion (as in Examples 4.1–4.3) that $(\tilde{q}(t), u(t)) \rightarrow \mathcal{E}$ as $t \rightarrow \infty$ may be interpreted as meaning “in a practical sense” (or “with probability one”) that $(\tilde{q}(t), u(t)) \rightarrow \mathcal{E}^s$ as $t \rightarrow \infty$, for the physical system under consideration. Hence, practically speaking, the board in Example 4.3 eventually moves in a straight line with the mass center C ahead of the wheels, regardless of its initial state.

Example 4.3 demonstrates why successful design of a tricycle landing-gear is very difficult when the swiveling wheel is to be at the tail of an aircraft [2]. If the tail-wheel assembly too closely resembles the ball-bearing of Example 4.3, the tail prefers to be in front and the rolling aircraft may ground-loop.

In all three of our examples we were able to conclude that the set $\mathcal{E}$ of all equilibria was a global attractor; i.e., $(\tilde{q}(t), u(t)) \rightarrow \mathcal{E}$ as $t \rightarrow \infty$, for every $(\tilde{q}^0, u^0) \in \mathcal{E}(t_0)$. This conclusion is not too surprising in Examples 4.1–4.2, wherein Q is strongly pseudodissipative. However, it seems a quite remarkable conclusion in Example 4.3 wherein Q is pseudoconservative.

[†] In each of Theorems 3.1–3.3 of [10], condition (a) is violated by Example 4.3.

A simple extension of Theorem 3.1 can be made. In some systems which are not strongly pseudodissipative, one may know of one or more "motion integrals" $W_i: \tilde{\mathcal{Q}} \times \mathcal{R}^n \rightarrow \mathcal{R}$; that is, one knows *a priori* that $W_i(\tilde{q}(t), u(t))$ is constant along each solution of the motion equations (3)–(4). Choosing arbitrary real numbers λ_i , we see that Theorem 3.1 remains valid when G is replaced therein by $\hat{G} \equiv G + \sum \lambda_i W_i$. This extension may be very useful for some systems.

No motion integrals existed in Examples 4.1–4.2, wherein Q was strongly pseudodissipative. In the pseudoconservative Example 4.3 there did exist one motion integral $W_1 = T$, but this also was a consequence of our analysis; see (22). In none of our examples was there a momentum-type integral.

A reviewer feels that this work is closely related to the results of [6] and [7]. The author can see no connection.

REFERENCES

- [1] G. D. Birkhoff, *Dynamical Systems*, American Mathematical Society Colloquium Publications, Vol. 9, Providence, R. I., 1927; revised edition, 1966
- [2] J. P. Den Hartog, *Mechanical Vibrations*, McGraw-Hill, New York and London, 1956
- [3] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Mathematical Surveys and Monographs, No. 25, American Mathematical Society, Providence, R. I., 1988
- [4] J. P. LaSalle, *The Stability of Dynamical Systems*, CBMS Regional Conference Series in Applied Mathematics, Society for Industrial and Applied Mathematics, Philadelphia, 1976
- [5] E. J. McShane, *Integration*, Princeton University Press, New Jersey, 1947
- [6] H. Roh, *Dissipative operators with finite-dimensional damping*, Proc. Roy. Soc. of Edinburgh Sect. A **91**, 243–263 (1982)
- [7] P. Ruge and J. Schnieder, *Reduction and iteration for computing the eigenvalues of discrete damped vibrating systems*, Ingenieur Archive **50**, 297–313 (1981)
- [8] J. A. Walker, *Dynamical systems and Evolution Equations*, Plenum Press, New York and London, 1980
- [9] J. A. Walker, *pseudodissipative systems I: stability of generalized equilibria*, ASME Journal of Applied Mechanics **55** 681–686 (1988)
- [10] J. A. Walker, *Pseudodissipative systems II: stability of reduced equilibria*, ASME Journal of Applied Mechanics **56** 466–473 (1989)