ROOTS OF $J_0(z) - iJ_1(z) = 0$ AS SADDLE POINTS OF THE REDUCED LOGARITHMIC DERIVATIVE OF $J_0(z)$

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Abstract. The roots of $J_0(z) - iJ_1(z) = 0$ are saddle points of the function $F_0(z) \equiv zJ'_0(z)/J_0(z)$. A very efficient algorithm allows one to obtain, by iteration, those roots to the desired accuracy.

Recently, several papers [1-3] have dealt with the zeros of the function $f(z) \equiv J_0(z) - iJ_1(z)$, which arises in problems of propagation of shallow-water waves. The solutions of the equation

$$J_0(z) - iJ_1(z) = 0 \quad (1)$$

are in fact the lower half-plane saddle points of the reduced logarithmic derivative of $J_0(z)$. This can be easily seen by checking that the solutions of Eq. (1) verify, for $\nu = 0$,

$$(zJ'_\nu(z)/J_\nu(z))^2 = \nu^2 - z^2, \quad (2)$$

that, as shown in [5] is the condition satisfied by the saddle points of the reduced logarithmic derivative of $J_\nu(z)$, that we denote by

$$F_\nu(z) \equiv zJ'_\nu(z)/J_\nu(z). \quad (3)$$

Some interesting results concerning those saddle points were reported in [5]. In particular, a recurrence was given that, by repeated application, allows one to determine the saddle points with the desired accuracy. In the case we are considering in this paper, that is, for $\nu = 0$, the recurrence becomes

$$z_{n+1}^2 = z_n^2 - b_1(z_n)t(z_n) - b_2(z_n)(t(z_n))^2 - b_3(z_n)(t(z_n))^3 - b_4(z_n)(t(z_n))^4, \quad (4)$$

where z_n and z_{n+1} represent successive approximations to the saddle points and the following notation has been used

$$b_1(z) = 2iz, \quad (5a)$$

$$b_2(z) = 1 - 2iz, \quad (5b)$$

$$b_3(z) = 8(iz - 1)/3, \quad (5c)$$

$$b_4(z) = i/2z + 17/3 - 4iz, \quad (5d)$$

$$t(z) = -z(J_1(z)/J_0(z) + i). \quad (6)$$

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The most efficient method of calculating $t(z)$ seems to be to make use of the continued fraction expansion [4, Eq. 9.1.73] for the quotient $J_1(z)/J_0(z)$. Of course, $t(z)$ vanishes at the roots of Eq. (1).

Recurrences of higher order in $t(z)$ can also be obtained by the procedure sketched in [5].

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