ON THE LARGE TIME BEHAVIOUR OF A THERMOVISCOUS FLOW WITH DISSIPATION

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Abstract. The large time behaviour of a dissipative, thermoviscous, Couette-Poiseuille flow is examined by asymptotic methods. Previous results on the adiabatic case are confirmed. In the case of a weakly conductive fluid the approach to a bounded, hot, steady state is shown to be caused by a forced linear diffusion mechanism.

1. Introduction. The large time behaviour of a thermoviscous Couette-Poiseuille flow with viscous dissipation has recently been considered by Boudourides and Charalambakis. This work considers a fluid with zero thermal conductivity and a viscosity which depends upon temperature in such a manner that as the fluid heats up the viscosity tends to a constant value. Using functional analysis methods the authors derive orders of magnitude for the fluid velocity and temperature at large times. In particular it is shown that in this adiabatic case the temperature is, in an asymptotic sense, proportional to time and that the velocity approaches the isoviscous steady state solution.

This paper generalises the above work by considering a dissipative thermoviscous fluid with weak thermal conductivity. By employing a perturbation method the adiabatic case above is found as a leading-order problem. The large time behaviour of this is examined by an asymptotic method and explicit, rather than order of magnitude, results are found which confirm previous work. Consideration of the first perturbation to the adiabatic problem reveals a nonuniformity in the perturbation series which suggests that the large time behaviour of the adiabatic and weakly conductive problems may be quite different. Finally, it is shown that the weakly conductive problem has a well defined steady state solution which is approached, at large times, by a forced linear diffusion of heat with a subsequent *bounded* heating of the fluid.

2. Mathematical analysis. The momentum and energy equations for a dissipative viscous, Couette-Poiseuille flow are

$$\frac{\partial u}{\partial t} = P + \frac{1}{\rho} \frac{\partial}{\partial x} \left(\mu \frac{\partial u}{\partial x} \right) \quad (2.1)$$

and

$$\rho c_v \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \mu \left(\frac{\partial u}{\partial x} \right)^2 \quad (2.2)$$

where x is a coordinate perpendicular to the pressure gradient and shear directions. It will be supposed that the fluid velocity is $u = 0$ at $x = 0$ and $u = U$ at $x = L$. The thermal conductivity k is taken as constant and the viscosity μ is dependent on temperature, i.e., $\mu = \mu_0 f(T)$ where μ_0 is a typical scale for the viscosity. Temperature boundary conditions of various forms can be prescribed at $x = 0$ and L .

These equations are nondimensionalized by the transformations

$$\bar{u} = \frac{u}{U}, \quad \bar{x} = \frac{x}{L}, \quad \bar{T} = \frac{T}{\mu_0 U / \rho c_v L}, \quad \bar{t} = \frac{t}{L/U} \quad (2.3)$$

to obtain (the bars are henceforth dropped)

$$\frac{\partial u}{\partial t} = P_0 + \frac{1}{R_0} \frac{\partial}{\partial x} \left(f(T) \frac{\partial u}{\partial x} \right) \quad (2.4)$$

$$\frac{\partial T}{\partial t} = \varepsilon \frac{\partial^2 T}{\partial x^2} + f(T) \left(\frac{\partial u}{\partial x} \right)^2. \quad (2.5)$$

In (2.4), (2.5) P_0 is a scaled pressure gradient, R_0 is a Reynolds number and $\varepsilon = k / \rho c_v L U$ is the reciprocal of the product of a Prantl number and Reynolds number. For weakly conducting fluids $\varepsilon \ll 1$; the adiabatic case considered by Boudourides and Charalambakis corresponds to $\varepsilon = 0$. The form of viscosity variation with temperature adopted by these authors is retained here, i.e., $f(T) = 1 + T^{-\gamma}$ as this allows an approach to constant viscosity as the fluid heats up.

Solutions to these equations are now sought in the form

$$u = u^0 + \varepsilon u^1 + O(\varepsilon^2), \quad T = T^0 + \varepsilon T^1 + O(\varepsilon^2) \quad (2.6)$$

and give rise to the following sequence of problems:

$$\begin{aligned} O(1): \quad u_t^0 &= P_0 + \frac{1}{R_0} ((1 + (T^0)^{-\gamma}) u_x^0)_x \\ T_t^0 &= (1 + (T^0)^{-\gamma}) (u_x^0)^2 \\ O(\varepsilon): \quad u_t^1 &= \frac{1}{R_0} (-\gamma T^1 (T^0)^{-(1+\gamma)} u_x^0 + (1 + (T^0)^{-\gamma}) u_x^1)_x \\ T_t^1 &= T_{xx}^0 + 2(1 + (T^0)^{-\gamma}) u_x^0 u_x^1 - \gamma T^1 (T^0)^{-(1+\gamma)} (u_x^0)^2. \end{aligned} \quad (2.7_{1,2})$$

Equations (2.7₁) are the adiabatic problem considered by Boudourides and Charalambakis. Equations (2.7₂) represent the first correction to the adiabatic state.

The large time behaviour of the solutions to (2.7₁) are most easily found by writing $\bar{t} = \delta t$ where $\delta \ll 1$. This makes $\bar{t} = O(1)$ when $t = O(\frac{1}{\delta}) \gg 1$. As δ is an artificial parameter any solutions found by this method cannot contain δ when they are written back in terms of the original variable. Equations (2.7₁) now take

the form

$$\begin{aligned}\delta u_{\bar{t}}^0 &= P_0 + \frac{1}{R_0}((1 + (T^0)^{-\gamma})u_x^0)_x \\ \delta T_{\bar{t}}^0 &= (1 + (T^0)^{-\gamma})(u_x^0)^2.\end{aligned}$$

Solutions to these are now sought in the form

$$\begin{aligned}u^0 &= \delta^a u_a^0 + \delta^b u_b^0 + \dots \\ T^0 &= \delta^c T_c^0 + \delta^d T_d^0 + \dots\end{aligned}$$

where $a, b, c, d, \dots$ are found by asymptotic balance. At leading order it is found that $a = 1, c = -1$, giving rise to

$$\begin{aligned}0 &= P_0 + \frac{1}{R_0}(u_a^0)_{xx} \\ (T_c^0)_{\bar{t}} &= \{(u_a^0)_x\}^2.\end{aligned}\tag{2.8}_{1,2}$$

These equations have simple solutions of the form

$$\begin{aligned}u_a^0 &= \frac{1}{2}P_0 R_0 x(1 - x) + x \\ T_c^0 &= \{(u_a^0)_x\}^2 \bar{t}\end{aligned}\tag{2.9}_{1,2}$$

where the constants of integration have been chosen to satisfy boundary conditions or to provide a solution that is free from the artificial parameter δ . Proceeding to the next order it is necessary to choose $b = \gamma$ and $d = \gamma - 1$, giving the equations

$$\begin{aligned}0 &= \{(u_b^0)_x + (T_c^0)^{-\gamma}(u_a^0)_x\}_x \\ (T_d^0)_{\bar{t}} &= (T_c^0)^{-\gamma}\{(u_a^0)_x\}^2 + 2(u_a^0)_x(u_b^0)_x.\end{aligned}\tag{2.10}$$

Although Eqs. (2.10) are easily solved exactly it is sufficient to note that upon integration they give rise to $u_b^0 = O(\bar{t}^{-\gamma})$, $T_d^0 = O(\bar{t}^{1-\gamma})$. Returning to the original variable we have the following expressions, valid for $t \gg 1$, to the adiabatic problem

$$\begin{aligned}u^0 &= \frac{1}{2}P_0 R_0 x(1 - x) + x + O(t^{-\gamma}) \\ T_0 &= \{1 + \frac{1}{2}P_0 R_0(1 - 2x)\}^2 t + O(t^{1-\gamma}).\end{aligned}\tag{2.11}$$

These results are in agreement with those of Boudourides and Charalambakis but are somewhat more explicit in that the x variation, as well as the t variation, is known.

If the above procedure is now applied to the first correction from the adiabatic state as given by Eqs. (2.7₂) and solutions are sought in the form

$$u^1 = \delta^e u_e^1 + \dots, \quad T^1 = \delta^f T_f^1 + \dots\tag{2.12}$$

then an asymptotic balance is found with $f = -2, e = \gamma - 1$ and the leading-order equations are

$$\begin{aligned}0 &= \{-\gamma T_f^1 \bar{t}^{-(1+\gamma)}\{(u_a^0)_x\}^{-(1+2\gamma)} + (u_e^1)_x\}_x \\ (T_f^1)_{\bar{t}} &= \{(u_a^0)_x\}_{xx} \bar{t}.\end{aligned}\tag{2.13}$$

The solutions to these are easy to find by direct integration. However it suffices to note that $u_e^1 = O(\bar{t}^{1-\gamma})$, $T_f^1 = O(\bar{t}^2)$. This shows that the large time behaviour of the original perturbation series (2.6) is of the form

$$\begin{aligned} u &= \frac{1}{2}P_0R_0x(1-x) + x + O(t^{-\gamma}, \varepsilon t^{1-\gamma}) \\ T &= \{1 + \frac{1}{2}P_0R_0(1-2x)\}^2 t + O(t^{1-\gamma}, \varepsilon t^2). \end{aligned} \quad (2.14)$$

The presence of the secular terms $\varepsilon t^{1-\gamma}$ and εt^2 in the expansions for u and T limits the validity of the series to times $\ll O(\varepsilon^{\gamma-1})$ and calls into question the validity of the adiabatic limit as the leading-order behaviour of the weakly conductive fluid.

In order to produce a uniformly valid description of the velocity and temperature at large times it is necessary to include *some* thermal conductivity in any perturbation to Eqs. (2.4, 2.5). This is most conveniently done by writing $\bar{t} = \varepsilon t$ in these equations and leads to

$$\begin{aligned} \varepsilon u_{\bar{t}} &= P_0 + \frac{1}{R_0}((1 + T^{-\gamma})u_x)_x \\ \varepsilon T_{\bar{t}} &= \varepsilon T_{xx} + (1 + T^{-\gamma})(u_x)^2. \end{aligned} \quad (2.15_{1,2})$$

If solutions are now sought in the form

$$\begin{aligned} u &= \varepsilon^a u_a + \varepsilon^b u_b + \dots \\ T &= \varepsilon^c T_c + \varepsilon^d T_d + \dots \end{aligned} \quad (2.16_{1,2})$$

then a leading-order balance requires $a = 0$, $c = -1$ and gives rise to

$$\begin{aligned} 0 &= P_0 + \frac{1}{R_0}(u_a)_{xx} \\ (T_c)_{\bar{t}} &= (T_c)_{xx} + \{(u_a)_x\}^2. \end{aligned} \quad (2.17_{1,2})$$

The choice $c = -1$ indicates that the fluid is hot, with a temperature of $O(\frac{1}{\varepsilon})$, and Eq. (2.17₂) shows that heat is being distributed in the fluid by a forced linear diffusion mechanism. The solution to these equations are taken as

$$\begin{aligned} u_a &= \frac{1}{2}P_0R_0x(1-x) + x \\ T_c &= Ax + B + e^{-k^2\bar{t}}(C \cos kx + D \sin kx) \\ &\quad - \frac{1}{12P_0^2R_0^2}\{1 + \frac{1}{2}P_0R_0(1-2x)\}^4 \end{aligned} \quad (2.18)$$

where the constants A, B, C, D, k are fixed by whatever thermal boundary conditions are imposed at $x = 0, 1$. At the next order it is necessary to choose $b = \gamma$, $d = \gamma - 1$ and the following equations are obtained:

$$\begin{aligned} 0 &= \{(u_b)_x + (T_c)^{-\gamma}(u_a)_x\}_x \\ (T_d)_{\bar{t}} &= (T_d)_{xx} + (T_c)^{-\gamma}\{(u_a)_x\}^2. \end{aligned} \quad (2.19)$$

Although it does not seem possible to find a closed form of solution to Eqs. (2.19),

solutions in the form of infinite series can be obtained and indicate that $u_b = u_b(x, T_c(x, t))$, $T_d = T_d(x, e^{-k^2 t}, T_c(x, t))$. As T_c only contains terms which decay in time it can be concluded that the series (2.16) is uniformly valid, at least in its first two terms.

Finally we note that Eqs. (2.4, 2.5) have a formal steady state solution ($\frac{\partial}{\partial t} \equiv 0$) which corresponds to $\bar{t} \rightarrow \infty$ in (2.18), indicating that the steady state is reached via the forced linear diffusion exhibited in the large time behaviour of the equations.

3. Conclusion. The large time behaviour of a dissipative thermoviscous Couette-Poiseuille flow has been examined in the case of weak conductivity. Previous results on the adiabatic limit of this type of flow are confirmed by a different method. It is suggested, however, that the large time behaviour of this system is not adiabatic but is dominated by a forced linear diffusion into a bounded, hot, steady state.

REFERENCE

- [1] M. A. Boudourides and N. Charalambakis, *Stabilization of an adiabatic Couette-Poiseuille flow by a temperature dependent viscosity*, Quart. Appl. Math. **47**, 747-751 (1989)