OPTIMAL DISTRIBUTION OF KNOTS FOR TENSOR-PRODUCT SPLINE APPROXIMATION

BY

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Abstract. This paper deals with optimal asymptotic distribution of knots for various types of spline approximations.

Known results on the optimal distribution of knots, which asymptotically minimizes the $L_p[0, 1]$ error in spline approximation of a sufficiently smooth function, are extended to the following cases:

- (1) An optimal distribution for a simultaneous $L_p[0, 1]$ approximation of a finite set of functions, or a set which depends on a continuous parameter, e.g., a bivariate function.
- (2) An optimal distribution on each axis for $L_2[0, 1]^2$ tensor-product spline approximation. This results extends to $L_2[0, 1]^n$, $n > 2$.

A method of knot placement for tensor-product spline approximation is proposed, based on the theoretical optimal distribution.

1. Introduction. The purpose of this paper is to extend the results obtained by Barrow and Smith [1], and by Pence and Smith [4], about optimal distribution of knots in univariate spline approximation, to various types of spline approximations and in particular to approximation by tensor-product splines. First, we briefly survey the results obtained for the univariate spline approximation in [1], [4], and then describe their various extensions.

Given a sequence of knots $t^N := \{0 = t_0 < t_1 \leq \dots \leq t_{N-1} < t_N = 1\}$, let $S_N^k(t^N)$ denote the space of polynomial splines of order k (degree less than k) with knots t^N . A function t is said to be a *distribution* if $t \in C^1[0, 1]$, $t(0) = 0$, $t(1) = 1$, and $t' > 0$ on $[0, 1]$. Let $S_N^k(t) := S_N^k(\{t(i/N), i = 0, 1 \dots N\})$, then the precise behavior of the $L_p[0, 1]$ distance of some $f \in C^k[0, 1]$ from $S_N^k(t)$ as $N \rightarrow \infty$ is given by

$$\lim_{N \rightarrow \infty} N^k \|(I - P_N^{k,p}(t))f\|_{L_p[0,1]} = C_{k,p}(J_{k,p}(f, t))^{1/p}, \quad 1 \leq p < \infty, \quad (1.1)$$

where $P_N^{k,p}(t)$ denotes the $L_p[0, 1]$ metric projection onto $S_N^k(t)$, $C_{k,p}$ is a known

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constant and

$$J_{k,p}(f, t) = \int_0^1 |f^{(k)}(t(x))|^p (t'(x))^{kp+1} dx. \quad (1.2)$$

A function u is said to be a *positive density* if $u \in C^0[0, 1]$, $u > 0$, and $\int_0^1 u(\tau) d\tau = 1$. Redefining (1.2) in terms of a positive density $u(\tau) = \frac{\partial}{\partial \tau} t^{-1}(\tau)$, with t^{-1} the inverse function of t , we have:

$$J_{k,p}(f, u) = \int_0^1 |f^{(k)}(\tau)|^p (u(\tau))^{-kp} d\tau. \quad (1.3)$$

Let D^+ denote the class of positive density functions, then the minimum of (1.3) over the closure of D^+ is attained uniquely by the function

$$u^*(\tau) = |f^{(k)}(\tau)|^\sigma / \int_0^1 |f^{(k)}(\xi)|^\sigma d\xi, \quad (1.4)$$

where

$$\sigma = \frac{p}{kp+1}, \quad (1.5)$$

and as is easily seen

$$J_{k,p}(f, u^*) = \|f^{(k)}\|_{L_\sigma[0,1]}^p. \quad (1.6)$$

In case $u^* \notin D^+$, the sequence $\{u_j\}$ defined by $u_j = (u^* + \varepsilon_j)/(1 + \varepsilon_j)$ with $\varepsilon_j > 0$, $\lim_{j \rightarrow \infty} \varepsilon_j = 0$ is a sequence of positive densities satisfying

$$\lim_{j \rightarrow \infty} J_{k,p}(f, u_j) = J_{k,p}(f, u^*).$$

Hence, u^* is said to be the optimal density of knots in the asymptotic sense of $N \rightarrow \infty$. For N large, the asymptotically optimal knots $(t^*)^N$ are determined by the equations

$$\int_{t_i^*}^{t_{i+1}^*} u^*(x) dx = \frac{1}{N}, \quad i = 0, 1, \dots, N-1.$$

These results also hold under weaker hypotheses on f and t (see [4]). Let S_N^k be the set of k th order splines on $[0, 1]$ having at most $N-1$ interior knots counting multiplicities, and let $\{f_N^k\}$ be the best approximants to f from S_N^k . An important result is that the knots of $\{f_N^k\}$ tend uniquely to the optimal distribution as $N \rightarrow \infty$. For the exact convergence sense see Theorem 3 in [1].

Our first extension of these results is to the case of approximation of a finite set of functions. Let $F := \{f_i\}_{i=1}^m \subset C^k[0, 1]$ be approximated by $S_N^k(t)$, where the simultaneous approximation error is defined by

$$E_N^{k,p}(t)F = \left(\sum_{i=1}^m \|(I - P_N^{k,p}(t))f_i\|_{L_p[0,1]}^p \right)^{1/p}. \quad (1.7)$$

By deriving an analogous result to (1.1), we obtain in Sec. 2 an optimal distribution of knots for this approximation. A similar extension to the simultaneous approximation

of a family of functions depending on a continuous parameter $F := \{f(\cdot, y) \mid 0 \leq y \leq 1\}$, is carried out in Sec. 3. Let $f(x, y)$ be a sufficiently smooth function defined on $[0, 1]^2$, and denote by $U_N^{k,p}(t)f(\cdot, y)$ the $L_p[0, 1]$ metric projection of $f(\cdot, y)$ onto $S_N^k(t)$ for each $0 \leq y \leq 1$. Defining the approximation error by

$$\check{E}_N^{k,p}(t)F = \left(\int_0^1 \left(\int_0^1 |(I - U_N^{k,p}(t))f|^p dx \right) dy \right)^{1/p}, \quad (1.8)$$

we obtain an analogous result to (1.1) which yields the optimal density of knots $\check{u}^*$. Similarly, by exchanging the roles of x and y we get the optimal density $\check{u}^*$ for the approximation by $V_N^{k,p}(s)f(x, \cdot)$ —the $L_p[0, 1]$ metric projection of $f(x, \cdot)$ onto $S_N^k(s)$, for each $0 \leq x \leq 1$.

In Sec. 4, we consider the operators $U := U_{N_1}^{k_1,2}$, $V := V_{N_2}^{k_2,2}$ and prove that $UVf = VUf$ is the $L_2[0, 1]^2$ projection of f onto $S_{N_1}^{k_1}(t) \otimes S_{N_2}^{k_2}(s)$. By using the orthogonal projection properties of U, V, UV we get:

$$\max(\|(I - U)f\|, \|(I - V)f\|) \leq \|(I - UV)f\| \leq (\|(I - U)f\|^2 + \|(I - V)f\|^2)^{1/2}, \quad (1.9)$$

where all the norms are those of $L_2[0, 1]^2$. Thus $\check{u}^*$ and $\check{u}^*$ are nearly optimal densities for approximation by tensor-product splines. Based on (1.9), a method for knot placement is proposed. The knots are placed according to the densities $\check{u}^*$ and $\check{u}^*$, with N_1, N_2 chosen so that $\|(I - U)f\| \sim \|(I - V)f\|$, as implied by (1.9).

In Sec. 5, we show that $\check{u}^*$ and $\check{u}^*$ are optimal densities in the following sense: For any fixed distribution s and N_2 knots on the Y -axis there is a density of knots on the X -axis $\check{u}_{s, N_2}^*$ which is optimal for the tensor-product approximation. We prove that $\{\check{u}_{s, N_2}^*\}$ converges to $\check{u}^*$ as $N_2 \rightarrow \infty$, in a sense to be made precise in Sec. 5. Thus, $\check{u}^*$ is optimal as a limit of optimal densities and the same applies to $\check{u}^*$.

Finally in Sec. 6 we note that our results extend easily to tensor-product splines in $\mathbb{R}^n$.

2. Optimal distribution of knots for approximation of a finite set of functions.

THEOREM 2.1. Let $F = \{f_i\}_{i=1}^m \subset C^k[0, 1]$, $1 \leq p < \infty$, t a distribution, and $E_N^{k,p}(t)F$ the error defined by (1.7). Then

$$\lim_{N \rightarrow \infty} N^k E_N^{k,p}(t)F = \lim_{N \rightarrow \infty} N^k \|(I - P_N^{k,p}(t))g\|_{L_p[0, 1]}, \quad (2.1)$$

where g is any function satisfying

$$g^{(k)} = \left(\sum_{i=1}^m |f_i^{(k)}|^p \right)^{1/p}. \quad (2.2)$$

In addition, minimization of (2.1) over D^+ yields

$$\inf_{v \in D^+} \lim_{N \rightarrow \infty} N^k E_N^{k,p}(u)F = C_{k,p} \|g^{(k)}\|_{L_p[0, 1]}. \quad (2.3)$$

Proof. In view of (1.1) and (1.2) the expression $\lim_{N \rightarrow \infty} N^{pk} \|(I - P_N^{k,p}(t))f_i\|_{L_p[0,1]}^p$ is additive in f_i . This together with (2.2) implies

$$\lim_{N \rightarrow \infty} (N^k E_N^{k,p}(t)F)^p = (C_{k,p})^p \sum_{i=1}^m J_{k,p}(f_i, t) = (C_{k,p})^p J_{k,p}(g, t),$$

which proves (2.1). Result (2.3) is a direct consequence of (1.6). $\square$

COROLLARY 2.2. The optimal density for the approximation of the set F is the optimal one for the approximation of g satisfying (2.2). This density is given by

$$u^*(x) = \left(\sum_{i=1}^m |f_i^{(k)}(x)|^p \right)^{\sigma/p} / \int_0^1 \left(\sum_{i=1}^m |f_i^{(k)}(\tau)|^p \right)^{\sigma/p} d\tau. \quad (2.4)$$

3. Optimal distribution of knots for approximation of a family of functions depending on a continuous parameter. In the following we use the notation $f_y(\cdot) := f(\cdot, y)$ for a function f defined on $[0, 1]^2$.

THEOREM 3.1. Let t be a distribution, $F = \{f(\cdot, y) \mid 0 \leq y \leq 1\}$, $1 \leq p < \infty$ and let $\tilde{E}_N^{k,p}(t)F$ be defined as in (1.8) for f with the property $(\partial^k / \partial x^k)f \in C[0, 1]^2$. Then

$$\lim_{N \rightarrow \infty} N^k \tilde{E}_N^{k,p}(t)F = \lim_{N \rightarrow \infty} N^k \|(I - P_N^{k,p}(t))g\|_{L_p[0,1]}, \quad (3.1)$$

where g is any function satisfying

$$g^{(k)}(x) = \left(\int_0^1 \left| \frac{\partial^k}{\partial x^k} f(x, y) \right|^p dy \right)^{1/p}. \quad (3.2)$$

In addition, minimization of (3.1) over D^+ yields:

$$\inf_{u \in D^+} \lim_{N \rightarrow \infty} N^k \tilde{E}_N^{k,p}(u)F = C_{k,p} \|g^{(k)}\|_{L_p[0,1]}. \quad (3.3)$$

Proof. By the definition of $\tilde{E}_N^{k,p}(t)F$:

$$\lim_{N \rightarrow \infty} N^k \tilde{E}_N^{k,p}(t)F = \lim_{N \rightarrow \infty} \left(\int_0^1 E_N(y)f dy \right)^{1/p}, \quad (3.4)$$

where

$$E_N(y)f = N^{pk} \int_0^1 |(I - P_N^{k,p}(t))f_y|^p dx. \quad (3.5)$$

By the hypotheses on f , $E_N(y)f \in C[0, 1]$ and the integral on the right side of (3.4) is well defined.

In view of (1.1)

$$\lim_{N \rightarrow \infty} E_N(y)f = (C_{k,p})^p \cdot J_{k,p}(f_y, t). \quad (3.6)$$

By a Jackson type theorem for splines (see [5], Theorem 6.27)

$$E_N(y)f \leq C_1(k, p) \cdot N^{kp} \cdot \tilde{t}^{kp} (\|f_y^{(k)}(x)\|_{L_p[0,1]})^p, \quad (3.7)$$

where $\bar{t} = \max_i(t_{i+1} - t_i)$ and $C_1(k, p)$ is a constant depending on k and p . In view of the quasi-uniformity condition $t' > 0$, $t \in C^1[0, 1]$ (see [1]), $\bar{t} = O(\frac{1}{N})$, and by the hypotheses on f , the sequence $\{E_N(y)f\}$ is therefore uniformly bounded in y and in N , hence by (3.6) and the Lebesgue convergence Theorem we have

$$\lim_{N \rightarrow \infty} \int_0^1 E_N(y)f dy = \int_0^1 (C_{k,p})^p \cdot J_{k,p}(f_y, t) dy. \quad (3.8)$$

Substituting $J_{k,p}(f_y, t)$ by (1.2), we get an iterated integral with an integrand in $C[0, 1]^2$, thus we may change the order of integration to get

$$\lim_{N \rightarrow \infty} N^k E_N^{k,p}(t)F = C_{k,p}(J_{k,p}(g, t))^{1/p}, \quad (3.9)$$

with $g^{(k)}$ defined by (3.2). This together with (1.1) and (1.6) completes the proof of the theorem. $\square$

COROLLARY 3.2. The optimal density for the approximation of $F = \{f(\cdot, y) \mid 0 \leq y \leq 1\}$ is given by

$$u^*(x) = \left(\int_0^1 \left| \frac{\partial^k f(x, y)}{\partial x^k} \right|^p dy \right)^{\sigma/p} / \int_0^1 d\tau \left(\int_0^1 \left| \frac{\partial^k f(\tau, y)}{\partial \tau^k} \right|^p dy \right)^{\sigma/p}. \quad (3.10)$$

4. Algebra of orthogonal projections in $L_2[0, 1]^2$. First we introduce the following notations: $t(x)$, $s(y)$ are distributions, and $\{u_i(x)\}_{i=1}^{n_1}$, $\{v_j(y)\}_{j=1}^{n_2}$ are bases of $S_{N_1}^{k_1}(t)$, $S_{N_2}^{k_2}(s)$, respectively, where $n_1 = k_1 + N_1 - 1$, $n_2 = k_2 + N_2 - 1$. In the following we assume

$$f \in L_2[0, 1]^2 \quad (4.1)$$

and by the Fubini Theorem we have

$$\begin{aligned} f_y(x) &\in L_2[0, 1] \quad \text{for almost every } 0 \leq y \leq 1 \\ f_x(y) &\in L_2[0, 1] \quad \text{for almost every } 0 \leq x \leq 1. \end{aligned} \quad (4.2)$$

Hence the functions Uf, Vf are defined on $[0, 1]^2$ a.e., where U, V are the operators defined in the Introduction.

Denote the $L_2[0, 1]^2$ metric projection of f onto $S_{N_1}^{k_1}(t) \otimes S_{N_2}^{k_2}(s)$ by Qf , then Q is clearly an orthogonal projection operator. For a detailed description of the tensor-product splines see [2], [5].

THEOREM 4.1. Let $f \in L_2[0, 1]^2$ then

$$Uf, Vf \in L_2[0, 1]^2 \quad (4.3)$$

$$UVf = VUf = Qf. \quad (4.4)$$

Proof. Suppose $\{u_i(x)\}_{i=1}^{n_1}, \{v_j(y)\}_{j=1}^{n_2}$ are orthonormal bases, then

$$Qf = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \left(\int_0^1 \int_0^1 f(\xi, \eta) u_i(\xi) v_j(\eta) d\xi d\eta \right) u_i(x) v_j(y), \quad (4.5)$$

$$Uf = \sum_{i=1}^{n_1} \left(\int_0^1 f(\xi, y) u_i(\xi) d\xi \right) u_i(x), \quad (4.6)$$

$$Vf = \sum_{j=1}^{n_2} \left(\int_0^1 f(x, \eta) v_j(\eta) d\eta \right) v_j(y). \quad (4.7)$$

An integral operator K defined by $Kg = \int_0^1 k(x, y)g(y)dy$ maps $L_2[0, 1]$ into $L_2[0, 1]$ if $k \in L_2[0, 1]^2$. Thus (4.3) follows from (4.6) and (4.7). Moreover $U(Vf)$ and $V(Uf)$ are well defined and by integration we get (4.4). $\square$

THEOREM 4.2. U and V are orthogonal projection operators with respect to the norm $L_2[0, 1]^2$.

Proof. A bounded linear operator P defined on a Hilbert space is an orthogonal projection if and only if P is self-adjoint and $P = P^2$. These properties of U, V are easily verified. $\square$

COROLLARY 4.3.

$$\begin{aligned} \|(I - U)f\|^2 &= \|f\|^2 - \|Uf\|^2, \\ \|(I - V)f\|^2 &= \|f\|^2 - \|Vf\|^2, \\ \|(I - UV)f\|^2 &= \|f\|^2 - \|UVf\|^2, \end{aligned} \quad (4.8)$$

$$\begin{aligned} \|UVf\| &\leq \|Uf\| \leq \|f\|, \\ \|UVf\| &\leq \|Vf\| \leq \|f\|, \end{aligned} \quad (4.9)$$

$$\begin{aligned} \|(I - UV)f\|^2 &= \|(I - V)f\|^2 + \|(I - U)Vf\|^2, \\ \|(I - UV)f\|^2 &= \|(I - U)f\|^2 + \|(I - V)Uf\|^2. \end{aligned} \quad (4.10)$$

THEOREM 4.4.

$$\begin{aligned} \max(\|(I - U)f\|, \|(I - V)f\|) &\leq \|(I - UV)f\| \\ &\leq (\|(I - U)f\|^2 + \|(I - V)f\|^2)^{1/2}. \end{aligned} \quad (4.11)$$

Proof. The left inequality follows from Corollary 4.3. Since U is an orthogonal projection we have $\|(I - V)Uf\|^2 = \|U(I - V)f\|^2 \leq \|(I - V)f\|^2$. This together with (4.10) yields the right inequality. $\square$

By the Fubini Theorem $\|(I - U)f\| = \mathbb{E}_{N_1}^{x_{k_1}, 2}(t)F$ where $F = \{f(\cdot, y) \mid 0 \leq y \leq 1\}$, and $\|(I - V)f\| = \mathbb{E}_{N_2}^{y_{k_2}, 2}(s)F$ where $F = \{f(x, \cdot) \mid 0 \leq x \leq 1\}$. Thus Theorem 4.4 indicates that a close to optimal distribution of knots for tensor-product $L_2[0, 1]^2$ spline approximation is given by the optimal knot distribution for U in the x direction and the optimal knot distribution for V in the y direction. By Corollary

3.2 these densities are

$$u^*(x) = \frac{\psi_1(x)}{\int_0^1 \psi_1(\tau) d\tau}, \quad \psi_1(x) = \left(\int_0^1 \left| \frac{\partial^{k_1}}{\partial x^{k_1}} f \right|^2 dy \right)^{\sigma_1/2}, \quad \sigma_1 = \frac{1}{k_1 + \frac{1}{2}}, \quad (4.12)$$

$$u^*(y) = \frac{\psi_2(y)}{\int_0^1 \psi_2(\tau) d\tau}, \quad \psi_2(y) = \left(\int_0^1 \left| \frac{\partial^{k_2}}{\partial y^{k_2}} f \right|^2 dx \right)^{\sigma_2/2}, \quad \sigma_2 = \frac{1}{k_2 + \frac{1}{2}}. \quad (4.13)$$

By (4.11), a method for knot placement based on these distributions should choose N_1 and N_2 so that

$$\|(I - U)f\| \cong \|(I - V)f\|,$$

namely

$$\frac{1}{N_1^{k_1}} C_{k_1, 2} \left(\int_0^1 \psi_1(x) dx \right)^{1/\sigma_1} \cong \frac{1}{N_2^{k_2}} C_{k_2, 2} \left(\int_0^1 \psi_2(y) dy \right)^{1/\sigma_2}, \quad (4.14)$$

as is easily verified by Theorem 3.1.

5. Optimal distributions in approximation by tensor-product splines. In this section we show that the close to optimal distributions of knots for $L_2[0, 1]^2$ tensor-product spline approximation presented in Sec. 4, are in fact optimal in an appropriate sense.

LEMMA 5.1. Let $f(x, y)$ with $(\partial^{k_1}/\partial x^{k_1})f(x, y) \in C[0, 1]^2$ then

$$\frac{\partial^{k_1}}{\partial x^{k_1}} (Vf) = V \left(\frac{\partial^{k_1}}{\partial x^{k_1}} f \right) = \sum_{j=1}^{n_2} \bar{v}_j(x) v_j(y), \quad (5.1)$$

where $\bar{v}_j \in C[0, 1]$, $j = 1, \dots, n_2$.

Proof. By (4.7) we have

$$\frac{\partial^{k_1}}{\partial x^{k_1}} Vf = \sum_{j=1}^{n_2} v_j(y) \bar{v}_j(x),$$

where $\bar{v}_j(x) = (\partial^{k_1}/\partial x^{k_1}) \int_0^1 v_j(y) f(x, y) dy$ and by the assumption on f , (5.1) follows. Since $v_j(y) \cdot (\partial^{k_1}/\partial x^{k_1}) f(x, y)$ is a continuous function in x , then according to [3] Theorem 4.16 $\bar{v}_j(x)$ is continuous. $\square$

THEOREM 5.2. Let $f(x, y)$ with $(\partial^{k_1}/\partial x^{k_1})f(x, y) \in C[0, 1]^2$, and let N_2 knots on the Y -axis be determined by a fixed distribution s . Then the optimal density on the X -axis for the approximation by tensor-product splines is given by

$$u_{s, N_2}^*(x) = \frac{|g^{(k_1)}(x)|^{\sigma_1}}{\int_0^1 |g^{(k_1)}(\tau)|^{\sigma_1} d\tau}, \quad (5.2)$$

where

$$g^{(k_1)} = \left(\int_0^1 \left(\frac{\partial^{k_1}}{\partial x^{k_1}} (Vf) \right)^2 dy \right)^{1/2}. \quad (5.3)$$

Proof. In view of (4.10) and the fact that $V = V_{N_2}^{k_1, 2}$ is fixed, the optimal distribution on the X -axis for $F = \{Vf(\cdot, y) \mid 0 \leq y \leq 1\}$ minimizes $\|(I - U)Vf\|$. According to Theorem 3.1, (5.2) is the optimal density if $(\partial^{k_1}/\partial x^{k_1})(Vf) \in C[0, 1]^2$. A close examination of the arguments in the proof of Theorem 3.1 shows that the theorem holds for Vf with the smoothness properties given by Lemma 5.1, since $\{v_j(y)\} \in S_{N_2}^{k_2}(s)$. $\square$

The next theorem claims that $\tilde{u}^*(x)$, $\tilde{u}^*(y)$ (see (4.12), (4.13)) are optimal densities in the following sense:

THEOREM 5.3. Under the conditions of Theorem 5.2

$$\lim_{N_2 \rightarrow \infty} \tilde{u}_{s, N_2}^*(x) = \tilde{u}^*(x), \quad (5.4)$$

$$\lim_{N_2 \rightarrow \infty} \int_0^x \tilde{u}_{s, N_2}^*(\tau) d\tau = \int_0^x \tilde{u}^*(\tau) d\tau, \quad (5.5)$$

for any fixed distribution s .

Proof. Let $\psi_{N_2}(x)$, $\psi(x)$ be given by

$$\psi_{N_2}(x) = \left(\int_0^1 \left| V \frac{\partial^{k_1}}{\partial x^{k_1}} f \right|^2 dy \right)^{\sigma_1/2}, \quad (5.6)$$

$$\psi(x) = \left(\int_0^1 \left| \frac{\partial^{k_1}}{\partial x^{k_1}} f \right|^2 dy \right)^{\sigma_1/2}. \quad (5.7)$$

For any $h \in L_2[0, 1]$ and any distribution $\tilde{t}$ it is clear that

$$\lim_{N \rightarrow \infty} \|h - P_N^{k_1, 2}(\tilde{t})h\|_{L_2[0, 1]} = 0.$$

Hence the pointwise convergence $\psi_{N_2} \xrightarrow{N_2 \rightarrow \infty} \psi$ is obvious. By orthogonal projection arguments we have

$$0 \leq \psi_{N_2}(x) \leq \psi(x), \quad 0 \leq x \leq 1,$$

and the continuity of an orthogonal projection operator together with the continuity of $(\partial^{k_1}/\partial x^{k_1})f$ yields that $\psi_{N_2} \in C[0, 1]$. By invoking the Lebesgue convergence Theorem we have $\lim_{N_2 \rightarrow \infty} \|\psi_{N_2}\|_{L_1[0, x]} = \|\psi\|_{L_1[0, x]}$, $0 \leq x \leq 1$. Since $\tilde{u}^*(x) = \psi(x)/\|\psi\|_{L_1[0, 1]}$, $\tilde{u}_{s, N_2}^*(x) = \psi_{N_2}(x)/\|\psi_{N_2}\|_{L_1[0, 1]}$ the proof is completed. $\square$

A similar result holds when the role of the axes is exchanged.

It is of interest to note that there exists a family of functions with the property $u_{s, N_2}^*(x) \equiv \tilde{u}^*(x)$ independently of s and N_2 , and a similar property with the roles of the axes exchanged. This family includes all functions of the form

$$F(x) \cdot G(y) + T(x, y), \quad (5.8)$$

where $F \in C^{k_1}[0, 1]$, $G \in C^{k_2}[0, 1]$, and $(\partial^{k_1}/\partial x^{k_1})(\partial^{k_2}/\partial y^{k_2})T(x, y) \equiv 0$ on $[0, 1]^2$. In particular, $T(x, y)$ can be written as

$$\sum_{i=1}^{r_1} p_i(x)G_i(y) + \sum_{j=1}^{r_2} F_j(x)q_j(y), \quad r_1 \geq 0, r_2 \geq 0, \quad (5.9)$$

where

$$\begin{aligned} G_i &\in C^{k_2}[0, 1], & p_i &\in \pi_{k_1}[0, 1], \quad i = 1 \dots r_1, \\ F_j &\in C^{k_1}[0, 1], & q_j &\in \pi_{k_2}[0, 1], \quad j = 1 \dots r_2. \end{aligned}$$

Here $\pi_k[0, 1]$ denotes the space of polynomials of order k on $[0, 1]$.

6. Extensions to $\mathbb{R}^n$. Let $f(x_1 \dots x_n) \in L_2[0, 1]^n$ be approximated by the tensor-product spline space $\bigotimes_{i=1}^n S_{N_i}^{k_i}(t^i)$, where t^i is a distribution on the x_i -axis, $i = 1 \dots n$. By defining $U_i f$ analogously to the 2D case, it is clear that $(\prod_{i=1}^n U_i)f$ is the $L_2[0, 1]^n$ metric projection of f onto $\bigotimes_{i=1}^n S_{N_i}^{k_i}(t^i)$, and the algebra of orthogonal projections presented in Sec. 4 is extended. In particular the following estimate holds:

$$\max_i \|(I - U_i)f\| \leq \left\| \left(I - \prod_{i=1}^n U_i \right) f \right\| \leq \left(\sum_{i=1}^n \|(I - U_i)f\|^2 \right)^{1/2}, \quad (6.1)$$

where all the norms are those of $L_2[0, 1]^n$.

The optimal density for the approximation of f by $U_i f$ is a direct extension of that in Sec. 3. This extension follows from the fact that the Lebesgue convergence Theorem holds on an n -dimensional cube. The sense of optimality given in Sec. 5 also extends easily.

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