QUARTERLY OF APPLIED MATHEMATICS
VOLUME XLIX, NUMBER 1
MARCH 1991, PAGES 87-105

ANTICROWDING POPULATION MODELS IN SEVERAL SPACE VARIABLES

By

GASTON E. HERNANDEZ

University of Connecticut, Storrs, Connecticut

1. Introduction. In this paper we discuss a model for diffusion of populations with age dependence in several space variables. The fundamental principle states that in a fixed region in space the population can change only through births, deaths, immigration, and emigration (see [12]). In a small interval of time dt the change in the population ρ is

$$D\rho = (B - D + I - E)\,dt. \tag{1.1}$$

1.1. *Age dependent populations.* Let $\rho(t, a)$ be the age population density, i.e., $\int_{a_1}^{a_2} \rho(t, a)\,da$ represents the number of individuals at time t of ages between a_1 and a_2. In particular $u(t) = \int_0^\infty \rho(t, a)\,da$ is the total population at time t. A change of h units in time implies a change of h units in age. Thus assuming differentiability $D\rho = \rho_t + \rho_a$. When using Malthus's law for births and deaths, with $I = E = 0$, eq. (1.1) becomes

$$\rho_t + \rho_a = -\mu\rho, \tag{1.2}$$

where μ might depend on a. Integrating along characteristics we arrive at the formal solution

$$\rho(t, a) = \begin{cases} \rho(0, a-t)e^{-\int_0^t \mu(a-t+s)\,ds} & t \le a, \\ \rho(t-a, 0)e^{-\int_0^t \mu(s)\,ds} & t \ge a. \end{cases} \tag{1.3}$$

Thus in addition to specifying the initial age distribution $\rho(0, a) = \rho_0(a)$, we also need to specify $\rho(t, 0)$, the number of newborns at time t. Assuming that the population sex ratio remains constant, the birth rate $\beta(a)$ is defined such that $\beta(a)\,da$ represents the average number of offsprings produced per unit time by an individual aged between a and $a + da$. In this way there is a birth law

$$B(t) = \rho(t, 0) = \int_0^\infty \beta(a)\rho(t, a)\,da. \tag{1.4}$$

The Lotka-Von Foerster model (also McKendrick-Von Foerster) consists of Eqs. (1.2) and (1.4) along with a nonnegative initial condition $\rho_0(a) \ge 0$. In [15, 18] Gurtin and MacCamy proposed a model in which the birth modulus and the death

Received February 12, 1990.

modulus depend also on the total population $u(t)$. The Gurtin-MacCamy model for age dependent populations without diffusion is:

$$\rho_t + \rho_a = -\mu(a, u)\rho,$$
$$\rho(t, 0) = B(t) = \int_0^\infty \beta(a, u)\rho(t, a)\, da,$$
$$\rho(0, a) = \rho_0(a) \geq 0, \qquad u(t) = \int_0^\infty \rho(t, a)\, da.$$

1.2. *Diffusion of populations.* Let $\rho(\mathbf{x}, t)$ be the number of individuals present at time t at position $\mathbf{x}$, $\mathbf{x} \in \mathbf{R}^N$. Immigration and emigration are modeled here by diffussion

$$\rho_t + \operatorname{div} \mathbf{v} = \sigma, \tag{1.6}$$

where $\mathbf{v}(\mathbf{x}, t)$ is the diffusion velocity and $\sigma(\mathbf{x}, t)$ the population supply. The first model of this type was given by Skellam [28] in 1951. Assuming random motion of the individuals, $\mathbf{v} = -k\nabla\rho$, he arrived at

$$\rho_t = k\Delta\rho + \sigma(t), \tag{1.7}$$

where k and μ are constants. It has been observed however that several species instead of dispersing at random actually disperse to avoid crowding (see for instance [7]). This corresponds to $\mathbf{v} = -k\rho\nabla\rho$ which gives

$$\rho_t = k \operatorname{div}(\rho\nabla\rho) + \sigma. \tag{1.8}$$

In [12] Gurney and Nibset arrived at this equation after considering a probabilistic walk in which individuals either stay at their present location or move in a direction of decreasing population. In [16] Gurtin and MacCamy considered

$$\rho_t = \Delta\varphi(\rho) + \sigma(\rho), \tag{1.9}$$

where φ has properties similar to $\varphi(\rho) = \rho^m$, $m \geq 1$.

When $\sigma = 0$ the equation

$$\rho_t = \Delta(\rho^m) \tag{1.10}$$

is the porous medium equation which models the flow of a homogeneous gas with density ρ flowing through a homogeneous porous medium. There is an extensive literature for this equation (see for instance [1, 26, 29] and the references contained there). The most striking difference between the solutions of (1.10) and those of the usual heat equation

$$\rho_t = \Delta\rho \tag{1.11}$$

is their speed of propagation: Assume the population is initially distributed in a bounded region Ω in space, i.e., $\operatorname{supp} \rho_0 \subseteq \Omega$. The solutions of (1.11) have an infinite speed of propagation: $\rho(\mathbf{x}, t) > 0$ for $\mathbf{x} \in \mathbf{R}^n$, $t > 0$, thus the population would spread immediately to all the space. On the other hand Eq. (1.10) has a finite speed of propagation. There are two fronts that separate the populated region $\rho(\mathbf{x}, t) > 0$ from the unpopulated region $\rho(\mathbf{x}, t) = 0$.

1.3. *Age dependence and diffusion.* In 1981 Gurtin and MacCamy [14, 17] proposed a complete model with age dependence and diffusion. They let $\rho(\mathbf{x}, t, a)$ be the population density at time t, age a and spatial position $\mathbf{x}$. The anticrowding model is:

$$\rho_t + \rho_a = \kappa \operatorname{div}(\rho \nabla u) - \mu(a, u)\rho, \tag{1.12}$$

$$\rho(\mathbf{x}, t, 0) = \int_0^\infty \beta(a, u)\rho(\mathbf{x}, t, a)\, da, \tag{1.13}$$

$$u(\mathbf{x}, t) = \int_0^\infty \rho(\mathbf{x}, t, a)\, da, \tag{1.14}$$

$$\rho(\mathbf{x}, 0, a) = \rho_0(\mathbf{x}, a) \geq 0. \tag{1.15}$$

This system is too general to be treated in this form and some simplifying assumptions are necessary.

The model (1.12)–(1.15) can be reduced to a system of partial differential equations when the birth modulus β has the form $\beta(a, u) = \beta(u)e^{-\alpha a}$ and the death modulus μ depends on u alone. These assumptions correspond to the case in which individuals are more fertile at younger age and age is not a significant cause of death. This is typically true both in the case of a population exposed to a harsh environment and that of a population in the presence of predators that do not discriminate by age. We introduce the auxiliary function $G(\mathbf{x}, t) = \int_0^\infty e^{-\alpha a}\rho(\mathbf{x}, t, a)\, da$ and the per capita distribution term

$$p(\mathbf{x}, t) = \frac{G(\mathbf{x}, t)}{u(\mathbf{x}, t)}$$

for $u \neq 0$, and $p = 0$ for $u = 0$.

Integrating (1.12) with respect to a from 0 to ∞ and multiplying by $e^{-\alpha a}$ and integrating with κ normalized to be 1, one arrives at the system

$$u_t = u\Delta u + |\nabla u|^2 + (\beta(u)p - \mu(u))u, \tag{1.16}$$

$$p_t - \nabla u \nabla p = (\beta(u) - \alpha)p - \beta(u)p^2, \tag{1.17}$$

and the initial conditions

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}) = \int_0^\infty \rho_0(\mathbf{x})\, d\mathbf{x}, \tag{1.18}$$

$$p(\mathbf{x}, 0) = p_0(\mathbf{x}) = \frac{\int_0^\infty e^{-\alpha a}\rho_0(\mathbf{x}, a)\, da}{\int_0^\infty \rho_0(\mathbf{x}, a)\, da}. \tag{1.19}$$

This is a mixed system. The first equation is of porous medium type. It is nonlinear parabolic for $u(\mathbf{x}, t) \geq 0$, but it degenerates to $u_t = |\nabla u|^2$ at the points $u(\mathbf{x}, t) = 0$. The second equation is of first order nonlinear hyperbolic type.

In [17] Gurtin and MacCamy treat the case in which $\beta(a, u) = \hat{\beta}e^{-\alpha a}$, $\hat{\beta}$ and μ constants, in the one-dimensional domain $0 \leq x \leq 1$, obtaining existence of a solution under these conditions.

Because of the similarity of (1.16) and (1.10) we should expect that the population $u(x, t)$ would disperse at a finite speed. Also under specific conditions on the birth and death modulus the population might remain localized for all times or under other assumptions extend to all of $\mathbf{R}^N$. In the one-dimensional case these results were proven in [20] and [19]. Namely, there exist two interfaces that separate the populated region from the unpopulated region, i.e., the support of $u(\mathbf{x}, t)$ is finite for all time. Also if

$$\sup_{0\le u\le M_1} \frac{\mu(u)}{\beta(u)} < \inf_{0\le u\le M_1} \frac{\beta(u)-\alpha}{\beta(u)}$$

then the support of u grows to $\mathbf{R}$ as $t \to \infty$. In this case all the region will be ultimately populated. On the other hand if

$$\sup_{0\le u\le M_1} \frac{\beta(u)-\alpha}{\beta(u)} < \inf_{0\le u\le M_1} \frac{\mu(u)}{\beta(u)}$$

then the population remains localized in an interval $[-L, L]$ for all times. Here the interaction between age dependence and diffusion is such that the population persists in a limited region.

1.4. *Weak solutions.* It is well known [23, 3] that the porous medium equation (1.10) even with real analytic data will not have classical solutions unless u_0 is strictly positive in $\mathbf{R}^N$. This is due to the fact that if $u_0(x)$ has compact support the solutions will not have a continuous first derivative when crossing the interfaces. Thus we need to introduce a suitable definition of weak solutions of (1.16)–(1.19). Assume u and p are classical solutions. Multiplying (1.16) by p, (1.17) by u and adding we arrive at

$$(up)_t = \operatorname{div}(p\nabla u^2) + (\beta(u) - \alpha - \mu)up. \tag{1.20}$$

We define a test function $\varphi(\mathbf{x}, t)$ as a continuously differentiable function in $\Omega_T = \mathbf{R}^N \times (0, T)$ with compact support in $\overline{\Omega}_T = \mathbf{R} \times [0, T]$ and that equals 0 near T. Multiplying (1.16) and (1.20) by $\varphi(\mathbf{x}, t)$, integrating and using the divergence theorem we arrive at

$$\begin{aligned} &\iint_{\Omega_T} \left(\frac{1}{2}\nabla u^2 \nabla\varphi - u\varphi_t\right) d\mathbf{x}\,dt \\ &\quad = \iint_{\Omega_T} (\beta(u)p - \mu(u))u\varphi\, d\mathbf{x}\,dt + \int_{\mathbf{R}^N} u_0(\mathbf{x})\varphi(\mathbf{x}, 0)\, d\mathbf{x}, \end{aligned} \tag{1.21}$$

$$\begin{aligned} &\iint_{\Omega_T} \left(\frac{1}{2}p\nabla u^2 \nabla\varphi - pu\varphi_t\right) d\mathbf{x}\,dt \\ &\quad = \iint_{\Omega_T} (\beta(u) - \alpha - \mu(u))pu\varphi\, d\mathbf{x}\,dt + \int_{\mathbf{R}^N} u_0(\mathbf{x})p_0(\mathbf{x})\varphi(\mathbf{x}, 0)\, d\mathbf{x}. \end{aligned} \tag{1.22}$$

We define a weak solution of (1.16)–(1.19) as a pair of functions $u(\mathbf{x}, t)$, $p(\mathbf{x}, t)$ such that $u \in \mathscr{C}(\Omega_T)$, u^2 has partial derivatives in the sense of distributions, $p \in \mathscr{L}^2_{\text{loc}}(\Omega_T)$ and (1.21), (1.22) are satisfied for any test function $\varphi(\mathbf{x}, t)$.

The proof of existence of solutions of (1.16)–(1.19) was given in [20] for the one-dimensional case. There are no results in anticrowding model (1.16)–(1.19) for $N > 1$. This is due to the fact that the porous medium equation (1.10) is much better understood in dimension 1 than in higher dimensions.

In the one-dimensional case of (1.10) Aronson [2] proved that if $u_0^{m-1}(x)$ is Lipschitzian then $v(x, t) = u^{m-1}(x, t)$ is also Lipschitzian with respect to $x \in \mathbf{R} \times (0, T)$. Benilam [4] and Aronson and Caffarelli [8] proved that v also satisfies a Lipschitz condition with respect to t in the same domain. In particular $u(x, t)$ is α-Hölder continuous for any $\alpha \in (0, 1)$, i.e., the α-norm of u

$$|u|_\alpha = \sup|u| + \sup \frac{|u(x, t) - u(y, \tau)|}{|x-y|^\alpha + |t-\tau|^{\alpha/2}} \tag{1.23}$$

is bounded by a constant K that depends only on u_0, m, and T. In higher dimensions Caffarelli and Friedman [5] proved that $u(\mathbf{x}, t)$ is continuous with modulus of continuity

$$w(\rho) = C|\log\rho|^{-\varepsilon}, \qquad N \geq 3, \ 0 < \varepsilon < \frac{2}{N},$$

and

$$w(\rho) = 2^{-c}|\log\rho|^{1/2}, \qquad N = 2,$$

where $\rho = (|x-y|^2 + |t-\tau|)^{1/2}$ is the parabolic distance between (x, t) and (y, τ). Thus if u_0 is α-Hölder continuous for some $\alpha \in (0, 1)$ then $|u(x, t) - u(y, \tau)| \leq w(\rho)$ uniformly in $\mathbf{R}^N \times [0, T]$. The same authors in [6] proved that $u(x, t)$ is actually α-Hölder continuous for some $\alpha \in (0, 1)$ but α is unknown. In [1] Aronson describes an example due to Graveleau which shows that if the support of u_0 has holes then it is possible for ∇u to blow up near the boundary. Hence $v(\mathbf{x}, t)$ cannot in general be Lipschitzian in $\mathbf{R}^N \times (\tau, T)$ for arbitrarily small τ. Specifically for the porous medium problem in the radially symmetric case, if the gas lies initially completely outside a ball around 0, as time increases the gas will fill the ball and ultimately reach its center. The Aronson-Graveleau example shows that at the moment $v(r, t)$ is like r^α, where $\alpha \in (0, 1)$ depends on the dimension N and the constant m. For $N = 1$, $\alpha(1, m) = 1$. For $N > 1$, α has only been estimated numerically. For example, it is given in [1] that $\alpha(2, 2) = .832221204\ldots$.

The one-dimensional porous medium type problem

$$\begin{aligned} u_t &= (u^m)_{xx} + h(x, t, u)u, \\ u(x, 0) &= u_0(x) \geq 0, \end{aligned} \tag{1.24}$$

was treated by the author in [21]. It is shown there that the corresponding $v(x, t)$ is α-Hölder continuous for any $\alpha \in (0, 1)$ provided v_0 is α-Hölder continuous and h is bounded. The proof of existence given in [20] is largely based in this fact. Related results are given by Di Benedetto [10], Paul Sacks [27], and others.

In this work we shall prove existence of weak solutions for radially symmetric initial distributions when $N \geq 3$. For the random dispersal model we refer the reader to Garroni-Langlais [11], Langlais [24], di Blasio [9] and the references contained there.

2. Main results.

2.1. *Statement of existence of solutions.* Let $r = |\mathbf{x}| = (\sum_{i=1}^N x_i^2)^{1/2}$ be the Euclidean norm in $\mathbf{R}^N$. Radial solutions $u(r, t), p(r, t)$ of (1.16)–(1.19) satisfy

$$u_t = uu_{rr} + u_r^2 + (N-1)\frac{uu_r}{r} + (\beta(u)p - \mu(u))u, \tag{2.1}$$

$$p_t - u_r p_r = (\beta(u) - \alpha)p - \beta(u)p^2, \tag{2.2}$$

$$u_r(0, t) = 0, \qquad u(r, 0) = u_0(r), \qquad p(r, 0) = p_0(r). \tag{2.3}$$

We shall assume that the population is initially concentrated in a ball $B(0, R_1)$ and it is strictly positive in some interior ball $B(0, R_2)$. From the definition of u and G it is easy to see that if $\rho_0(\mathbf{x}, a) = 0$ a.e. with respect to a then $u_0(\mathbf{x}) = G_0(\mathbf{x}) = 0$. On the other hand if $\rho_0(\mathbf{x}, a) > 0$ in a set of positive measure, then $0 < G_0(\mathbf{x}) < u_0(\mathbf{x})$ and $p_0(\mathbf{x}) < 1$. The birth and death modulus β and μ are assumed to be smooth and bounded along with their derivatives. We introduce a final change of dependent variable $q(r, t) = e^{\alpha t}p(r, t)$, and the problem to be considered is

$$u_t = uu_{rr} + u_r^2 + (N-1)\frac{uu_r}{r} + (\beta(u)e^{-\alpha t}q - \mu(u))u, \tag{2.4}$$

$$q_t - u_r q_r = \beta(u)q(1 - e^{-\alpha t}q), \tag{2.5}$$

$$u_r(0, t) = 0, \qquad u(r, 0) = u_0(r), \qquad q(r, 0) = q_0(r), \tag{2.6}$$

where it is assumed $0 < \overline{m} \le q_0(r) \le 1$, $0 < \eta_0 \le u_0(r) \le M_0$ on $[0, R_2)$ $u_0(r) \equiv 0$ for $r > R_1$, and $\beta, \mu, |\beta'|, |\mu'| \le M_0$, $\beta, \mu \ge 0$.

We let $\mathscr{C}^\alpha(\Omega)$ be the Banach space consisting of functions whose α-norms (1.23) are bounded in Ω. Similarly define the spaces $\mathscr{C}^{1+\alpha}(\Omega, \mathscr{C}^{2+\alpha}(\Omega)$ with the corresponding norms $|u|_{1+\alpha} = |u|_\alpha + |u_r|_\alpha$ and $|u|_{2+\alpha} = |u|_{1+\alpha} + |u_r|_{1+\alpha} + |u_t|_{1+\alpha}$.

The corresponding weak solutions are given by:

Definition. A weak solution of (2.4)–(2.6) is a pair of bounded functions u, q such that u is continuous in $\Omega_T = (0, \infty) \times (0, T)$, u^2 is differentiable in the sense of distributions, $r^{N-1}(u^2)_r \in \mathscr{L}_{\text{Loc}}$, $q \in \mathscr{L}^2_{\text{Loc}}$ and for any $\varphi(r, t)$ which is 0 near T and for r large, the following two equalities are satisfied

$$\begin{aligned}\iint_{\Omega_T} & r^{N-1}\left(\frac{1}{2}(u^2)_r\varphi_r - u\varphi_t\right) dr\,dt \\ &= \iint_{\Omega_T} r^{N-1}(\beta(u)e^{-\alpha t}q - \mu(u))u\varphi\, dr\,dt + \int_0^\infty r^{N-1}\varphi(r, 0)u_0(r)\,dr,\end{aligned} \tag{2.7}$$

$$\begin{aligned}\iint_{\Omega_T} & r^{N-1}\left(\frac{1}{2}q(u^2)_r\varphi_r - qu\varphi_t\right) dr\,dt \\ &= \iint_{\Omega_T} r^{N-1}(\beta(u) - \mu(u))qu\varphi\, dr\,dt + \int_0^\infty r^{N-1}u_0(r)q_0(r)\varphi(r, 0)\,dr.\end{aligned} \tag{2.8}$$

Our first result establishes the existence of solutions for the radially symmetric Gurtin-MacCamy model.

Theorem 1. Under the previous hypothesis there exists a weak solution for the system (2.4)–(2.6).

Proof. We shall follow the approach presented in [20] using a fixed point technique. The proof is long and it is done in several steps: First we prove existence of solutions of a smoother version of (2.4)–(2.6) depending on two parameters ε and n. Then we let $n \to \infty$ and $\varepsilon \to 0$ after proving a priori bound for these solutions. Our main tool is an appropriate estimate for the Hölder modulus of continuity of radial solutions of (2.4).

Let $y = (r, t)$. In $\mathbf{R}^2$ we consider a mollifier $J(y)$, a symmetric $\mathscr{C}^\infty$-function such that $J(y) \geq 0$ if $|y| \leq 1$ and $\int_{\mathbf{R}^2} J(y)\, dy = 1$. (For instance

$$J(y) = \begin{cases} ke^{-1/(1-|y|^2)} & |y| < 1, \\ 0 & |y| > 1, \end{cases}$$

for appropriate constant k.) We let $J_n(y) = (1/n^2)J(ny)$. It is then clear that if $q \in \mathscr{L}^2$, $\{q * J_n\}$ is a $\mathscr{C}^\infty$-sequence that converges to q in $\mathscr{L}^2$, and if q is continuous $\{q * J_n\}$ converges to q uniformly on compact sets. We will apply Schauder's fixed point theorem to the following ε-n-approximating systems:

$$u_t = uu_{rr} + u_r^2 + (N-1)\frac{(u-\varepsilon)u_r}{r+\varepsilon} + (\beta(u)e^{-\alpha t}z - \mu(u))(u-\varepsilon), \tag{2.9}$$

$$q_t - u_r q_r = \beta(u)q(1 - e^{-\alpha t}q), \tag{2.10}$$

$$z = \frac{1}{n^2}\int_{|y-y'|\leq(1/n)} J(n(y-y'))q(y')\, dy', \tag{2.11}$$

$$u_r(0, t) = 0, \qquad u(r, 0) = u_0(r) + \varepsilon, \qquad q(r, 0) = q_0(r). \tag{2.12}$$

Here ε is introduced to regularize the porous medium part and n is introduced to smooth out the term $h(r, t, u) = (\beta(u)e^{-\alpha t}z - \mu(u))(u-\varepsilon)$ in (2.4), $z = q * J_n$.

2.2. *Solution of the equation for* q. We begin by studying the existence and regularity of solutions of (2.10) for u given. In this case the equation is nonlinear hyperbolic first order and can be solved by integration along characteristics.

LEMMA 1. Assume $u \in \mathscr{C}^{2+\alpha}(\Omega_T)$. Then (2.10) has a unique solution $q \in \mathscr{C}^{1+\alpha}(\Omega_T)$. This solution has $q(r, 0) = q_0(r)$ and $\overline{m} \leq q \leq e^{\alpha t}$.

Proof. We define characteristic curves $r(t; x, \tau)$ by

$$\frac{\partial r}{\partial t} = -u_r(r(t), t), \tag{2.13}$$

$$r(\tau; x, \tau) = x. \tag{2.14}$$

Since $\overline{m} \leq q(r(0), 0) \leq e^{\alpha t}$, it follows that any solution q increases along characteristics and $\overline{m} \leq q \leq e^{\alpha t}$. In particular we expect q to be positive. Let $Q(r, t) = q^{-1}(r, t)$. Along characteristics we have:

$$\frac{\partial Q}{\partial t} + \beta(u)Q = \beta(u)e^{-\alpha t}, \tag{2.15}$$

$$Q(r, 0) = q_0^{-1}(r(0)).$$

Upon integration (2.15) yields

$$q(x, \tau) = e^{\int_0^\tau \beta(u)\, ds}\left[q_0^{-1}(r(0)) + \int_0^\xi \beta(u)e^{\int_0^\sigma (b(u)-\alpha)\, d\sigma}\, d\xi\right]^{-1}. \tag{2.16}$$

Since u_r is Lipschitzian there is always a local solution of (2.13)–(2.14). Since u_r is bounded this solution can be extended to the boundary of Ω_T.

Direct differentiation of (2.16) shows that $q(x, \tau)$ is a solution of (2.10), (2.12). Haar's lemma implies uniqueness. Since $u \in \mathscr{C}^{2+\alpha}(\Omega_T)$ then $q \in \mathscr{C}^{1+\alpha}(\Omega_T)$.

LEMMA 2. There exists a solution of (2.9)–(2.12).

Proof. Let K_1 be a constant and denote $V = \{w \in \mathscr{C}^{2+\alpha}(\Omega_T)/|w|_{2+\alpha} \leq k_1, w \geq \varepsilon\}$. Define $T\colon V \mapsto \mathscr{C}^{2+\alpha}(\Omega_T)$ in the following way: Given $w \in V$, by the previous lemma there exists a unique solution $q(w) \in \mathscr{C}^{1+\alpha}(\Omega_T)$ and $\overline{m} \leq q \leq e^{\alpha t}$. Then $z \in \mathscr{C}^{\infty}(\Omega_T)$ and $|z|_\sigma \leq 2n$ for any $\sigma \in (0, 1)$, so by standard results in parabolic differential equations coupled with the fact that $u(r, 0) \geq \varepsilon$, there exists a unique solution $u \in \mathscr{C}^{2+\sigma}(\Omega_T)$ with $|u|_{2+\sigma} \leq k_2$, where k_2 depends only on n and ε; let $u = T(w)$.

Taking $k_1 \leq k_2$ and $\sigma < \alpha$ we have that T maps V into V. Further, since bounded sets in $\mathscr{C}^{2+\sigma}(\Omega_T)$ are precompact in $\mathscr{C}^{2+\alpha}$ for $\sigma < \alpha$ we also have that $T(V)$ is precompact. It is clear that T is continuous since the equations and functions involved are smooth (depending on n and ε). It follows then by Schauder's fixed point theorem that T has a fixed point $u = T(u)$. This u and the corresponding q and z form a solution of (2.9)–(2.12).

Next we show that after letting n tends to infinity we obtain a solution of the following ε-approximation problems:

$$u_t = uu_{rr} + (N-1)\frac{(u-\varepsilon)u_r}{r+\varepsilon} + (\beta(u)e^{-\alpha t}q - \mu(u))(u-\varepsilon), \tag{2.17}$$

$$q_t - u_r q_r = \beta(u)q(1 - e^{-\alpha t}q), \tag{2.18}$$

$$u_r(0, t) = 0, \qquad u(r, 0) = u_0(r) + \varepsilon, \qquad q(r, 0) = q_0(r). \tag{2.19}$$

2.3. *Estimates for* u. The solutions of (2.9) with the initial conditions in (2.12) can be obtained as limits when $R \to \infty$ of the solution of the problems

$$\begin{gathered} u_t = E(u)u_{rr} + u_r^2 + (N-1)\frac{(u-\varepsilon)u_r}{r+\varepsilon} \\ + (\beta(u)e^{-\alpha t}z - \mu(u))(u-\varepsilon), \\ u(r, 0) = u_0(r) + \varepsilon, \qquad u_r(0, t) = 0, \qquad u(R, t) = \varepsilon, \end{gathered} \tag{2.20}$$

where $E(z)$ is a $\mathscr{C}^\infty$-function satisfying $E(z) = z$ for $z \geq \varepsilon$, $E(z) = \frac{\varepsilon}{2}$ for $z \leq \frac{\varepsilon}{2}$ and $E(z)$ increases from $\frac{\varepsilon}{2}$ to ε in $\frac{\varepsilon}{2} \leq z \leq \varepsilon$.

The function $v = e^{-M_0 t}(u - \varepsilon)$ satisfies

$$\begin{gathered} v_t = E(u)v_{rr} + e^{M_0 t}v_r^2 + (N-1)e^{M_0 t}\frac{vv_r}{r+\varepsilon} + (h - M_0)v, \\ v(r, 0) = u_0(r), \qquad v_r(0, t) = 0, \qquad u(R, t) = 0. \end{gathered} \tag{2.21}$$

Since $h \leq M_0$, the maximum principle implies that $0 \leq v(r, t) \leq M_0$, independently of R. Therefore $\varepsilon \leq u(r, t) \leq M_1 = M_0 e^{M_0 T}$ in Ω_T.

Now we derive an appropriate estimate for the gradient of u and the gradient of q.

LEMMA 3. Let $u \in \mathscr{C}^{2+\alpha}(\Omega_T)$ be a solution of (2.9), $z \in \mathscr{C}^{1+\alpha}(\Omega_T)$. Then there exist constants K_1, K_2 (depending on ε) such that for any $c_1 > 0$

$$|u_r|_\infty^2 \le K_1 + \frac{K_2}{c_1} + c_1 |z_r|_\infty^2 . \tag{2.22}$$

Proof. This is an application of Bershtein's technique as in Oleinik [25] and Aronson [2]. The details involve some straightforward calculations that are only sketched.

Let $\varphi(y) = (M_1/3)y(4-y)$. $\varphi\colon [0, 1] \mapsto [0, M_1]$ with positive first derivatives bounded away from 0 and negative second derivative. Let $w = \varphi^{-1}(u)$, then w satisfies

$$w_t = \varphi w_{rr} + \left[\varphi' + \frac{\varphi\varphi''}{\varphi'} + \frac{N-1}{r+\varepsilon}\varphi\right] w_r$$
$$+ (\beta(\varphi)e^{-\alpha t} z - \mu)\left(\frac{\varphi - \varepsilon}{\varphi}\right) . \tag{2.23}$$

Differentiating with respect to r and letting $v = w_r$, we obtain

$$\frac{1}{2}(v_t^2 - \varphi v v_{rr}) = A v^2 v_r + \frac{N-1}{r+\varepsilon}\varphi' v^3 + B v^4 + \frac{N-1}{r+\varepsilon}(\varphi - \varepsilon) v v_r$$
$$- \frac{N-1}{(r+\varepsilon)^2}(\varphi - \varepsilon) v^2 + C v^2 + D v z_r , \tag{2.24}$$

where the coefficients A, B, C, D depend only on φ, φ', φ''. Next take a cutoff function $\xi(r) \in \mathscr{C}^\infty(\mathbf{R})$, $\xi(r) = 0$ for $r \ge m_1 + 1$ and $\xi(r) \equiv 1$ for $0 \le r \le m_1$ and let $p = \xi^2 v^2$. At an interior maximum of p we should have

$$p_r = 0, \qquad p_t - \varphi p_{rr} \ge 0 . \tag{2.25}$$

Using (2.23) and (2.25) we arrive at

$$-2\xi^2 B v^4 \le \left[2\frac{N-1}{r+\varepsilon}\varphi'\xi - 2A\xi_r\right]\xi v^3 \tag{2.26}$$
$$+ \left[2\frac{N-1}{r+\varepsilon}(\varphi - \varepsilon)\xi\xi_r - 2\frac{N-1}{(r+\varepsilon)^2}(\varphi-\varepsilon)\xi^2\right] v^2 + 2D\xi^2 |z_r| v .$$

Let E and F be the coefficients of v^3 and v^2, respectively. Note that

$$-B = -\left(\varphi' + \frac{\varphi\varphi''}{\varphi'}\right)' \ge \frac{3}{2} M_1$$

and the triangle inequality gives

$$\xi v^3 \le \frac{v^2 E^2}{4M} + M\xi^2 v^4$$

thus for any $c_2 > 0$

$$2|D|\xi^2 v^4 \le \left(\frac{E^2}{4m} + F + \frac{D^2}{c_2^2}\right) v^2 + c_2^2 |z_r|_\infty^2 .$$

From here it follows that there exist constants K_1', K_2' such that

$$\xi^2 v^2 \le K_1' + \frac{K_2'}{c_2^2} + c_2^2 |z_r|_\infty^2$$

for $c_2 > 0$ arbitrary.

Since

$$u_r^2 = \varphi'^2 w_r^2 = \varphi'^2 \xi^2(r) w_r^2,$$

for $r \in (0, m_1)$ we have

$$u_r^2 \le K_1 \varphi'^2 + \frac{K_2' \varphi'^2}{c_2^2} + c_2^2 \varphi'^2 |z_r|_\infty^2 . \tag{2.27}$$

Since m_1 is arbitrary and $\varphi' \le (2M_1/3)$ this proves the lemma.

2.4. *Estimates for* q.

LEMMA 4. Let $q \in \mathscr{C}^{1+\alpha}(\Omega_T)$ be a solution of (2.10) where u is a solution of (2.9). Then there exist constants K_3, K_4 (depending on ε) such that

$$|q_r| \le K_3 |u_r|_\infty \quad \text{and} \quad |q_t| \le K_4 |u_r|_\infty^2 . \tag{2.28}$$

Proof. Differentiating with respect to the parameter x in (2.15) we obtain

$$\frac{\partial r_x}{\partial t} = -u_{rr} r_x , \qquad r_x(\tau) = 1 . \tag{2.29}$$

Thus

$$\begin{aligned} r_x &= e^{\int_\tau^t u_{rr}(r(s), s)\, ds} \\ &\le e^{\int_\tau^t (u_t - u_r^2)/u - [(N-1)(u_r/(r+\varepsilon)) - (\beta e^{-\alpha t} z - \mu)]((u-\varepsilon)/u)\, ds} \\ &\le \frac{u(r(t), t)}{u(r(\tau), \tau)} + \left(\frac{r(t)+\varepsilon}{r(\tau)+\varepsilon}\right)^{N-1} + e^{|h|T} \\ &\le K_5(\varepsilon). \end{aligned}$$

The function $V(t) = r_\tau(t) - u_r(r, \tau) r_x(\tau)$ satisfies $V'(t) = u_{rr} V(t)$ and $V(\tau) = 0$. Thus $V \equiv 0$ and $r_\tau = u_r(r, x) r_x$. It follows that $r_\tau \le K_5 |u_r|_\infty$.

Differentiating expression (2.16) we have that q_x and q_τ are bounded by multiples of β', u_r, r_x, q_0^{-1}, and q_0', and using the previous estimates we obtain that $|q_x|_\infty$ and $|q_\tau|_\infty$ are bounded by multiples of $|u_r|_\infty$ and $|u_r|_\infty^2$, respectively.

With the aid of the previous two lemmas we obtain bounds for u_r^n and q_r^n independently of n, for a solution u^n, q^n of (2.9)–(2.12). By Lemma 4 we have $|q_r^n|_\infty \le K_3 |u_r^n|_\infty$, thus $|z_r^n|_\infty \le K_3 |u_r^n|_\infty$ and by Lemma 3, with $c_1 = 1/2K_3^2$ we have

$$|u_r^n|_\infty^2 \le K_1 + 2K_2 K_3^2 + \frac{1}{2} |u_r^n|_\infty^2 ,$$

i.e.,

$$|u_r^n|_\infty^2 \le 2(K_1 + 2K_2 K_1^2)$$

independently of n.

Also

$$|q_r^n|_\infty \le K_3(2(K_1 + 2K_2K_1^2))^{1/2}$$

is uniformly bounded.

Since $|q^n|_\alpha$ and $|z^n|_\alpha$ are bounded independently of n we can extract a subsequence $\{q^{n_k}\}$ of $\{q^n\}$ that converges uniformly to a function q_ε on compact subsets of Ω_T. The corresponding sequence $\{z^{n_k}\}$ converges uniformly to q_ε. We rename these sequences $\{q^n\}$ and $\{z^n\}$. Since $u^n \ge \varepsilon$, Eqs. (2.9) are uniformly parabolic in n. It follows then from the standard theory that $|u^n|_{2+\alpha} \le K_6$, a constant independent of n, and for $\alpha' < \alpha$ there exists a subsequence $\{u^{n_k}\}$ that converges to a function u_ε in $\mathscr{C}^{2+\alpha'}(\Omega_T)$. In particular $\{u^{n_k}\}$ and $\{u_r^{n_k}\}$ converge uniformly in compact sets to u_ε and u_r^ε, respectively.

The uniform boundedness of $|u_r^{n_k}|_\alpha$ implies that also q_ε satisfies (2.16), thus the pair u_ε, q_ε is a solution of (2.17)–(2.19).

2.5. *The α-norm of u*. Next it is shown that $\{u_\varepsilon\}$, $\{q_\varepsilon\}$ converge to a weak solution of (2.4)–(2.6). We start with the following fundamental result which is important in its own right.

THEOREM 2. The α-norms of the solutions $\{u_\varepsilon\}$ are uniformly bounded in Ω_T for $0 < \alpha < 1$, i.e., there exists $K^* > 0$ such that

$$\frac{|u(x, t) - u(y, \tau)|}{|x - y|^\alpha + |t - \tau|^{\alpha/2}} \le K^*,$$

and K^* depends only on the data M_0, α, and T.

Proof. A more general result with $u_0(r) = 0$ in a neighborhood of 0 has been proved in [22]. Here, the fact that $u_0 > 0$ on $[0, R_2]$ will simplify the proof and also will allow us to obtain bounded α-norm of any $\alpha \in (0, 1)$. In the general case treated in [22] one only gets the α-norm bounded for $\alpha \le 1/2(N-2)$, $N \ge 3$.

Let $h(r, t, u) = \beta(u)e^{-\alpha t}q - \mu(u)$, $u = u_\varepsilon$. Then (2.9) is

$$u_t = uu_{rr} + u_r^2 + (N-1)\frac{(u-\varepsilon)u_r}{r+\varepsilon} + h(r, t, u)(u - \varepsilon), \tag{2.30}$$
$$u_r(0, t) = 0, \qquad u(r, 0) = u_0(r) + \varepsilon.$$

We begin with the Hölder continuity with respect to r. Since u is a classical solution in Ω_T we can choose δ small such that $|u_t|_\infty$, $|u_r|_\infty \le \delta^{\alpha-1}$ in Ω_T and $\delta \le r_0 = R_2/2$.

LEMMA 5. Let $S_{\delta, R} = \{(r, s) \in \mathbf{R}^2 : \delta < s < s + \delta < r < R\}$, $B_{\delta, R} = S_\delta \times (0, T)$ and define for $\alpha \in (0, 1)$ fixed

$$g(r, s, t) = \frac{|u(r, t) - u(s, t)|^\lambda}{(r-s)^2},$$

$\lambda = \frac{2}{\alpha}$. Then g is bounded in B_δ (indendently of δ and R).

Proof. Clearly g is continuous, so it must attain its maximum at a point $Q_1 = (r_1, s_1, t_1) \in B_\delta$. Either Q_1 is an interior point at which g is differentiable or Q_1

is a point on the boundary of $B_{\delta,R}$ (g is not differentiable only at those points (r, t, s) where $g(r, t, s) = 0$. We begin with the former possibility.

In this case we must have

$$g_r = g_s = 0, \qquad g_{rr}, g_{ss} \le 0, \qquad g_t \ge 0,$$

and

$$E = v_1 g_{rr} + v_2 g_{ss} - g_t \le 0 \quad \text{at } Q_1, \tag{2.31}$$

where $v_1 = v(r, t)$, $v_2 = v(s, t)$. Let $S = |v_1 - v_2|$, $\sigma = \operatorname{sgn}(S)$. The first derivatives are

$$\begin{aligned} g_r &= \sigma\lambda|S|^{\lambda-1} v_{1r} R^{-2} - 2|S|^{\lambda} R^{-3}, \\ g_s &= -\sigma\lambda|S|^{\lambda-1} v_{2s} R^{-2} + 2|S|^{\lambda} R^{-3}, \\ g_t &= \sigma\lambda|S|^{\lambda-1}(v_{1t} - v_{2t}) R^{-2}. \end{aligned}$$

Thus (2.31) implies

$$v_{1r} = \frac{2\sigma}{\lambda}|S|R^{-1} = v_{2s} \tag{2.32}$$

and

$$\begin{aligned} g_{rr} &= 2|S|^{\lambda} R^{-4}(1-\alpha) + \lambda|S|^{\lambda-1}\sigma R^{-2} v_{1rr}, \\ g_{ss} &= 2|S|^{\lambda} R^{-4}(1-\alpha) - \lambda|S|^{\lambda-1}\sigma R^{-2} v_{2ss}. \end{aligned} \tag{2.33}$$

Replacing in E we obtain

$$2|S|^{\lambda} R^{-4}(1-\alpha)(v_1 + v_2) + \lambda\sigma|S|^{\lambda-1} R^{-2}[(v_1 v_{1rr} - v_{1t}) - (v_2 v_{2ss} - v_{2t})] \le 0. \tag{2.34}$$

Using the differential equation (2.30) in (r, t) and (s, t) in the last term we have

$$\begin{aligned} &2(1-\alpha)|S|^{\lambda} R^{-4}(v_1 + v_2) \\ &\quad + \lambda\sigma|S|^{\lambda-1} R^{-2}\left[-v_{1r}^2 - (N-1)\frac{(v_1 - \varepsilon)v_{1r}}{r+\varepsilon} - h(r, t, v_1)(v_1 - \varepsilon)\right. \\ &\quad\quad \left. + v_{2s}^2 + (N-1)\frac{(v_2 - \varepsilon)v_{2s}}{s+\varepsilon} + h(s, t, v_2)(v_2 - \varepsilon)\right] \le 0 \end{aligned} \tag{2.35}$$

and using (2.32) with $h_1 = h(r, t, v_1)$, $h_2 = h(s, t, v_2)$

$$\begin{aligned} &2\frac{|S|^{\lambda}}{R^2}\left[(1-\alpha)(v_1 + v_2) + (N-1)(r-s)\left(\frac{(v_2 - \varepsilon)}{s+\varepsilon} - \frac{(v_1 - \varepsilon)}{r+\varepsilon}\right)\right] \\ &\quad + \lambda\sigma|S|^{\lambda-1}(h_2(v_2 - \varepsilon) - h_1(v_1 - \varepsilon)) \le 0. \end{aligned}$$

We drop the positive term in $(v_2 - \varepsilon)/(s + \varepsilon)$. Since $|S| \le 2M_1$ we obtain,

$$\begin{aligned} &2\frac{|S|^{\lambda}}{R^2}\left[(1-\alpha)(v_1 + v_2) - (N-1)(r-s)\frac{v_1 - \varepsilon}{r+\varepsilon}\right] \\ &\quad \le \lambda(2M_1)^{\lambda-1}(h_2(v_2 - \varepsilon) - h_1(v_1 - \varepsilon)). \end{aligned} \tag{2.36}$$

Let $\hat{\delta} = ((1-\alpha)r_0)/(2(N-1))$. If $r - s < \hat{\delta}$ the coefficient on the left of (2.36) is bounded below by $\frac{1}{2}(1-\alpha)(v_1+v_2)$ and

$$g_{\max} = \frac{|S|^\lambda}{R^2} \leq \frac{\lambda}{2}\frac{(2M_1)^{\lambda-1}}{1-\alpha}\left|\frac{h_2(v_2-\varepsilon)}{v_1+v_2} - \frac{h_1(v_1-\varepsilon)}{v_1+v_2}\right| \tag{2.37}$$

$$\leq \frac{(2M_1)^{\lambda-1}}{\alpha(1-\alpha)}(2M_0). \tag{2.38}$$

On the other hand if $r - s > \hat{\delta}$, then directly from the definition of g

$$g_{\max} \leq \frac{(2M_1)^{\lambda-1}}{\hat{\delta}^2} < \frac{4(N-1)^2(2M_1)^{\lambda-1}}{(1-\alpha)^2 r_0^2}.$$

A bound for the α-Hölder quotient of u at interior points of B_δ has thus been obtained.

If Q_1 lies on the boundary of B_δ we are led to consider four possibilities: (i) Q_1 lies on the interior boundary $r - s = \delta$, (ii) Q_1 lies on the interior boundary $s = \delta$, (iii) Q_1 lies on the lower boundary $t = 0$, and (iv) Q_1 lies on the lateral boundary $r = R$. We treat each of these cases separately.

(i) If $r = s + \delta$ by the mean value theorem and the assumption that $|u_r|_\infty \leq \delta^{\alpha-1}$ we get

$$g(Q_1) \leq \frac{|u_r(\cdot\,, t)|^\lambda \delta^\lambda}{\delta^2} \leq \delta^{\alpha-1}\delta^{\lambda-2} = 1.$$

(ii) If $s = \delta$ by the standard regularity results for the porous medium equation with positive data there is a constant K_4 (independent of δ, ε) such that $|u_t|_\infty$, $|u_r|_\infty \leq K_4$ on $[0, R_2]$. Thus, if $|r-s| \leq R_2/2$, again by the mean value theorem

$$g(Q_1) = \frac{|u_2 - u_1|^\lambda}{(r-s)^2} \leq K_4^\lambda (r-s)^{\lambda-2} \leq K_4^\lambda \left(\frac{R_2}{2}\right)^{\lambda-2}.$$

On the other hand if $|r-s| \geq R_2/2$ then

$$g(Q_1) \leq \frac{(2M)^\lambda}{(R_2/2)^2}.$$

(iii) If the maximum occurs at $t = 0$ we use the initial data to get that

$$g(Q_1) \leq M_0 \quad \text{when } |r-s| \leq 1$$

and

$$g(Q_1) \leq (2M_0)^\lambda \quad \text{when } |r-s| \geq 1.$$

(iv) The case $r = R$ needs special consideration. We start by proving that the oscillation of u tends to 0 as r tends to infinity.

LEMMA 6. Let u be a solution of (2.30). Then for any $\eta > 0$ there exists R_η such that $|u(r,t) - u(s,t)| \leq \eta$ for $r, s \geq R_\eta$, $t \in [0, T]$.

Proof. We choose R large such that $(N-1)u_r/(r+\varepsilon) \leq M_0$ and $u_0(r) \equiv 0$ for $r > R$. Write (2.30) as

$$u_t = uu_{rr} + u_r^2 + \hat{h}(u - \varepsilon) \tag{2.39}$$

where

$$|\hat{h}(u, r, t)| = |(N-1)\frac{u_r}{r+\varepsilon} + \beta(u)q - \mu(u)| \leq 2M_0 .$$

Without loss of generality we assume $M_0 \geq 1$.

Let $t_2 = 1/(24M_0)$, $\sigma < 1$, and $m = \frac{1}{\sigma}$. Let $g(t) = 2t_2/(2t_2 - t)$ and consider

$$v(r, t) = \sigma g(t)(r - r_1)^2 + \varepsilon g^{\sigma}(t)$$

on $B_1 = [r_1 - m, r_1 + m] \times [0, t_2]$ and r_1 is such that $r_1 - m \geq R$. We will use v as an upper bound for u and obtain $u(r, t) \leq 2^{\sigma}\varepsilon$ on B_1.

Let $\mathscr{L}[z] = z_t - zz_{rr} - z_r^2 - \hat{h}(z, r, t)(z - \varepsilon)$. Then $\mathscr{L}[u] = 0$ and

$$\begin{aligned}\mathscr{L}[v] = {} & \sigma\frac{(r-r_1)^2 g^2(t)}{2t_2}(1 - 12\sigma t_2 - (2t_2 - t)\hat{h}) \\ & + \varepsilon\left(\sigma\frac{g^{\sigma+1}(t)}{2t_2} - 2ag^{\sigma+1}(t) - \hat{h}(u, r, t)(g^{\sigma}(t) - 1)\right) .\end{aligned}$$

By the mean value theorem

$$g^{\sigma}(t) - 1 = \sigma g^{\sigma-1}(s)g'(s)t = \frac{\sigma t_2}{2t_2}g^{\sigma+1}(s) \quad \text{for some } s \in (0, t) .$$

Since $|\hat{h}| \leq 2M_0$, $0 \leq t \leq t_2$, and $g(s)/g(t) = (2t_2 - t)/(2t_2 - s)$ lies between $\frac{1}{2}$ and 1 we have

$$\begin{aligned}\mathscr{L}[v] \geq {} & \sigma\frac{(r-r_1)^2 g(t)^2}{2t_2}(1 - 12\sigma t_2 - 4t_2M_0) \\ & + \varepsilon\sigma\frac{g(t)^{\sigma+1}}{2t_2}(1 - 4t_2 - 2M_0t_2) \\ \geq {} & 0\end{aligned}$$

by the choice of t_2 and σ.

Next

$$v(r, 0) = \sigma(r - r_1)^2 + \varepsilon \geq \varepsilon = u(r, 0) \quad \text{on } [r_2 - m, r_2 + m]$$

and

$$v(r_2 \pm m, t) = \sigma\frac{2t_2m^2}{2t_2 - t} + \varepsilon g^{\sigma}(t) \geq \sigma m^2 = m .$$

Thus $v(r_2 \pm m, t) \geq u(r_2 \pm m, t)$ for $m \geq M_1$. It follows then by the maximum principle that $u \leq v$ on $[r_1 - m, r_1 + m]$. In particular

$$u(r_1, t) \leq \varepsilon\left(\frac{2t_2}{2t_2 - t}\right)^{\sigma} \leq 2^{1/m}\varepsilon . \tag{2.40}$$

Since the only restriction for r_1 is to be larger than $R + m$, Eq. (2.40) is valid for any $r \geq r_1$. We repeat the argument with $r_2 \geq r_1 + m$ on $[t_2, 2t_2]$ with initial datum $u(r, t_2) \leq 2^{1/m}\varepsilon$ and obtain that $u(r, t) \leq 2^{2/m}\varepsilon$ for $r \geq r_2 \geq R + 2m$. After $k = [T/(t_2 + 1)]$ steps we arrive at

$$u(r, t) \leq 2^{k/m}\varepsilon \quad \text{for } r \geq r_k \geq R + km , \qquad 0 \leq t \leq T .$$

Now, given $\eta > 0$, choose m so large that also $2^{k/m} \leq 1+\eta/\varepsilon$. Then for $R_\eta = r_k$ we have $u(r, t) \leq \varepsilon + \eta$ for all $r \geq R_\eta$, $t \in [0, T]$. This proves the lemma.

We use this result in the proof of (iv). Choose R so large so as to have $u(r, t) \leq \varepsilon + \delta$ for $r \geq R - 1$. Then if $|r - s| \geq 1$ we have

$$g(Q_1) \leq (2M_1)^\lambda$$

and if $|r - s| \leq 1$ we have

$$g(Q_1) = \frac{|u_1 - u_2|^\lambda}{(r-s)^2} \leq \frac{\delta^\lambda}{\delta^2} \leq 1 .$$

Therefore $g(r, s, t)$ is bounded in B_δ independently of δ and ε. Letting $\delta \to 0$ and $R \to \infty$, we obtain that $g(r, s, t)$ is bounded in Ω_T. Lemma 6 implies that $g(r, s, t)^{\alpha/2}$, the α-quotient of u, is bounded in Ω_T.

The result for t follows in a similar way by considering the function

$$k(r, s, t, \tau) = \frac{|u(r, t) - u(s, \tau)|^\lambda}{(r-s)^2 + A|t - \tau|}$$

at a point of maximum (we omit the details). This proves Theorem 2

2.6. *Convergence of* $\{u_\varepsilon\}$. Since $\{u_\varepsilon\}$, $\{q_\varepsilon\}$ satisfy (2.17)–(2.19) we also have

$$\begin{aligned}(q_\varepsilon u_\varepsilon)_t = {} & \frac{1}{2}(q_\varepsilon((u_\varepsilon)^2)_r)_r + \frac{1}{2}\frac{N-1}{r+\varepsilon} q_\varepsilon((u_\varepsilon)^2)_r \\ & + \frac{1}{2}\frac{N-1}{r+\varepsilon} q_\varepsilon(((u_\varepsilon - \varepsilon)^2)_r - ((u_\varepsilon^2)_r)) \\ & + [\beta(u_\varepsilon) - \mu(u_\varepsilon)] q_\varepsilon u_\varepsilon + \varepsilon q_\varepsilon[\mu(u_\varepsilon) - \beta(u_\varepsilon) e^{-\alpha t} q_\varepsilon] .\end{aligned} \tag{2.41}$$

Multiplying (2.17), (2.41) by $(r+\varepsilon)^{N-1}$, $\varphi(r, t)$ and integrating over Ω_T we obtain

$$\begin{aligned}&\iint_{\Omega_T} (r+\varepsilon)^{N-1}\left[\frac{1}{2}((u_\varepsilon)^2)_r\varphi_r - u_\varepsilon\varphi_t\right] dr\, dt \\ &\quad = \iint_{\Omega_T} (r+\varepsilon)^{N-1}[\beta(u_\varepsilon)e^{-\alpha t} q_\varepsilon - \mu_\varepsilon(u_\varepsilon)](u_\varepsilon - \varepsilon)\varphi\, dr\, dt \\ &\quad + \int_0^\infty (r+\varepsilon)^{N-1}\varphi(r, 0)(u_0(r) + \varepsilon)\, dr ,\end{aligned} \tag{2.42}$$

$$\begin{aligned}&\iint_{\Omega_T} (r+\varepsilon)^{N-1}\left[\frac{1}{2} q((u_\varepsilon)^2)_r\varphi_r - q_\varepsilon u_\varepsilon\varphi_t\right] dr\, dt \\ &\quad = \iint_{\Omega_T} (r+\varepsilon)^{N-1}(\beta(u_\varepsilon) - \mu(u_\varepsilon)) q u_\varepsilon \varphi\, dr\, dt \\ &\quad + \int_0^\infty (r+\varepsilon)^{N-1}(u_0(r) + \varepsilon) q_0(r)\varphi(r, 0)\, dr \\ &\quad + \varepsilon \iint_{\Omega_T} (r+\varepsilon)^{N-1} q(\mu(u_\varepsilon) - \beta(u_\varepsilon)e^{-\alpha t} q)\varphi\, dr\, dt \\ &\quad + \iint_{\Omega_T} \frac{1}{2}(N-1)[((u_\varepsilon + \varepsilon)^2)_r - ((u_\varepsilon)^2)_r]\, dr\, dt .\end{aligned} \tag{2.43}$$

Since $|u_\varepsilon|_\alpha \le K^*$ by the Arzela-Ascoli theorem we can extract a subsequence $\{u^{\varepsilon_l}\}$ that converges to an α-Hölder continuous function $u(r, t)$ uniformly on compact sets. The corresponding subsequence $\{q^{\varepsilon_l}\}$ satisfies $|q^{\varepsilon_l}|_\alpha \le e^{\alpha T}$. Hence it contains a subsequence $\{q^{\varepsilon_{l,k}}\}$ that converges weakly to a function $q(r, t)$, $|q|_\infty \le e^{\alpha t}$. We rename both $\{u^{\varepsilon_{l,k}}\}$ and $\{q^{\varepsilon_{l,k}}\}$ as $\{u_k\}$ and $\{q_k\}$, respectively. We shall prove that u^2 is differentiable in Ω_T and that $\{(u_k^2)_r\}$ converges pointwise to $(u^2)_r$. Then all the integrals will converge in (2.42)–(2.43).

The proof of the differentiability of u^2 is divided into two cases:

First, if $u(r_0, t_0) = \eta_1 > 0$ by the uniform convergence on compact sets we must have that for $k \ge k_0$, $u_k(r, t) \le \frac{\eta}{2}$ in a neighborhood N of (r_0, t_0). It follows then by the standard theory of the porous medium equation that u_k 's are classical solutions on N and $|u_k|_{2+\alpha} \le K_6$, where K_6 depends only on η_1 and N. Thus there exists a subsequence $\{u_{k,n}\}$ such that $\{(u^2_{k,n})_r\}$ converges uniformly to $(u^2)_r$ in N. In particular u^2 is differentiable at (r_0, t_0).

Second, if $u(r_0, t_0) = 0$, since u is α-Hölder continuous for $\alpha > \frac{1}{2}$ we have

$$\begin{aligned} \frac{u^2(r_0+h, t) - u^2(r_0, t)}{h} &= \frac{u^2(r_0+h, t)}{h} \\ &= \left(\frac{u(r_0+h, t)}{h^\alpha}\right)^2 h^{2\alpha-1} \\ &\le (K^*)^2 h^{2\alpha-1}. \end{aligned}$$

Thus u^2 is differentiable at (r_0, t_0) and $(u^2)_r(r_0, t_0) = 0$. The proof of the pointwise convergence of a subsequence of $\{(u_k^2)_r\}$ to $(u^2)_r$ is the same as the one-dimensional case given in [20] and we shall omit it here.

3. Qualitative behavior of solutions.

3.1. *Populated and unpopulated regions.* The existence of the interface that separates the populated region from the unpopulated region will be investigated first.

THEOREM 3. If $u(r_0, t_0) \ge \eta > 0$ then $u(r_0, t) > 0$ for all $t \le t_0$. Thus if a region becomes populated at time t_0 it remains populated for all later times. In particular the initial region $[0, R_2]$ remains populated for all times (but it might tend to 0 as $t \to \infty$).

Proof. Assume first $t_0 = 0$ and $r_0 > 0$. Using the continuity of $u_0(r)$ choose $\delta > 0$ small, $\delta < r_0/2$, such that $\delta^2 \le \frac{\eta}{2}$ and $u_0(r) \le \frac{\eta}{2}$ on $(r_0 - \delta, r_0 + \delta)$. Define $v(r, t) = e^{-kt}(\delta^2 - (r - r_0)^2) + \frac{\varepsilon}{2}$, over $B = (r_0 - \delta, r_0 + \delta) \times (0, T]$, $k = 4N + M_0$.

If

$$\mathscr{L}[z] = z_t - zz_{rr} - z_r^2 - \frac{(N-1)}{r+\varepsilon}(z - \varepsilon) + (M_0 + 2N)(z - \varepsilon)$$

we have that $\mathscr{L}[u_\varepsilon] = (h + M_0 + 2N)(u_\varepsilon - \varepsilon) \ge 0$ and since $|(r - r_0)/(r + \varepsilon)| \le 1$, $e^{-kt} \le 1$, thus

$$\begin{aligned} \mathscr{L}[v] &\le (\delta^2 - (r - r_0)^2)(4N + M_0 - k) - 4e^{-kt}(r - r_0)^2 - (M_0 + 2N)\frac{\varepsilon}{2} \\ &\le 0. \end{aligned}$$

Also

$$v(r, 0) = \delta^2 - (r - r_0)^2 + \frac{\varepsilon}{2} \le \delta^2 + \frac{\varepsilon}{2} \le \frac{\eta}{2} + \varepsilon \le u_\varepsilon(r, 0)$$

and

$$v(r_0 \pm \delta, t) = \frac{\varepsilon}{2} \le \varepsilon \le u_\varepsilon(r_0 \pm \delta, t).$$

The maximum principle now implies that $v(r, t) \le u(r, t)$ on $\overline{B} = [r_0 - \delta, r_0 + \delta] \times [0, T]$. In particular

$$u_\varepsilon(r_0, t) \ge e^{-kt}\delta^2 + \frac{\varepsilon}{2}.$$

Letting $\varepsilon \to 0$ we obtain $u(r_0, t) \ge \delta^2 e^{-(4N+M_0)t}$, for all $t \ge 0$.

If $t_0 > 0$ we use the Hölder continuity of u and the uniform convergence of u_ε to u to find a δ such that

$$u_\varepsilon(r, t_0) \ge \frac{\eta}{2} + \varepsilon \quad \text{on } [r_0 - \delta, r_0 + \delta].$$

Then the function $v^\varepsilon(r, t) = u_\varepsilon(r, t+t_0)$ satisfies all the requirements of the previous argument.

If $r = 0$ or actually $r \in [0, R_2]$, the result follow directly from the regularity of the porous medium equation with positive initial data.

3.2. *Interfaces.* We have shown in the proof of Lemma 7 that $\varepsilon \le u_\varepsilon(r, t) \le 2^{k/m}\varepsilon$ for $r \ge R_m$. As $\varepsilon \to 0$ we obtain that $u(r, t) = 0$ for $r \ge R_m$. Thus the support of $u(r, t)$ is finite for all t. On the other hand, by Theorem 3 we know that once u becomes positive it stays positive for all later times; hence the support of $u(r, t)$ increases with t. In particular, if $\operatorname{supp} u_0(r)$ is an interval, another application of the maximum principle will show that $\operatorname{supp} u(r, t)$ is also an interval. We state the following result without proof.

THEOREM 4. If $u_0(r) > 0$ on $[0, R_2]$, $u_0(r) \equiv 0$ on $[R_2, \infty)$, then there exists a continuous increasing interface curve $r = \xi(t)$, with $R_2 = \xi(0)$, separating the populated region $\{(r, t): u(r, t) > 0\}$ from the unpopulated region $\{(r, t): u(r, t) = 0\}$, i.e., $\operatorname{supp} u(r, t) = [0, \xi(t)]$ or all t.

3.3. *Localization.* We now turn to the question of localization. As $t \to \infty$, does $\operatorname{supp} u(r, t)$ increase to a limiting domain $[0, L]$ or does $\operatorname{supp} u(r, t)$ increase to $\mathbf{R}^+ = [0, \infty)$?

A first observation is that if $\beta(s) \le \mu(s) - \delta$ for all s, then

$$h(r, t, u) = \beta(u)e^{-\alpha t}q - \mu(u) \le \beta(u) - \mu(u) \le \delta < 0.$$

In this case by the usual comparison arguments $u(r, t)$ is bounded above by the solution $v(r, t)$ of

$$\begin{aligned} v_t &= vv_{rr} + v_r^2 + (N-1)\frac{vv_r}{r} - \delta v, \\ v(r, 0) &= u_0(r). \end{aligned} \tag{3.1}$$

In turn this equation is simplified by letting $t(\tau) = -\frac{1}{\delta}\log(1 - \delta\tau)$ and $z(r, \tau) = \frac{1}{1-\delta\tau}v(r, t(\tau))$. Then $z(r, \tau)$ satisfies

$$\begin{aligned} z_\tau &= zz_{rr} + z_r^2 + (N-1)\frac{zz_r}{r}, \\ z(r, 0) &= v(r, 0) = u_0(r). \end{aligned} \tag{3.2}$$

This is a porous medium equation and thus it has finite support for every τ. There exists $L > 0$ such that $\operatorname{supp} z(r, \tau) \subseteq [0, L]$ for $0 \le \tau \le \frac{1}{\delta}$. Hence $\operatorname{supp} v(r, \tau) = \operatorname{supp} z(r, \tau) \subseteq [0, L]$, because $u(r, t) \le v(r, t)$, $\operatorname{supp} u(r, t) \subseteq [0, L]$ for all times.

Next, if $h(r, t, u) = \beta(u)e^{-\alpha t}q - \mu(u) \ge 0$, the comparison principle implies that $u(r, t) \ge w(r, t)$, the solution of the porous medium equation

$$w_t = ww_{rr} + w_r^2 + (N-1)\frac{ww_r}{r},$$
$$w(r, 0) = u_0(r).$$

It is known, [23] that $\operatorname{supp} w(r, t) \to [0, \infty)$ as $t \to \infty$. Therefore $\operatorname{supp} u(r, t) \to [0, \infty)$ in this case.

Thus, if the birth module is "clearly" less than the death module, the population will not diffuse further than a fixed interval. On the other hand if $h(r, t, u) \ge 0$, the population will eventually cover all $[0, \infty)$. Unfortunately the condition $\beta(u) \le \mu(u) - \delta$ is too restrictive and the condition $h \ge 0$ cannot be checked based on the data alone. More general conditions for localization were given in [19] for the one-dimensional problem. Those results and examples can be obtained here with only the obvious changes that are omitted.

Let

$$a_1 = \inf_{0 \le s \le M_1} \frac{\beta(s) - \alpha}{\beta(s)}, \qquad a_2 = \sup_{0 \le s \le M_1} \frac{\beta(s) - \alpha}{\beta(s)},$$
$$b_1 = \inf_{0 \le s \le M_1} \frac{\mu(s)}{\beta(s)}, \qquad b_2 = \sup_{1 \le s \le M_1} \frac{\mu(s)}{\beta(s)}.$$

THEOREM 5. (i) Assume $a_2 < b_1$ and $0 < \overline{m} \le q_0(r) \le c_1$ for some $c_1 \in (a_2, b_1)$. Then there exists $L > 0$ such that the solution $u(r, t)$ is identically 0 outside $[0, L]$ for all t.

(ii) Assume $b_2 < a_1$ and $c \le q_0(r) \le 1$ for some $c \in (b_2, a_1)$. Then the support of $u(r, t)$ tends to $[0, \infty)$ as $t \to \infty$.

3.4. *Examples.* Let $\beta^* = \sup_{0 \le s \le M_1} \beta(s)$, $\beta_* = \inf_{0 \le s \le M_1} \beta(s)$ and μ^*, μ_* defined similarly.

If $\beta^* \le \mu_* + \alpha$ and $q_0(r) \le c_1$ for some $c_1 \in ((\beta^* - \alpha)/\beta^*, \mu_*/\beta^*)$, then Theorem 5 implies that $u(r, t)$ is localized. On the other hand if $\beta_* \ge \mu^* + \alpha$ and $c \le q_0(r) \le 1$ for some $c \in (\mu^*/\beta_*, (\beta_* - \alpha)/\beta_*)$, then $\operatorname{supp} u(r, t)$ tends to $[0, \infty)$ as $t \to \infty$. These results were first conjectured by Gurtin [13], for β and μ constants.

REFERENCES

[1] D. G. Aronson, *The porous medium equation*, CIME Lectures Montecatini Terme, 10–18 (June 1985)

[2] D. G. Aronson, *Regularity properties of flows through porous media*, SIAM J. Appl. Math. **17** 461–467 (1969)

[3] D. G. Aronson, *Regularity properties of flows through porous media: a counterexample*, SIAM J. Appl. Math. **19**, 299–307 (1970)

[4] Ph. Benilam, *A strong regularity Lp for solutions of the porous media equations*, Contributions to Nonlinear Partial Differential Equations, C. Bardos, A. Damlamian, J. I. Diaz and J. Hernandez, Editors, London, 1983, pp. 39–58

[5] L. A. Caffarelli and A. Friedman, *Continuity of the density of a gas flow in a porous medium*, Trans. Amer. Math. Soc. **252**, 99–113 (1979)
[6] L. A. Caffarelli and A. Friedman, *Regularity of the free boundary of a gas flow in an n-dimensional porous medium*, Indiana Univ. Math. J. **29**, 361–391 (1980)
[7] E. A. Carl, *Population control in arctic ground squirrels*, Ecology **52**, 395–413 (1971)
[8] D. G. Aronson and L. A. Caffarelli, *Optimal regularity for one-dimensional porous medium flow*, Rev. Math. Iberoamericana **2**, 357–366 (1986)
[9] Gabriella Di Blasio, *Nonlinear age-dependent population diffusion*, J. Math. Biology **8**, 265–284 (1979)
[10] E. di Benedetto, *Continuity of weak solutions to a general Porous medium equation*, Indiana Univ. Math. J. **32**, 83–118 (1983)
[11] Maria Garroni and Michel Langlais, *Age dependent population diffusion with external constraints*, J. Math. Biol. **14**, 77–94 (1982)
[12] W. S. C. Gurney and R. M. Nibset, *The regulation of inhomogeneous populations*, J. Theoretical Biology **52**, 441–457 (1975)
[13] M. E. Gurtin, *Some questions and open problems in continuum mechanics and population dynamics*, J. Differential Equations **48**, 293–312 (1983)
[14] M. E. Gurtin and R. C. MacCamy, *Diffusion models for age-structured populations*, Math. Biosciences **54**, 49–59 (1981)
[15] M. E. Gurtin and R. C. MacCamy, *Nonlinear age-dependent population dynamics*, Arch. Rational Mech. Anal. **54**, 281–300 (1974)
[16] M. E. Gurtin and R. C. MacCamy, *On the diffusion of biological populations*, Math. Biosciences **33**, 35–49 (1977)
[17] M. E. Gurtin and R. C. MacCamy, *A population model with nonlinear diffusion*, J. Differential Equations **39**, 52–72 (1981)
[18] M. E. Gurtin and R. C. MacCamy, *Some simple models for nonlinear age dependence population dynamics*, Math. Biosciences **43**, 199–211 (1979)
[19] Gaston E. Hernandez, *Dynamics of populations with age dependence and diffusion: localization*, Appl. Anal. **29**, 143–163 (1988)
[20] Gaston E. Hernandez, *Existence of solutions in a population dynamic problem*, Quart. Appl. Math. **43**, 509–521 (1986)
[21] Gaston E. Hernandez, *The Hölder property in some degenerate parabolic problems*, J. Differential Equations **65**, 240–249 (1986)
[22] Gaston E. Hernandez and Ioannis Roussos, *The Hölder exponent for radially symmetric solutions of porous medium type equations*, to appear, in Canad. J. Math., 1990.
[23] A. S. Kalashnikov, *On the occurrence of singularities in the solutions of the equation of nonstationary filtration*, Zh. Vychisl. Mat. i Mat. Fiz. **14**, 891–905 (1974)
[24] Michel Langlais, *A nonlinear problem in age dependent population diffusion*, Preprint, 1983
[25] O. A. Oleinik and S. N. Kruzhkov, *Quasilinear second order parabolic equations with many independent variables*, Russian Math. Surveys, **16**, 106–146 (1961)
[26] L. A. Peletier, *The porous media equation*, Applications of Nonlinear Analysis in the physical Sciences, H. Amann et al. editors, Pitman, London, 1981, pp. 229–241
[27] Paul Sacks, *Existence and regularity of solutions of inhomogeneous porous medium type equations*, T.S.R. No. 2214 M.R.C. Madison, Wisconsin, 1985
[28] J. G. Skellam, *Random dispersal in theoretical populations*, Biometrika **38**, 196–218 (1951)
[29] J. L. Vazquez, *La ecuacion de los medios porosos.*, Actas 1ra Jornadas de Mecanica y Fisica Matematicas, El Escorial, Sept. 1983