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SPATIALLY PERIODIC STOKES FLOW STIRRED BY A ROTLET INTERIOR TO A CLOSED CORRUGATED BOUNDARY

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Abstract. Spatially periodic solutions of the creeping flow equations are found for the stream function in which the motion is stirred by a two-dimensional rotlet in the region interior to a closed corrugated boundary. Streamlines are given for different geometrical configurations. In some cases there is separation of the streamlines in the crevices of the corrugation.

§**1. Introduction.** Fluid motion in the presence of a periodic or corrugated boundary is of interest in connection with flow in a porous medium. The reader is referred to [1] for earlier work in this area.

The present paper considers the two-dimensional fluid motion interior to a corrugated cylinder stirred by a rotlet or equivalently a circular cylinder of small radius λ rotating with angular velocity ω. The corrugated boundary is modeled by the interior fluted column transformation in one case and by the inverse of the exterior fluted column in the second case. Exact explicit spatially periodic solutions of the Stokes equations are given for the stream function in the two flows. The main result of the paper is to demonstrate the sensitivity of the boundary geometry to the production of streamline separation in the two cases. The first case was reported earlier in [2] but the second case is a new analytical solution for spatially periodic flow. The results are of possible relevance to contamination at periodic boundaries induced by reverse flow.

§**2. The transformation.** The exterior fluted column transformation is defined by

$$z = \zeta + \frac{\varepsilon}{\zeta^{n-1}}, \qquad z = x + iy, \ \ \zeta = \rho e^{i\phi}, \tag{2.1}$$

where ε is a constant, and $n \geq 1$ is an integer. The mapping is conformal in the region $\rho \geq 1$, provided

$$0 \leq |\varepsilon|(n-1) < 1. \tag{2.2}$$

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In terms of x and y

$$x = \rho\cos\phi + \frac{\varepsilon}{\rho^{n-1}}\cos(n-1)\phi\,, \tag{2.3}$$

$$y = \rho\cos\phi - \frac{\varepsilon}{\rho^{n-1}}\sin(n-1)\phi\,, \tag{2.4}$$

and in particular the circle $|\zeta| = \rho = 1$, in the ζ-plane, maps into the closed curve $\mathscr{C}$ which can be expressed parametrically as

$$x = \cos\phi + \varepsilon\cos(n-1)\phi\,, \tag{2.5}$$

$$y = \sin\phi - \varepsilon\sin(n-1)\phi\,, \tag{2.6}$$

for $0 \le \phi \le 2\pi$. In fact, from (2.5) and (2.6)

$$r^2 = z\bar{z} = x^2 + y^2 = 1 + 2\varepsilon\cos n\phi + \varepsilon^2\,, \tag{2.7}$$

so that $\mathscr{C}$ is a closed curve with n peaks lying between the circles $r = 1 \pm \varepsilon$. The interior fluted column transformation has been described in [2] and will be further discussed in Sec. 4. Now define analytic inversion by the transformation

$$z' = \frac{1}{z} \tag{2.8}$$

which essentially represents geometric inversion plus a reflection in the real axis. The region exterior to $\rho = 1$ maps into the region interior to the inverted curve $\mathscr{C}'$ of $\mathscr{C}$. The curve $\mathscr{C}'$ may be parametrically described by

$$x = \frac{\cos\phi + \varepsilon\cos(n-1)\phi}{1 + 2\varepsilon\cos n\phi + \varepsilon^2}\,, \tag{2.9}$$

$$y = \frac{\sin\phi - \varepsilon\sin(n-1)\phi}{1 + 2\varepsilon\cos n\phi + \varepsilon^2}\,, \tag{2.10}$$

for $0 \le \phi \le 2\pi$. Also since

$$r'^2 = z'\overline{z'} = x'^2 + y'^2 = \frac{1}{1 + 2\varepsilon\cos n\phi + \varepsilon^2}\,, \tag{2.11}$$

the curve $\mathscr{C}'$ is a closed corrugated boundary with n peaks lying between the circles

$$r' = (x'^2 + y'^2)^{1/2} = \frac{1}{1 \pm \varepsilon}\,. \tag{2.12}$$

Note here that the transformation will only be used for $n \ge 3$.

§3. **The flow problem.** The flow problem to be considered in this section is the motion interior to the closed curve $\mathscr{C}'$ in the z'-plane which is stirred by a rotlet at $z' = 0$. It will be shown by considering the asymptotic flow as $z' \to 0$ that this flow corresponds to the motion produced by a circular cylinder of small radius centered at $z' = 0$, in the presence of the fixed boundary $\mathscr{C}'$. In the limit of zero Reynolds numbers the creeping flow equations describing the flow are

$$\underline{q}' = \operatorname{curl}'(-\psi'\hat{k}) = -\frac{\partial\psi'}{\partial y'}\hat{\imath} + \frac{\partial\psi'}{\partial x'}\hat{\jmath}\,, \tag{3.1}$$

where ψ' is the stream function, $\underline{q}'$ the fluid velocity and the nondimensional Stokes flow equations are

$$\frac{\partial p'}{\partial x'} = -\frac{\partial}{\partial y'}\nabla'^2\psi', \qquad \frac{\partial p'}{\partial y'} = \frac{\partial}{\partial x'}\nabla'^2\psi', \tag{3.2}$$

where p' is the pressure and ∇'^2 is the two-dimensional Laplacian defined by

$$\nabla'^2 \equiv \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} = 4\frac{\partial}{\partial z'}\frac{\partial}{\partial \bar{z}'}. \tag{3.3}$$

If p' is eliminated from (3.2) then ψ' will satisfy the biharmonic equation given by

$$\nabla'^4\psi' = 0. \tag{3.4}$$

In the z-plane the fluid velocity is defined by

$$\underline{q} = \mathrm{curl}(-\psi\hat{k}) = -\frac{\partial\psi}{\partial y}\hat{i} + \frac{\partial\psi}{\partial x}\hat{j}, \tag{3.5}$$

and the equations of motion are

$$\frac{\partial p}{\partial x} = -\frac{\partial}{\partial y}(\nabla^2\psi), \qquad \frac{\partial p}{\partial y} = \frac{\partial}{\partial x}(\nabla^2\psi), \tag{3.6}$$

where p is the pressure and $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$. Elimination of the pressure p from (3.6) again produces the biharmonic equation

$$\nabla^4\psi = 0. \tag{3.7}$$

It is a known result that if ψ is a solution of (3.7), then

$$\psi' \equiv \frac{\psi}{x^2+y^2} = \frac{\psi}{[\rho^2 + (2\varepsilon/\rho^{n-2})\cos n\phi + \varepsilon^2/\rho^{2n-2}]}, \tag{3.8}$$

is a solution of (3.4). In the present case it is more convenient to consider the function ψ for which a suitable solution is expressed by

$$\begin{aligned} \psi = &\left[\rho^2 + \frac{2\varepsilon}{\rho^{n-2}}\cos n\phi + \frac{\varepsilon^2}{\rho^{2n-2}}\right]\log\rho \\ &+ A\left[\rho^2 + \frac{2\varepsilon}{\rho^{n-2}}\cos n\phi + \frac{\varepsilon^2}{\rho^{2n-2}}\right] + B\left[\rho^2 + \frac{\varepsilon}{\rho^{n-2}}\cos n\phi\right] \\ &+ C\frac{\cos n\phi}{\rho^n} - A(1+\varepsilon^2) - B, \end{aligned} \tag{3.9}$$

where A, B, and C are constants to be determined by the boundary conditions. These are expressed by

$$\psi = \frac{\partial\psi}{\partial\rho} = 0 \quad \text{at } \rho = 1, \tag{3.10}$$

and the constants are determined from the equations

$$\begin{aligned} 1 + \varepsilon^2 + 2A[1-(n-1)\varepsilon^2] + 2B &= 0, \\ 2\varepsilon A + B\varepsilon + C &= 0, \\ 2\varepsilon - 2\varepsilon A(n-2) - B\varepsilon(n-2) - nC &= 0, \end{aligned} \tag{3.11}$$

for which the solutions are

$$A = -\frac{1}{2}\frac{(1-\varepsilon^2)}{[1+(n-1)\varepsilon^2]}, \qquad B = \frac{-n\varepsilon^2}{[1+(n-1)\varepsilon^2]}, \qquad C = \varepsilon. \tag{3.12}$$

ψ may now be written as

$$\begin{aligned}\psi = {} & \left[\rho^2 + \frac{2\varepsilon}{\rho^{n-2}}\cos n\phi + \frac{\varepsilon^2}{\rho^{2n-2}}\right]\log\rho \\ & - \frac{1}{2}\frac{(1-\varepsilon^2)}{[1+(n-1)\varepsilon^2]}\left[\rho^2 + \frac{2\varepsilon}{\rho^{n-2}}\cos n\phi + \frac{\varepsilon^2}{\rho^{2n-2}}\right] \\ & - \frac{n\varepsilon^2}{[1+(n-1)\varepsilon^2]}\left[\rho^2 + \frac{\varepsilon}{\rho^{n-2}}\cos n\phi\right] + \frac{\varepsilon\cos n\phi}{\rho^n} \\ & + \frac{\frac{1}{2}(1-\varepsilon^4) + n\varepsilon^2}{1+(n-1)\varepsilon^2}.\end{aligned} \tag{3.13}$$

Note that this solution is only valid for $n \geq 3$ and ε satisfying (2.2). It is of interest to point out that the forcing term for this flow in the z-plane is represented by

$$\psi_0 = \left(\rho^2 + \frac{2\varepsilon}{\rho^{n-2}}\cos n\phi + \frac{\varepsilon^2}{\rho^{2n-2}}\right)\log\rho, \tag{3.14}$$

which produces a pressure drop in the region $\rho \geq 1$, $0 \leq \phi \leq 2\pi$. The pressure p is discontinuous along $\phi = 0$, $\phi = 2\pi$ and this is the reason that this solution is not normally used in Stokes flow. The corresponding forcing flow in the z'-plane is expressed by

$$\psi_0' = \frac{\psi_0}{r^2} = \log\rho, \tag{3.15}$$

which represents rotlet behaviour near $z' = 0$. From (3.8) and (3.10) the boundary conditions for ψ' are satisfied and

$$\psi' = \frac{\partial\psi'}{\partial\rho} = 0 \quad \text{at } \rho = 1. \tag{3.16}$$

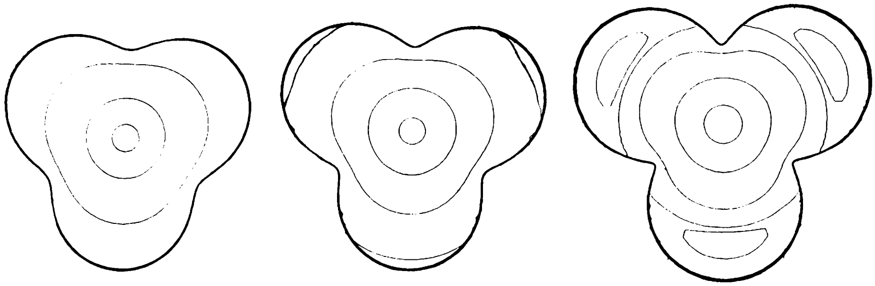
The stream function ψ' is then given by

$$\begin{aligned}\psi' = {} & \log\rho - \frac{1}{2}\frac{(1-\varepsilon^2)}{[1+(n-1)\varepsilon^2]} + \frac{1}{[\rho^2 + (2\varepsilon/\rho^{n-2})\cos n\phi + \varepsilon^2/\rho^{2n-2}]} \\ & \times\left\{-\frac{n\varepsilon^2}{[1+(n-1)\varepsilon^2]}\left[\rho^2 + \frac{\varepsilon}{\rho^{n-2}}\cos n\phi\right] + \frac{\varepsilon\cos n\phi}{\rho^n} + \frac{1}{2}\frac{1-\varepsilon^4+2n\varepsilon^2}{1+(n-1)\varepsilon^2}\right\}.\end{aligned} \tag{3.17}$$

The transition to separated flow for the $n = 3$ case is shown in Fig. 1. This phenomena also occurs for all $n \geq 3$, as seen in Fig. 2 and Fig. 3.

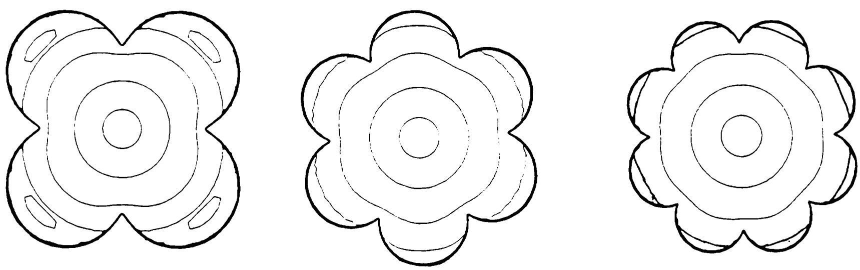
The vorticity ω' is defined by

$$\omega' = \nabla'^2\psi' = 4\partial_{z'}\partial_{\bar{z}'}\psi' = r^2\nabla^2\psi - 4z\psi_z - 4\bar{z}\psi_{\bar{z}} + 4\psi. \tag{3.18}$$



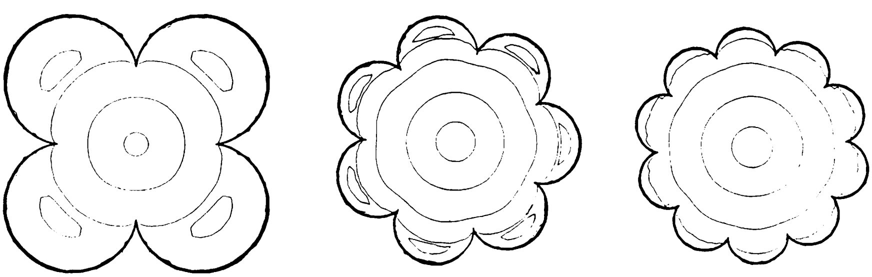
$\epsilon = 0.20$ $\epsilon = 0.25$ $\epsilon = 0.33$

FIG. 1. Some solutions for $n = 3$, and $|\varepsilon| < 0.5$.



$n = 4$, $\epsilon = 0.25$ $n = 6$, $\epsilon = 0.167$ $n = 8$, $\epsilon = 0.125$

FIG. 2. Some solutions for $\varepsilon = 1/n$.



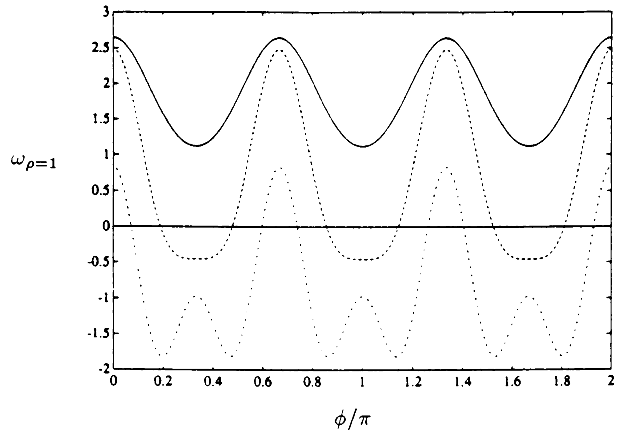
$n = 4$, $\epsilon = 0.33$ $n = 7$, $\epsilon = 0.167$ $n = 10$, $\epsilon = 0.111$

FIG. 3. Some solutions for $\varepsilon \simeq 1/(n-1)$.

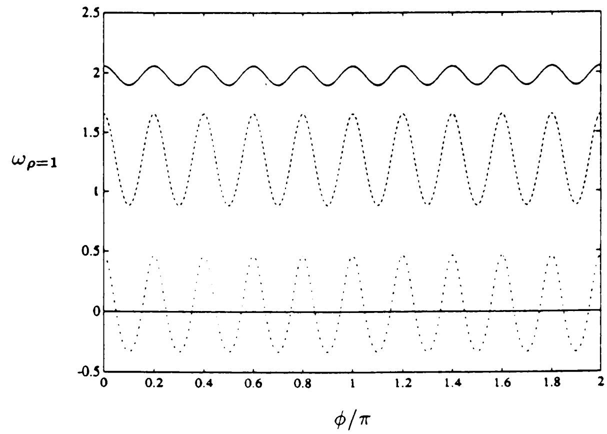
Using (2.8), (3.8), and (3.10) the boundary vorticity can be written as

$$\begin{aligned}(\omega')_{\rho=1} &= 4(|z|^2 \partial_z \partial_{\bar{z}} \psi)_{\rho=1} \\ &= (1 + 2\varepsilon\cos(n\phi) + \varepsilon^2)\left(4\varepsilon\cos(n\phi) + 2\frac{1 - n^2\varepsilon^2 - (n-1)^2\varepsilon^4}{1 + (n-1)\varepsilon^2}\right). \quad (3.19)\end{aligned}$$

Fig. 4 shows that there is boundary separation for ε sufficiently large, and satisfying (2.2).



$n = 3$, $\epsilon = 0.1, 0.3, 0.45$.



$n = 10$, $\epsilon = 0.01, 0.06, 0.10$.

FIG. 4. Variation of boundary vorticity for some cases of n . Read ε from top to bottom.

In the vicinity of $z' = 0$

$$\psi' \sim -\log r' + \frac{1 - \varepsilon^2 + 2n\varepsilon^2}{2[1 + (n-1)\varepsilon^2]} r'^2 \quad \text{as } r' \to 0, \tag{3.20}$$

which corresponds to the motion of a circular cylinder of small radius λ (< 1) rotating with constant angular velocity Ω . The angular velocity is defined by

$$\Omega = -\frac{1}{\lambda^2} + \frac{1 + 2n\varepsilon^2 - \varepsilon^4}{1 + (n-1)\varepsilon^2} . \tag{3.21}$$

If the outer boundary is replaced by a circular cylinder of radius $(1+\varepsilon)^{-1}$ the stream function ψ_1' for the flow is given by

$$\psi_1' = -\log r' + \frac{1}{2}(1 + \varepsilon)^2 r'^2 \tag{3.22}$$

and the angular velocity of the boundary $r' = \lambda$ is

$$\Omega' = -\frac{1}{\lambda^2} + (1+\varepsilon)^2 . \tag{3.23}$$

In both cases the torques on the boundaries are the same and the ratio

$$\frac{\Omega}{\Omega'} = \frac{1 - \lambda^2(1 - \varepsilon^4 + 2n\varepsilon^2/1 + (n-1)\varepsilon^2)}{1 - \lambda^2(1+\varepsilon)^2} > 1 . \tag{3.24}$$

This equation indicates that for a given torque there is a small increase in angular velocity produced by the corrugations.

§**4. Comparison with solution of** [2]. In [2], the transformation considered was the (interior) fluted column transformation

$$z = \zeta + \varepsilon\zeta^{n+1}, \qquad z = x + iy, \; \zeta = \rho e^{i\phi} . \tag{4.1}$$

It is analytic in the interior of $|\zeta| = 1$, when $n \geq 1$ is an integer and

$$0 \leq |\varepsilon|(n+1) < 1 . \tag{4.2}$$

The Stokes flow problem for a rotlet at the origin and no-slip boundary conditions was solved. No separation was found, and in fact the stream function was a function of ρ only, given by

$$\psi_i = \log\rho - \frac{\rho^2 - \varepsilon^2\rho^{2n+2}}{2[1 - (n+1)\varepsilon^2]} . \tag{4.3}$$

Hence the streamlines coincided with the fluted columns $\rho =$ constant, as shown in Fig. 5. Although the mapping in (4.1) gives a boundary similar to the one produced by the transformation introduced In Sec. 2, there is a difference. In fact if the points on $\mathscr{C}$ furthest from the rotlet are considered (see (2.7)), then the vorticity is $\omega' = 2$ for $\varepsilon = 0$, $n \geq 3$, and for $\varepsilon = \frac{1}{n}$, $n \geq 3$, $\omega' = -2(n^2 - 2n + 1)(1 - 4n + 3n^2 + 2n^3)/n^4(n^2 + n - 1) < 0$. Hence separation always occurs for ε sufficiently large, and satisfying (2.2) (see Fig. 2 and Fig. 4). More specifically, taking $\cos(n\phi) = -1$ shows that separation will begin for $n \geq 3$, when ε satisfies $\varepsilon_0 \leq |\varepsilon| < (n+1)^{-1}$,

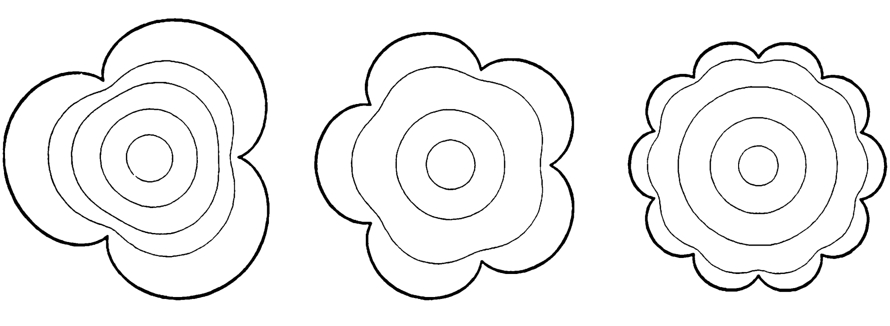


FIG. 5. Some solutions from [8].

and ε_0 satisfies

$$n = \frac{(\varepsilon_0^2 - 1)(\varepsilon_0 - 1)}{\varepsilon_0(\varepsilon_0^2 + 1)}. \tag{4.4}$$

The above is a cubic equation in ε_0, but the exact solution is not very informative. Instead note that as $n \to \infty$, $\varepsilon_0 \to 0$ since it must be bounded by $\frac{1}{n-1}$. One sees that

$$n + 1 = \frac{1}{\varepsilon_0}\frac{2\varepsilon_0^3 - \varepsilon_0^2 + 1}{\varepsilon_0^2 + 1} = \frac{1}{\varepsilon_0} + \mathscr{O}(\varepsilon_0), \tag{4.5}$$

or

$$\varepsilon_0 = \frac{1}{n+1} + \mathscr{O}\left(\frac{1}{n^3}\right). \tag{4.6}$$

The term $\mathscr{O}(1/n^3)$ is negative $\forall n \geq 3$. Thus separation starts for ε_0 slightly less than $\frac{1}{n+1}$.

As a way of measuring the interaction of fluid flow with a distorted boundary, consider the curvatures at the furthest points on the boundaries. The formula used, for a curve $z(\phi)$ in the complex plane, is

$$K = \frac{\operatorname{Im}(\dot{\bar{z}}\ddot{z})}{|z\bar{z}|^{3/2}}. \tag{4.7}$$

For the interior fluted column, the curvature is found to be, when $\phi = 2\pi/n$ (i.e., a furthest point from the center)

$$K_i(\phi = 2\pi/n) = \frac{1 + \varepsilon(n+1)(n+2) + \varepsilon^2(n+1)^3}{(1 + \varepsilon(n+1))^3} \tag{4.8}$$

The exterior fluted column transformation gives, at $\phi = \pi/n$

$$K_e(\phi = \pi/n) = \frac{(1-\varepsilon)^4 + 3n\varepsilon(1-\varepsilon)^3 + n^2\varepsilon(1-\varepsilon)^2(1+3\varepsilon) + n^3\varepsilon^2(1-\varepsilon^2)}{(1 + \varepsilon(n-1))^3}. \tag{4.9}$$

The most extreme values that K_i can take, for fixed n at $\phi = 2\pi/n$, occurs when $\varepsilon = 1/(n+1)$; i.e.,

$$K_i^{ex} = \frac{n+2}{4}. \tag{4.10}$$

At this same value for ε, it is found that

$$K_e^{ex} = \frac{n(n+5)}{4(n+1)}, \tag{4.11}$$

thus for each $n \geq 3$, there is a critical curvature $K^c(n)$, such that

$$K_i^{ex} < K^c(n) < K_e^{ex} \tag{4.12}$$

at which separation begins. It is found that the amount of curvature required for separation, given n, is

$$\Delta K(n) = K_e^{ex} - K_i^{ex} = \frac{1}{2}\frac{(n-1)}{(n+1)}, \tag{4.13}$$

more than K_i^{ex}. ΔK ranges from $\frac{1}{4}$ at $n = 3$, to $\frac{1}{2}$ as $n \to \infty$.

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