ASYMPTOTIC FORMULAS FOR THE SHOCK WAVE OF THE SCALAR CONSERVATION LAW WITH SMOOTH INITIAL DATA

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1. Introduction. We consider the scalar conservation law in a single space variable

$$u_t + f(u)_x = 0, \quad t > 0, \quad x \in \mathbf{R}, \quad (1.1)$$

with initial data

$$u(x, 0) = u_0(x), \quad x \in \mathbf{R}. \quad (1.2)$$

We assume that f is C^2 and uniformly convex: $f''(u) \geq \varepsilon > 0$, $u \in \mathbf{R}$. To select the physically relevant solution we impose the entropy condition

$$\frac{u(x+a, t) - u(x, t)}{a} \leq \frac{E}{t}, \quad a > 0, \quad t > 0, \quad (1.3)$$

where $E > 0$ is a constant independent of x , t , and a .

It is well known that for arbitrary bounded measurable initial data there exists uniquely a global weak solution to (1.1), (1.2), and (1.3), which is called the entropy solution. Generally, this solution develops discontinuities in a finite time, even though $u_0(x)$ is smooth, called shock waves (see Smoller [7]). Let the initial datum be a smooth function. In this paper we shall give the asymptotic formulas for the solution at a shock generation point, at a shock interaction point, and at time infinity.

Lax [4] gives an explicit formula for the entropy solution for all time (see also Lax [5]). Put $f'(u) = a(u)$ and $a^{-1}(u) = b(u)$. We define the conjugate function $g(z)$ by $g(z) = zb(z) - f(b(z))$. Let

$$G(x, y, t) = \int_0^y u_0(z) dz + tg \left(\frac{x-y}{t} \right). \quad (1.4)$$

Then for fixed $t > 0$, for all but a countable set of x the minimum of (1.4) is taken at exactly one point $y = y(x, t)$. The entropy solution is given by

$$u(x, t) = b \left(\frac{x - y(x, t)}{t} \right). \quad (1.5)$$

For fixed $t > 0$, we call the point $x = x(t)$ a shock point if the solution $u(\cdot, t)$ has a jump discontinuity across $x(t)$. The entropy condition (1.3) requires that at

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the shock point $u(x(t)-0, t) > u(x(t)+0, t)$. Also we call the locus of $x(t)$ a shock curve.

Let

$$t^* = \inf\{t > 0 \mid \exists x = x(t) \text{ such that } u(x(t)-0, t) > u(x(t)+0, t)\},$$

$$x^* = \lim_{t \rightarrow t^*+0} x(t).$$

We call a point (x^*, t^*) a shock generation point.

In Sec. 2, using Lax's formula (1.5) we shall give a practical method for determining the location of the shock point $x(t)$ for all $t > t^*$ and the behavior of the solution at the point of discontinuity. In Sec. 3, using the method in Sec. 2 we can find the shock curve and the entropy solution at a shock generation point in terms of asymptotic expansions. We shall give an explicit method of constructing these asymptotic expansions. In Sec. 4, we shall apply the method in Sec. 2 to the case where shock curves $x_1(t), x_2(t), \dots, x_n(t)$, $n \geq 2$ ($t < t_c$), interact at $t = t_c$ and give rise to one shock curve $x(t)$ ($t > t_c$). We shall give the asymptotic formulas for the shock curves and the solution at the shock interaction time, t_c . Finally we shall give the asymptotic formulas in the case where the initial data have compact support. For the study of the regularity of the entropy solution and the shock curves based on the theory of singularities of differentiable mappings, we refer to the work of Guckenheimer [3] and Schaeffer [6]. Our goal is to give a quantitative statement about the shock wave.

2. A method of determining the shock point $x(t)$. We may assume without loss of generality that $f(0) = f'(0) = 0$, since in (1.1) we may replace $f(u)$ by $f(u) - f(0) - f'(0)u$ and $u(x, t)$ by $u(x + f'(0)t, t)$. Now let the initial data $u_0(x)$ be a bounded C^2 function satisfying the following condition (H):

- (H) There exists a unique point m such that $u'_0(x) > 0$ for $x < m$, $u'_0(m) = 0$ and $u'_0(x) < 0$ for $x > m$. There exists a unique point $\tau > m$ such that $(a \circ u_0)''(x) < 0$ for $m \leq x < \tau$, $(a \circ u_0)''(\tau) = 0$, and $(a \circ u_0)''(x) > 0$ for $\tau < x$.

Put $t^* = -\{(a \circ u_0)'(\tau)\}^{-1}$. Note that since $a(u)$ is an increasing function,

$$u_0(y) - b\left(\frac{x-y}{t}\right) = 0$$

is equivalent to

$$a \circ u_0(y) - \frac{x-y}{t} = 0.$$

If $0 < t \leq t^*$, then for each $x \in \mathbf{R}$ the two graphs $z = b((x-y)/t)$ and $z = u_0(y)$ intersect at a unique point $y = y_0(x, t)$.

If $t^* < t$, then there exist two functions $\beta_1(t), \beta_2(t) \in C^\infty((t^*, \infty))$ such that $\beta_1(t) < \tau < \beta_2(t)$ and $-1/t = (a \circ u_0)'(\beta_1(t)) = (a \circ u_0)'(\beta_2(t))$. Let $\alpha_i(t) = (a \circ u_0)(\beta_i(t))t + \beta_i(t)$, $i = 1, 2$ (see Fig. 1). Then we can easily check that $\alpha_1(t) > \alpha_2(t)$

FIG. 1

FIG. 2

and

$$\lim_{t \rightarrow t^*+0} \alpha_i(t) = \tau - \frac{(a \circ u_0)(\tau)}{(a \circ u_0)'(\tau)}, \quad i = 1, 2. \quad (2.1)$$

Now we observe that for each $x \in (-\infty, \alpha_2(t)]$ the two graphs $z = b((x-y)/t)$ and $z = u_0(y)$ intersect at a unique point $y = y_1(x, t)$, for each $x \in [\alpha_1(t), \infty)$ at a unique point $y = y_2(x, t)$, and for each $x \in (\alpha_2(t), \alpha_1(t))$ at three distinct points $y = y_1(x, t)$, $\hat{y}(x, t)$, $y_2(x, t)$ ($y_1(x, t) < \hat{y}(x, t) < y_2(x, t)$). Note that for fixed $t > t^*$, $y_1(x, t)$ is a continuous function of $x \in (-\infty, \alpha_1(t))$ and $y_2(x, t)$ is a continuous function of $x \in (\alpha_2(t), +\infty)$. For $x \in (\alpha_2(t), \alpha_1(t))$ let D_1 and D_2 be the left and right regions surrounded by the two graphs respectively (see Fig. 2). We denote the area of D_i by $|D_i|$ ($i = 1, 2$).

THEOREM 2.1. (1) If $0 < t \leq t^*$, then $u(x, t) = u_0(y_0(x, t))$.

(2) Let $t^* < t$ and $x = x(t) \in (\alpha_2(t), \alpha_1(t))$ be the point where $|D_1| = |D_2|$ holds.

Then we have

- (a) if $x \in (-\infty, x(t))$, then $u(x, t) = u_0(y_1(x, t))$;
- (b) if $x \in (x(t), +\infty)$, then $u(x, t) = u_0(y_2(x, t))$;
- (c) $x(t)$ is the shock point where $u(x(t)-0, t) > u(x(t)+0, t)$, $u_0(y_1(x(t), t)) = u(x(t)-0, t)$, and $u_0(y_2(x(t), t)) = u(x(t)+0, t)$.

(3) The limit of the shock curve $x^* = \lim_{t \rightarrow t^*+0} x(t)$ is given by

$$x^* = \tau - \frac{(a \circ u_0)(\tau)}{(a \circ u_0)'(\tau)}.$$

Proof. We make use of Lax's formula (1.5). Since $y(x, t)$ minimizes $G(x, y, t)$, we have

$$\frac{\partial}{\partial y} G(x, y(x, t), t) = u_0(y(x, t)) - b\left(\frac{x - y(x, t)}{t}\right) = 0. \quad (2.2)$$

Here we have used the equality

$$g'(z) = b(z) + zb'(z) - f'(b(z))b'(z) = b(z).$$

Let us consider the case where the two graphs $z = b((x - y)/t)$ and $z = u_0(y)$ intersect at a unique point $y = \xi(x, t)$. Then we can easily observe that $\partial^2 G(x, \xi, t)/\partial y^2 > 0$ and therefore $\xi = y(x, t)$. Now let us consider the case where the two graphs intersect at three points, that is, $t^* < t$ and $x \in (\alpha_2(t), \alpha_1(t))$. Then $y_1(x, t)$ and $y_2(x, t)$ are the points where $G(x, y, t)$ takes its relative minimum. To select the point which minimizes $G(x, y, t)$ we rewrite (1.4) as

$$G(x, y, t) = \int_y^x u_0(z) dz + \int_y^x b\left(\frac{x - z}{t}\right) dz.$$

Now let $x \in (\alpha_2(t), x(t))$. Then from Fig. 2 we can easily observe that $|D_1| > |D_2|$, which implies $G(x, y_1(x, t), t) < G(x, y_2(x, t), t)$. Hence we have $y(x, t) = y_1(x, t)$. Similarly, for $x \in (x(t), \alpha_1(t))$ we have $y(x, t) = y_2(x, t)$. Therefore, combining with (1.5) and (2.2) we obtain (1) and (2).

(3) is immediately obtained from $\alpha_1(t) > x(t) > \alpha_2(t)$ and (2.1). Q.E.D.

REMARK. Whitham obtained a result similar to the above theorem in [8, Secs. 2.8, 2.9], but his proof seems to be incomplete.

3. Construction of the asymptotic expansions at the shock generation point. Relying upon the arguments in Sec. 2 we shall give the asymptotic formulas for the shock curve $x(t)$ and the entropy solution $u(x, t)$ at the shock generation point

$$(x^*, t^*) = \left(\tau - \frac{(a \circ u_0)(\tau)}{(a \circ u_0)'(\tau)}, \frac{-1}{(a \circ u_0)'(\tau)} \right).$$

THEOREM 3.1. Let $N \geq 5$, and assume that

1. the initial data $u_0(y)$ is C^{N+1} and satisfies condition (H),
2. $(a \circ u_0)^{(3)}(\tau) > 0$,
3. $f(u)$ is $C^{N+2}(\mathbf{R})$.

FIG. 3

Then the shock curve $x(t)$ and the solution $u(x, t)$ admit the following asymptotic expansions as $t \rightarrow t^* + 0$:

$$x(t) = x^* + \sum_{k=2}^N c_k (t - t^*)^{k/2} + O((t - t^*)^{(N+1)/2}),$$

$$u(x(t) - 0, t) = u_0(\tau) + \sum_{k=1}^{N-2} A_k (t - t^*)^{k/2} + O((t - t^*)^{(N-1)/2}),$$

$$u(x(t) + 0, t) = u_0(\tau) + \sum_{k=1}^{N-2} B_k (t - t^*)^{k/2} + O((t - t^*)^{(N-1)/2}).$$

The coefficients c_k ($k \geq 2$) and A_k, B_k ($k \geq 1$) can be determined successively by solving certain recursive equations. In particular, $c_k = 0$, $A_k + B_k = 0$ for odd k , $A_k = B_k$ for even k , and

$$c_2 = a \circ u_0(\tau),$$

$$c_4 = \frac{-3}{10} \{ (a \circ u_0)'(\tau) \}^3 (a \circ u_0)^{(4)}(\tau) \{ (a \circ u_0)^{(3)}(\tau) \}^{-2} \\ - \frac{4}{5} f^{(3)}(u_0(\tau)) \{ u_0'(\tau) \}^2 \{ (a \circ u_0)'(\tau) \}^2 \{ (a \circ u_0)^{(3)}(\tau) \}^{-1},$$

$$A_1 = 6^{1/2} u_0'(\tau) (a \circ u_0)'(\tau) \{ (a \circ u_0)^{(3)}(\tau) \}^{-1/2},$$

$$B_1 = -A_1.$$

REMARK. The shock curve $x(t)$ has higher-order derivatives up to $t = t^*$. On the other hand, since $u(x(t) - 0, t)$ and $u(x(t) + 0, t)$ contain the terms $A_1(t - t^*)^{1/2}$ and $-A_1(t - t^*)^{1/2}$ ($A_1 > 0$), respectively, the two graphs $u = u(x(t) - 0, t)$ and $u = u(x(t) + 0, t)$ in the (t, u) plane are tangent to $t = t^*$ (see Fig. 3).

Proof. For simplicity we may assume that $f(0) = a(0) = 0$ and $u_0(\tau) = 0$, since in (1.1) we may replace $u(x, t)$ by $u(x + a(u_0(\tau))t, t) - u_0(\tau)$ and $f(u)$ by $f(u + u_0(\tau)) - f(u_0(\tau)) - a(u_0(\tau))u$. Let (x, t) ($t > t^*$) be sufficiently close to $(x^*, t^*) = (\tau, -1/(a \circ u_0)'(\tau))$ so that the two graphs $z = u_0(y)$ and $z = b((x - y)/t)$ intersect at three distinct points $y = y_1(x, t), \tilde{y}(x, t), y_2(x, t)$ ($y_1(x, t) < \tilde{y}(x, t) < y_2(x, t)$)

FIG. 4

(see Fig. 4). Let D_1 and D_2 be the left and right regions surrounded by these two graphs respectively.

According to Theorem 2.1 the shock curve $x = x(t)$ satisfies $|D_1| = |D_2|$, that is,

$$\int_{y_2(x(t), t)}^{y_1(x(t), t)} \left\{ u_0(y) - b \left(\frac{x(t) - y}{t} \right) \right\} dy = 0. \quad (3.1)$$

Now put $y_i(t) = y_i(x(t), t)$, $i = 1, 2$, and $\varepsilon = t - t^* > 0$. We seek the asymptotic expansions of $x(t)$, $y_1(t)$, and $y_2(t)$ in the following forms:

$$\begin{aligned} x(t) &= x(\varepsilon) \sim \tau + \sum_{k \geq 1} c_k \varepsilon^{k/2}, \\ y_1(t) &= y_1(\varepsilon) \sim \tau + \sum_{k \geq 1} a_k \varepsilon^{k/2}, \quad y_2(t) = y_2(\varepsilon) \sim \tau + \sum_{k \geq 1} b_k \varepsilon^{k/2}. \end{aligned} \quad (3.2)$$

In the first step we shall give an explicit method for determining the coefficients in each expansion of (3.2). The second step is devoted to proving that each expansion of (3.2) actually approximates $x(t)$, $y_1(t)$, and $y_2(t)$ respectively as $t \rightarrow t^* + 0$. Finally, in the third step we shall give asymptotic expansions in Theorem 3.1.

(First Step) From Fig. 4

$$b \left(\frac{x(t) - y_1(t)}{t} \right) = u_0(y_1(t)), \quad b \left(\frac{x(t) - y_2(t)}{t} \right) = u_0(y_2(t)).$$

Then

$$\frac{x(t) - y_1(t)}{t} = a \circ u_0(y_1(t)), \quad \frac{x(t) - y_2(t)}{t} = a \circ u_0(y_2(t)). \quad (3.3)$$

Moreover, from (3.1) and $g'(z) = b(z)$ it follows that

$$\int_{y_2(t)}^{y_1(t)} u_0(y) dy + t g \left(\frac{x(t) - y_1(t)}{t} \right) - t g \left(\frac{x(t) - y_2(t)}{t} \right) = 0,$$

so that by (3.3)

$$\int_{y_2(t)}^{y_1(t)} u_0(y) dy + t g \circ a \circ u_0(y_1(t)) - t g \circ a \circ u_0(y_2(t)) = 0. \quad (3.4)$$

Hence we obtain a system of three equations for $x(\varepsilon) = x(t)$, $y_1(\varepsilon) = y_1(t)$, and $y_2(\varepsilon) = y_2(t)$:

$$x(\varepsilon) - y_1(\varepsilon) = (t^* + \varepsilon)a \circ u_0(y_1(\varepsilon)), \quad (3.5)$$

$$x(\varepsilon) - y_2(\varepsilon) = (t^* + \varepsilon)a \circ u_0(y_2(\varepsilon)), \quad (3.6)$$

$$\int_{y_2(\varepsilon)}^{y_1(\varepsilon)} u_0(y) dy + (t^* + \varepsilon)\{g \circ a \circ u_0(y_1(\varepsilon)) - g \circ a \circ u_0(y_2(\varepsilon))\} = 0. \quad (3.7)$$

LEMMA 3.2. For $t > t^*$, $x(t)$ is differentiable and (3.4) is equivalent to

$$\frac{dx(t)}{dt} = \frac{f(u_0(y_1(t))) - f(u_0(y_2(t)))}{u_0(y_1(t)) - u_0(y_2(t))}. \quad (3.8)$$

We shall prove this lemma in the Appendix.

By virtue of this lemma we may replace (3.7) by

$$\frac{dx(\varepsilon)}{d\varepsilon} = \frac{f(u_0(y_1(\varepsilon))) - f(u_0(y_2(\varepsilon)))}{u_0(y_1(\varepsilon)) - u_0(y_2(\varepsilon))}. \quad (3.9)$$

From the assumptions we write

$$\begin{aligned} u_0(y) &= \sum_{k=1}^N u_k(y - \tau)^k + O((y - \tau)^{N+1}) \quad (y \rightarrow \tau), \\ f(u) &= \sum_{k=2}^{N+1} f_k u^k + O(u^{N+2}) \quad (u \rightarrow 0), \\ a \circ u_0(y) &= \sum_{k=1}^N F_k(y - \tau)^k + O((y - \tau)^{N+1}) \quad (y \rightarrow \tau), \end{aligned} \quad (3.10)$$

where

$$u_k = \frac{u_0^{(k)}(\tau)}{k!}, \quad f_k = \frac{f^{(k)}(0)}{k!}, \quad \text{and} \quad F_k = \frac{(a \circ u_0)^{(k)}(\tau)}{k!}. \quad (3.11)$$

Note that

$$t^* = \frac{-1}{F_1}, \quad F_1 = 2f_2 u_1 < 0, \quad F_2 = 0, \quad F_3 > 0. \quad (3.12)$$

Now expanding $a(u)$, $u_0(y)$, $f(u)$ in (3.5), (3.6), and (3.9) by (3.10) we substitute (3.2) into them. Equating all the coefficients in each term of the same power of ε to zero, we obtain a recurrent system of equations for c_k , a_k , and b_k ($k \geq 1$). We seek them under the condition $a_1 < 0 < b_1$. We shall justify this condition in the remark at the end of this step. From (3.5) and (3.6) we obtain

$$\sum_{k \geq 1} (c_k - a_k) \varepsilon^{k/2} = (t^* + \varepsilon) \sum_{k=1}^N F_k \left(\sum_{l \geq 1} a_l \varepsilon^{l/2} \right)^k + O(\varepsilon^{(N+1)/2}), \quad (3.13)$$

$$\sum_{k \geq 1} (c_k - b_k) \varepsilon^{k/2} = (t^* + \varepsilon) \sum_{k=1}^N F_k \left(\sum_{l \geq 1} b_l \varepsilon^{l/2} \right)^k + O(\varepsilon^{(N+1)/2}). \quad (3.14)$$

Now substituting (3.2) into (3.9) and taking $a_1 < 0 < b_1$ into account we obtain

$$\begin{aligned}
 \sum_{k \geq 1} \frac{k}{2} c_k \varepsilon^{k/2-1} = & f_2 \left\{ \sum_{l=1}^N u_l \left(\sum_{m \geq 1} a_m \varepsilon^{m/2} \right)^l + \sum_{l=1}^N u_l \left(\sum_{m \geq 1} b_m \varepsilon^{m/2} \right)^l \right\} \\
 & + f_3 \left\{ \left[\sum_{l=1}^N u_l \left(\sum_{m \geq 1} a_m \varepsilon^{m/2} \right)^l \right]^2 + \sum_{l=1}^N u_l \left(\sum_{m \geq 1} a_m \varepsilon^{m/2} \right)^l \right. \\
 & \cdot \sum_{l=1}^N u_l \left(\sum_{m \geq 1} b_m \varepsilon^{m/2} \right)^l + \left. \left[\sum_{l=1}^N u_l \left(\sum_{m \geq 1} b_m \varepsilon^{m/2} \right)^l \right]^2 \right\} + \dots \\
 & + f_{N+1} \sum_{k=0}^N \left[\sum_{l=1}^N u_l \left(\sum_{m \geq 1} a_m \varepsilon^{m/2} \right)^l \right]^{N-k} \\
 & \cdot \left[\sum_{l=1}^N u_l \left(\sum_{m \geq 1} b_m \varepsilon^{m/2} \right)^l \right]^k + O(\varepsilon^{(N+1)/2}). \tag{3.15}
 \end{aligned}$$

Equating the terms of order $\varepsilon^{-1/2}$ and ε^0 in (3.15) to zero, we have

$$c_1 = 0, \quad c_2 = 0. \tag{3.16}$$

Equating the terms of order $\varepsilon^{1/2}$ and ε in (3.13) and (3.14) to zero, we obtain

$$c_1 - a_1 = t^* F_1 a_1, \quad c_1 - b_1 = t^* F_1 b_1$$

and

$$c_2 - a_2 = t^* (F_1 a_2 + F_2 a_1^2), \quad c_2 - b_2 = t^* (F_1 b_2 + F_2 b_1^2).$$

Combined with (3.12) and (3.16), these are automatically satisfied. Equating the terms of order $\varepsilon^{1/2}$ in (3.15) and the terms of order $\varepsilon^{3/2}$ in (3.13) and (3.14) to zero we have

$$\begin{aligned}
 \frac{3}{2} c_3 &= f_2 u_1 (a_1 + b_1), \\
 c_3 - a_3 &= t^* (F_1 a_3 + 2F_2 a_1 a_2 + F_3 a_1^3) + F_1 a_1, \\
 c_3 - b_3 &= t^* (F_1 b_3 + 2F_2 b_1 b_2 + F_3 b_1^3) + F_1 b_1.
 \end{aligned}$$

Combining with (3.12) we have a system of cubic equations for a_1 , b_1 , and c_3 :

$$3c_3 = F_1(a_1 + b_1), \quad c_3 = t^* F_3 a_1^3 + F_1 a_1, \quad c_3 = t^* F_3 b_1^3 + F_1 b_1.$$

Thus, we obtain the solution (a_1, b_1, c_3) such that $a_1 < 0 < b_1$:

$$c_3 = 0, \quad a_1 = - \left(\frac{F_1^2}{F_3} \right)^{1/2}, \quad b_1 = \left(\frac{F_1^2}{F_3} \right)^{1/2}. \tag{3.17}$$

Similarly, equating the terms of order ε in (3.15) and the terms of order ε^2 in (3.13) and (3.14) to zero, and combining with (3.12) we have a system of linear equations for a_2 , b_2 , and c_4 :

$$\begin{aligned} 2c_4 &= f_2(u_1 a_2 + u_2 a_1^2 + u_1 b_2 + u_2 b_1^2) + f_3 u_1^2 (a_1^2 + a_1 b_1 + b_1^2), \\ c_4 &= t^* (3F_3 a_1^2 a_2 + F_4 a_1^4) + F_1 a_2, \\ c_4 &= t^* (3F_3 b_1^2 b_2 + F_4 b_1^4) + F_1 b_2. \end{aligned}$$

Then from (3.17) we obtain

$$\begin{aligned} c_4 &= \frac{-1}{5} \frac{F_1^3 F_4}{F_3^2} - \frac{4}{5} f_3 u_1^2 \frac{F_1^2}{F_3}, \\ a_2 = b_2 &= \frac{2}{5} \left(f_3 u_1^2 \frac{F_1}{F_3} - \frac{F_1^2 F_4}{F_3^2} \right). \end{aligned}$$

The same argument can also be applied to determine a_k , b_k ($k \geq 3$) and c_k ($k \geq 5$) and we shall also prove

$$c_k = 0, \quad a_k + b_k = 0 \quad \text{for odd } k, \quad a_k = b_k \quad \text{for even } k \quad (3.18)$$

by induction.

Suppose that we have obtained a_k , b_k ($k \leq 2n$, $n \geq 1$) and c_k ($k \leq 2n+2$, $n \geq 1$) such that

$$\begin{aligned} a_{2k+1} + b_{2k+1} &= 0, \quad c_{2k+3} = 0 \quad (k \leq n-1, \quad n \geq 1), \\ a_{2k+2} &= b_{2k+2} \quad (k \leq n-1, \quad n \geq 1). \end{aligned} \quad (3.19)$$

Equating to zero the terms of order $\varepsilon^{(2n+1)/2}$ in (3.15) and the terms of order $\varepsilon^{(2n+3)/2}$ in (3.13) and (3.14) we obtain a system of linear equations for a_{2n+1} , b_{2n+1} , and c_{2n+3} :

$$\begin{aligned} \frac{2n+3}{2} c_{2n+3} &= f_2 u_1 (a_{2n+1} + b_{2n+1}) + S_{2n}, \\ c_{2n+3} &= t^* \cdot 3F_3 a_1^2 a_{2n+1} + F_1 a_{2n+1} + R_{2n}(a), \\ c_{2n+3} &= t^* \cdot 3F_3 b_1^2 b_{2n+1} + F_1 b_{2n+1} + R_{2n}(b), \end{aligned} \quad (3.20)$$

where

$$\begin{aligned} R_m(a) &= t^* \sum_{k=3}^{m+3} \sum_{\substack{i_1+i_2+\dots+i_m=k, \\ i_1+2i_2+\dots+mi_m=m+3, \quad i_j \geq 0}} \frac{k!}{i_1! i_2! \dots i_m!} F_k(a_1)^{i_1} (a_2)^{i_2} \dots (a_m)^{i_m} \\ &+ \sum_{k=3}^{m+1} \sum_{\substack{i_1+i_2+\dots+i_m=k, \\ i_1+2i_2+\dots+mi_m=m+1, \quad i_j \geq 0}} \frac{k!}{i_1! i_2! \dots i_m!} F_k(a_1)^{i_1} (a_2)^{i_2} \dots (a_m)^{i_m}, \end{aligned}$$

$$R_m(b) = t^* \sum_{k=3}^{m+2} \sum_{\substack{i_1+i_2+\dots+i_m=k, \\ i_1+2i_2+\dots+mi_m=m+3, \ i_j \geq 0}} \frac{k!}{i_1!i_2!\dots i_m!} F_k(b_1)^{i_1}(b_2)^{i_2}\dots(b_m)^{i_m} \\ + \sum_{k=3}^{m+1} \sum_{\substack{i_1+i_2+\dots+i_m=k, \\ i_1+2i_2+\dots+mi_m=m+1, \ i_j \geq 0}} \frac{k!}{i_1!i_2!\dots i_m!} F_k(b_1)^{i_1}(b_2)^{i_2}\dots(b_m)^{i_m},$$

$$S_m = \sum_{k=2}^{m+2} f_k \sum_{\substack{\alpha_1+\dots+\alpha_{k-1}=m+1 \\ \alpha_j \geq 1}} \sum_{l=0}^{k-1} P_a^m(\alpha_1) \dots P_a^m(\alpha_l) P_b^m(\alpha_{l+1}) \dots P_b^m(\alpha_{k-1}),$$

$$P_a^m(\alpha) = \sum_{k=1}^{\alpha} \sum_{\substack{\xi_1+\xi_2+\dots+\xi_{\alpha}=k, \\ \xi_1+2\xi_2+\dots+\alpha\xi_{\alpha}=\alpha, \ \xi_i \geq 0, \ \xi_{m+1}=0}} \frac{k!}{\xi_1!\xi_2!\dots\xi_{\alpha}!} u_k(a_1)^{\xi_1}(a_2)^{\xi_2}\dots(a_{\alpha})^{\xi_{\alpha}},$$

$$P_b^m(\alpha) = \sum_{k=1}^{\alpha} \sum_{\substack{\xi_1+\xi_2+\dots+\xi_{\alpha}=k, \\ \xi_1+2\xi_2+\dots+\alpha\xi_{\alpha}=\alpha, \ \xi_i \geq 0, \ \xi_{m+1}=0}} \frac{k!}{\xi_1!\xi_2!\dots\xi_{\alpha}!} u_k(b_1)^{\xi_1}(b_2)^{\xi_2}\dots(b_{\alpha})^{\xi_{\alpha}}.$$

Note that $R_m(a)$, $R_m(b)$, S_m , $P_a^m(\alpha)$, and $P_b^m(\alpha)$ are determined from a_k and b_k ($1 \leq k \leq m$).

From (3.19) we can easily check that $S_{2n} = 0$ and $R_{2n}(a) + R_{2n}(b) = 0$. Hence combining with (3.17) we have

$$c_{2n+3} = 0, \quad a_{2n+1} = \frac{1}{2F_1} R_{2n}(a), \quad b_{2n+1} = \frac{1}{2F_1} R_{2n}(b).$$

Similarly, equating to zero the terms of order $\varepsilon^{(2n+2)/2}$ in (3.15) and the terms of order $\varepsilon^{(2n+4)/2}$ in (3.13) and (3.14) we obtain

$$\begin{aligned} \frac{2n+4}{2} c_{2n+4} &= f_2 u_1(a_{2n+2} + b_{2n+2}) + S_{2n+1}, \\ c_{2n+4} &= t^* \cdot 3F_3 a_1^2 a_{2n+2} + F_1 a_{2n+2} + R_{2n+1}(a), \\ c_{2n+4} &= t^* \cdot 3F_3 b_1^2 b_{2n+2} + F_1 b_{2n+2} + R_{2n+1}(b). \end{aligned} \quad (3.21)$$

From (3.19) and $a_{2n+1} + b_{2n+1} = 0$ it follows that $R_{2n+1}(a) = R_{2n+1}(b)$. Therefore we obtain

$$c_{2n+4} = \frac{1}{2n+5} \{R_{2n+1}(a) + 2S_{2n+1}\},$$

$$a_{2n+2} = b_{2n+2} = \frac{1}{(2n+5)F_1} \{(n+2)R_{2n+1}(a) - S_{2n+1}\}.$$

This completes the induction.

REMARK. Suppose $a_1 = b_1 = 0$. Then from the second and the third equations in (3.20) and (3.21) it follows $a_k = b_k$, $k \geq 2$ inductively, which contradicts $y_1(t) < \tau < y_2(t)$.

(Second Step) Put

$$\begin{aligned} y_1^{N-2}(\varepsilon) &= \tau + \sum_{k=1}^{N-2} a_k \varepsilon^{k/2}, & y_2^{N-2}(\varepsilon) &= \tau + \sum_{k=1}^{N-2} b_k \varepsilon^{k/2}, \\ x^N(\varepsilon) &= \tau + \sum_{k=1}^N c_k \varepsilon^{k/2}. \end{aligned} \quad (3.22)$$

Let $y_i(\varepsilon)$ ($i = 1, 2$), $x(\varepsilon)$ be an exact solution to a system (3.5), (3.6), (3.7). In this step we shall apply the method of Berger and Fraenkel [1] to prove

$$\begin{aligned} \delta_i^{N-2}(\varepsilon) &= y_i(\varepsilon) - y_i^{N-2}(\varepsilon) = O(\varepsilon^{(N-1)/2}), & i &= 1, 2, \\ \rho^N(\varepsilon) &= x(\varepsilon) - x^N(\varepsilon) = O(\varepsilon^{(N+1)/2}). \end{aligned} \quad (3.23)$$

Now we define G_i ($i = 1, 2$) and H by

$$\begin{aligned} G_i(y_i, x) &= (t^* + \varepsilon)(a \circ u_0)(y_i) - x + y_i, & i &= 1, 2, \\ H(y_1, y_2) &= \int_{y_2}^{y_1} u_0(y) dy + (t^* + \varepsilon)\{g \circ a \circ u_0(y_1) - g \circ a \circ u_0(y_2)\}. \end{aligned} \quad (3.24)$$

From (3.13) and (3.14) it follows that

$$G_i(y_i^{N-2}(\varepsilon), x^N(\varepsilon)) = g_i^N(\varepsilon) \cdot \varepsilon^{(N+1)/2}, \quad i = 1, 2.$$

Moreover from (3.15), (3.20), (3.21), and Lemma 3.2 we can easily observe that

$$H(y_1^{N-2}(\varepsilon), y_2^{N-2}(\varepsilon)) = h^N(\varepsilon) \cdot \varepsilon^{(N+2)/2}.$$

Here $|g_i^N(\varepsilon)| \leq C_N$ ($i = 1, 2$) and $|h^N(\varepsilon)| \leq C_N$ for some constant $C_N > 0$ independent of ε .

By the mean value theorem there exist $R_i(\delta_i^{N-2}(\varepsilon), \rho^N(\varepsilon))$, $i = 1, 2$, and $S(\delta_1^{N-2}(\varepsilon), \delta_2^{N-2}(\varepsilon))$ such that

$$\begin{aligned} \begin{pmatrix} -g_1^N(\varepsilon) \cdot \varepsilon^{(N+1)/2} \\ -g_2^N(\varepsilon) \cdot \varepsilon^{(N+1)/2} \\ -h^N(\varepsilon) \cdot \varepsilon^{(N+2)/2} \end{pmatrix} &= \begin{pmatrix} G_1(y_1(\varepsilon), x(\varepsilon)) - G_1(y_1^{N-2}(\varepsilon), x^N(\varepsilon)) \\ G_2(y_2(\varepsilon), x(\varepsilon)) - G_2(y_2^{N-2}(\varepsilon), x^N(\varepsilon)) \\ H(y_1(\varepsilon), y_2(\varepsilon)) - H(y_1^{N-2}(\varepsilon), y_2^{N-2}(\varepsilon)) \end{pmatrix} \\ &= \begin{pmatrix} \partial G_1 / \partial y_1 & 0 & \partial G_1 / \partial x \\ 0 & \partial G_2 / \partial y_2 & \partial G_2 / \partial x \\ \partial H / \partial y_1 & \partial H / \partial y_2 & 0 \end{pmatrix} \begin{pmatrix} y_1^{N-2}(\varepsilon) \\ y_2^{N-2}(\varepsilon) \\ x^N(\varepsilon) \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} \delta_1^{N-2}(\varepsilon) \\ \delta_2^{N-2}(\varepsilon) \\ \rho^N(\varepsilon) \end{pmatrix} + \begin{pmatrix} R_1(\delta_1^{N-2}(\varepsilon), \rho^N(\varepsilon)) \\ R_2(\delta_2^{N-2}(\varepsilon), \rho^N(\varepsilon)) \\ S(\delta_1^{N-2}(\varepsilon), \delta_2^{N-2}(\varepsilon)) \end{pmatrix} \end{aligned} \quad (3.25)$$

and

$$\begin{aligned} |R_i(\delta_i^{N-2}(\varepsilon), \rho^N(\varepsilon))| &\leq C'_N \{(\delta_i^{N-2}(\varepsilon))^2 + (\rho^N(\varepsilon))^2\}, & i &= 1, 2, \\ |S(\delta_1^{N-2}(\varepsilon), \delta_2^{N-2}(\varepsilon))| &\leq C'_N \{(\delta_1^{N-2}(\varepsilon))^2 + (\delta_2^{N-2}(\varepsilon))^2\} \end{aligned} \quad (3.26)$$

for some constant $C'_N > 0$.

Now we put

$$J = \begin{pmatrix} \partial G_1/\partial y_1 & 0 & \partial G_1/\partial x \\ 0 & \partial G_2/\partial y_2 & \partial G_2/\partial x \\ \partial H/\partial y_1 & \partial H/\partial y_2 & 0 \end{pmatrix} (y_1^{N-2}(\varepsilon), y_2^{N-2}(\varepsilon), x^N(\varepsilon))$$

From (3.24) we can easily check that

$$J^{-1} = \frac{1}{\det J} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \quad (3.27)$$

where $\det J$, the determinant of the matrix J , is given by

$$\det J = \{u_0(y_1^{N-2}(\varepsilon)) - u_0(y_2^{N-2}(\varepsilon))\} \\ \cdot \{(t^* + \varepsilon)(a \circ u_0)'(y_1^{N-2}(\varepsilon)) + 1\} \{(t^* + \varepsilon)(a \circ u_0)'(y_2^{N-2}(\varepsilon)) + 1\}$$

and b_{ij} , the cofactor of J , is given by

$$\begin{aligned} b_{11} &= -u_0(y_2^{N-2}(\varepsilon))\{1 + (t^* + \varepsilon)(a \circ u_0)'(y_2^{N-2}(\varepsilon))\}, \\ b_{12} &= -u_0(y_1^{N-2}(\varepsilon))\{1 + (t^* + \varepsilon)(a \circ u_0)'(y_1^{N-2}(\varepsilon))\}, \\ b_{13} &= -u_0(y_1^{N-2}(\varepsilon))\{1 + (t^* + \varepsilon)(a \circ u_0)'(y_1^{N-2}(\varepsilon))\} \\ &\quad \cdot \{1 + (t^* + \varepsilon)(a \circ u_0)'(y_2^{N-2}(\varepsilon))\}, \\ b_{21} &= u_0(y_2^{N-2}(\varepsilon))\{1 + (t^* + \varepsilon)(a \circ u_0)'(y_2^{N-2}(\varepsilon))\}, \\ b_{22} &= u_0(y_1^{N-2}(\varepsilon))\{1 + (t^* + \varepsilon)(a \circ u_0)'(y_1^{N-2}(\varepsilon))\}, \\ b_{23} &= u_0(y_2^{N-2}(\varepsilon))\{1 + (t^* + \varepsilon)(a \circ u_0)'(y_2^{N-2}(\varepsilon))\} \\ &\quad \cdot \{1 + (t^* + \varepsilon)(a \circ u_0)'(y_1^{N-2}(\varepsilon))\}, \\ b_{31} &= 1 + (t^* + \varepsilon)(a \circ u_0)'(y_2^{N-2}(\varepsilon)), \\ b_{32} &= 1 + (t^* + \varepsilon)(a \circ u_0)'(y_1^{N-2}(\varepsilon)), \\ b_{33} &= \{1 + (t^* + \varepsilon)(a \circ u_0)'(y_1^{N-2}(\varepsilon))\}\{1 + (t^* + \varepsilon)(a \circ u_0)'(y_2^{N-2}(\varepsilon))\}. \end{aligned}$$

Now we shall prove

$$\det J \geq C\varepsilon^{5/2} \quad \text{for some constant } C > 0. \quad (3.28)$$

From (3.10), (3.12), (3.17), and (3.22) it follows that

$$u_0(y_1^{N-2}(\varepsilon)) - u_0(y_2^{N-2}(\varepsilon)) = u_1(a_1 - b_1)\varepsilon^{1/2} + O(\varepsilon),$$

$$\begin{aligned} (t^* + \varepsilon)(a \circ u_0)'(y_1^{N-2}(\varepsilon)) + 1 &= \left(\frac{-1}{F_1} + \varepsilon \right) \{F_1 + 3F_3a_1^2\varepsilon + O(\varepsilon^{3/2})\} + 1 \\ &= -2F_1\varepsilon + O(\varepsilon^{3/2}), \end{aligned}$$

and similarly

$$(t^* + \varepsilon)(a \circ u_0)'(y_2^{N-2}(\varepsilon)) + 1 = -2F_1\varepsilon + O(\varepsilon^{3/2}).$$

This proves (3.28).

In the same way we obtain

$$b_{ij} = O(\varepsilon^{3/2}), \quad i, j = 1, 2, \quad b_{i3} = O(\varepsilon^{5/2}), \quad i = 1, 2, \\ b_{3j} = O(\varepsilon), \quad j = 1, 2, \quad b_{33} = O(\varepsilon^2).$$

Hence substituting them and (3.28) into (3.27) we have

$$J^{-1} = \begin{pmatrix} O(\varepsilon^{-1}) & O(\varepsilon^{-1}) & O(\varepsilon^{-3/2}) \\ O(\varepsilon^{-1}) & O(\varepsilon^{-1}) & O(\varepsilon^{-3/2}) \\ O(1) & O(1) & O(\varepsilon^{-1/2}) \end{pmatrix}. \quad (3.29)$$

Multiplying (3.25) with J^{-1} , and taking (3.29) into account leads to

$$\begin{pmatrix} \delta_1^{N-2}(\varepsilon) \\ \delta_2^{N-2}(\varepsilon) \\ \rho^N(\varepsilon) \end{pmatrix} = \begin{pmatrix} \tilde{g}_1^N(\varepsilon) \cdot \varepsilon^{(N-1)/2} \\ \tilde{g}_2^N(\varepsilon) \cdot \varepsilon^{(N-1)/2} \\ \tilde{h}^N(\varepsilon) \cdot \varepsilon^{(N+1)/2} \end{pmatrix} - J^{-1} \begin{pmatrix} R_1(\delta_1^{N-2}(\varepsilon), \rho^N(\varepsilon)) \\ R_2(\delta_2^{N-2}(\varepsilon), \rho^N(\varepsilon)) \\ S(\delta_1^{N-2}(\varepsilon), \delta_2^{N-2}(\varepsilon)) \end{pmatrix},$$

where $|\tilde{g}_i^N(\varepsilon)| \leq \tilde{C}_N$ ($i = 1, 2$) and $|\tilde{h}^N(\varepsilon)| \leq \tilde{C}_N$ for some constant $\tilde{C}_N > 0$ independent of ε .

Now we define the map $T: \mathbf{R}^3 \rightarrow \mathbf{R}^3$ by

$$T \begin{pmatrix} \delta_1 \\ \delta_2 \\ \rho \end{pmatrix} = \begin{pmatrix} \tilde{g}_1^N(\varepsilon) \cdot \varepsilon^{(N-1)/2} \\ \tilde{g}_2^N(\varepsilon) \cdot \varepsilon^{(N-1)/2} \\ \tilde{h}^N(\varepsilon) \cdot \varepsilon^{(N+1)/2} \end{pmatrix} - J^{-1} \begin{pmatrix} R_1(\delta_1, \rho) \\ R_2(\delta_2, \rho) \\ S(\delta_1, \delta_2) \end{pmatrix} \quad (3.30)$$

and the closed subset S of $\mathbf{R}^3$ by

$$S = \left\{ \begin{pmatrix} \delta_1 \\ \delta_2 \\ \rho \end{pmatrix}; |\delta_i| \leq K\varepsilon^{(N-1)/2} \ (i = 1, 2), \ |\rho| \leq K\varepsilon^{(N+1)/2} \right\},$$

where K is a constant such that

$$K > \tilde{C}_N + 1. \quad (3.31)$$

To prove (3.23) it is sufficient to show that T is a contraction map of S into itself. From (3.26) and (3.29) it follows that

$$J^{-1} \begin{pmatrix} R_1(\delta_1, \rho) \\ R_2(\delta_2, \rho) \\ S(\delta_1, \delta_2) \end{pmatrix} = \begin{pmatrix} O(\varepsilon^{N-1-3/2}) \\ O(\varepsilon^{N-1-3/2}) \\ O(\varepsilon^{N-1-1/2}) \end{pmatrix} \quad \text{for any } \begin{pmatrix} \delta_1 \\ \delta_2 \\ \rho \end{pmatrix} \in S.$$

Since $N \geq 5$ we have $N - 1 - \frac{3}{2} > \frac{N-1}{2}$, $N - 1 - \frac{1}{2} > \frac{N+1}{2}$. Hence for sufficiently small ε and with K given by (3.31) T maps S into itself.

Now we shall prove T is a contraction map on S . From (3.30)

$$T \begin{pmatrix} \delta_1 \\ \delta_2 \\ \rho \end{pmatrix} - T \begin{pmatrix} \delta'_1 \\ \delta'_2 \\ \rho' \end{pmatrix} = J^{-1} \begin{pmatrix} R_1(\delta'_1, \rho') - R_1(\delta_1, \rho) \\ R_2(\delta'_2, \rho') - R_2(\delta_2, \rho) \\ S(\delta'_1, \delta'_2) - S(\delta_1, \delta_2) \end{pmatrix} \quad \text{for } \begin{pmatrix} \delta_1 \\ \delta_2 \\ \rho \end{pmatrix}, \begin{pmatrix} \delta'_1 \\ \delta'_2 \\ \rho' \end{pmatrix} \in S. \quad (3.32)$$

On the other hand, (3.25) implies

$$|R_i(\delta'_i, \rho') - R_i(\delta_i, \rho)| \leq C(|\delta_i| + |\delta'_i| + |\rho| + |\rho'|)(|\delta'_i - \delta_i| + |\rho' - \rho|), \quad i = 1, 2,$$

and

$$|S(\delta'_1, \delta'_2) - S(\delta_1, \delta_2)| \leq C(|\delta_1| + |\delta_2| + |\delta'_1| + |\delta'_2|)(|\delta'_1 - \delta_1| + |\delta'_2 - \delta_2|)$$

for some constant C . Thus,

$$|R_i(\delta'_i, \rho') - R_i(\delta_i, \rho)| \leq 4CK\varepsilon^{(N-1)/2}(|\delta'_i - \delta_i| + |\rho' - \rho|), \quad i = 1, 2,$$

and

$$|S(\delta'_1, \delta'_2) - S(\delta_1, \delta_2)| \leq 4CK\varepsilon^{(N-1)/2}(|\delta'_1 - \delta_1| + |\delta'_2 - \delta_2|).$$

Substituting them into (3.32) and combining with (3.29) we have

$$\left\| T \begin{pmatrix} \delta_1 \\ \delta_2 \\ \rho \end{pmatrix} - T \begin{pmatrix} \delta'_1 \\ \delta'_2 \\ \rho' \end{pmatrix} \right\| = O(\varepsilon^{(N-1-3)/2}) \left\| \begin{pmatrix} \delta_1 \\ \delta_2 \\ \rho \end{pmatrix} - \begin{pmatrix} \delta'_1 \\ \delta'_2 \\ \rho' \end{pmatrix} \right\|,$$

where $\|\cdot\|$ denotes the usual $\mathbf{R}^3$ norm. Since $N \geq 5$, for sufficiently small ε , T is a contraction map on S .

(Third Step) From (3.10), (3.22), and (3.23) it follows that

$$\begin{aligned} u_0(y_1(\varepsilon)) &= \sum_{k=1}^N u_k \left(\sum_{m=1}^{N-2} a_m \varepsilon^{m/2} \right)^k + O(\varepsilon^{(N-1)/2}) \\ &= \sum_{k=1}^{N-2} A_k \varepsilon^{k/2} + O(\varepsilon^{(N-1)/2}), \end{aligned}$$

where

$$A_1 = u_1 a_1, \quad A_2 = u_1 a_2 + u_2 a_1^2, \dots,$$

$$A_k = \sum_{m=1}^k \sum_{\substack{i_1+i_2+\dots+i_k=m, \\ i_1+2i_2+\dots+ki_k=k, \quad i_j \geq 0}} \frac{m!}{i_1! i_2! \dots i_k!} u_m (a_1)^{i_1} (a_2)^{i_2} \dots (a_k)^{i_k}.$$

Similarly,

$$u_0(y_2(\varepsilon)) = \sum_{k=1}^{N-2} B_k \varepsilon^{k/2} + O(\varepsilon^{(N-1)/2}),$$

where

$$B_1 = u_1 b_1, \quad B_2 = u_1 b_2 + u_2 b_1^2, \dots,$$

$$B_k = \sum_{m=1}^k \sum_{\substack{i_1+i_2+\dots+i_k=m, \\ i_1+2i_2+\dots+ki_k=k, \quad i_j \geq 0}} \frac{m!}{i_1! i_2! \dots i_k!} u_m (b_1)^{i_1} (b_2)^{i_2} \dots (b_k)^{i_k}.$$

From (3.18) we can easily observe that $A_k + B_k = 0$ for odd k and $A_k = B_k$ for even k . Moreover from Theorem 2.1 we have

$$u(x(\varepsilon) - 0, t) = u_0(y_1(\varepsilon)), \quad u(x(\varepsilon) + 0, t) = u_0(y_2(\varepsilon)).$$

Finally, recalling that we have replaced $u(x, t)$ by $u(x + f'(u_0(\tau))t, t) - u_0(\tau)$ at the beginning of the proof, we obtain the result. Q.E.D.

4. Asymptotic formulas at a shock interaction point and at time infinity. In this section we shall assume that $f(0) = a(0) = 0$.

A. Suppose that the two graphs $z = u_0(y)$ and $z = b((x_c - y)/t_c)$ intersect at the $n + 1$ points $y = p_1, p_2, \dots, p_{n+1}$ ($p_1 < p_2 < \dots < p_{n+1}$, $n \geq 2$) as depicted in Fig. 5. Let D_i and $\tilde{D}_i$ be the left and right regions surrounded by the two graphs on $y \in [p_i, p_{i+1}]$, respectively. Moreover, we suppose that $|D_i| = |\tilde{D}_i|$ for all i ($1 \leq i \leq n$), where $|D|$ denotes the area of D .

THEOREM 4.1. Let $u_0(y) \in C^{N+1}(\mathbf{R})$ and $f(u) \in C^{N+2}(\mathbf{R})$ ($N \geq 1$).

(1) In a neighborhood of $(x, t) = (x_c, t_c)$ n shock curves $x_1(t), x_2(t), \dots, x_n(t)$ ($t \leq t_c$) collide at the one point (x_c, t_c) and give rise to one shock curve $x(t)$ ($t \geq t_c$).

(2) Each shock curve and the solution admit the following asymptotic expansions at the shocks interaction time, t_c :

$$x_i(t) = x_c + \sum_{k=1}^N c_k^i (t - t_c)^k + O((t - t_c)^{N+1}),$$

$$u(x_i(t) - 0, t) = u_0(p_i) + \sum_{k=1}^N A_k^i (t - t_c)^k + O((t - t_c)^{N+1}),$$

$$u(x_i(t) + 0, t) = u_0(p_{i+1}) + \sum_{k=1}^N B_k^i (t - t_c)^k + O((t - t_c)^{N+1}),$$

$$t \rightarrow t_c - 0, \quad 1 \leq i \leq n.$$

FIG. 5

FIG. 6

The coefficients c_k^i , A_k^i , B_k^i can be determined successively. In particular,

$$c_1^i = \frac{f \circ u_0(p_{i+1}) - f \circ u_0(p_i)}{u_0(p_{i+1}) - u_0(p_i)}, \quad A_1^i = \frac{-u_0'(p_i)\{f'(u_0(p_i)) - c_1^i\}}{t_c f^{(2)}(u_0(p_i))u_0'(p_i) + 1},$$

$$B_1^i = \frac{-u_0'(p_{i+1})\{f'(u_0(p_{i+1})) - c_1^i\}}{t_c f^{(2)}(u_0(p_{i+1}))u_0'(p_{i+1}) + 1},$$

$$x(t) = x_c + \sum_{k=1}^N c_k(t - t_c)^k + O((t - t_c)^{N+1}),$$

$$u(x(t) - 0, t) = u_0(p_1) + \sum_{k=1}^N A_k(t - t_c)^k + O((t - t_c)^{N+1}),$$

$$u(x(t) + 0, t) = u_0(p_{n+1}) + \sum_{k=1}^N B_k(t - t_c)^k + O((t - t_c)^{N+1}),$$

$$t \rightarrow t_c + 0.$$

In particular,

$$c_1 = \frac{f \circ u_0(p_{n+1}) - f \circ u_0(p_1)}{u_0(p_{n+1}) - u_0(p_1)}, \quad A_1 = \frac{-u_0'(p_1)\{f'(u_0(p_1)) - c_1\}}{t_c f^{(2)}(u_0(p_1))u_0'(p_1) + 1},$$

$$B_1 = \frac{-u_0'(p_{n+1})\{f'(u_0(p_{n+1})) - c_1\}}{t_c f^{(2)}(u_0(p_{n+1}))u_0'(p_{n+1}) + 1}.$$

REMARK. 1. c_1^i ($1 \leq i \leq n$), c_1 , the first derivative of each shock curve at $t = t_c$, is equal to the slope of each chord indicated in Fig. 6.

2. In a neighborhood of $t = t_c$ the graphs $u = u(x_i(t) \pm 0, t)$, $t \leq t_c$ ($1 \leq i \leq n$), and $u = u(x(t) \pm 0, t)$, $t_c \leq t$, in the (t, u) plane can be depicted as in Fig. 7.

Proof. (1) is a simple extension of Theorem 2.1. Now we shall give the outline of the proof of (2). Let t be sufficiently close to t_c and $t < t_c$. Suppose that $y_i(t)$,

FIG. 7

$y_{i+1}(t)$, and $x_i(t)$ are sufficiently close to p_i , p_{i+1} , and x_c , respectively, and satisfy

$$\int_{y_i(t)}^{y_{i+1}(t)} \left\{ u_0(y) - b \left(\frac{x_i(t) - y}{t} \right) \right\} dy = 0.$$

$$b \left(\frac{x_i(t) - y_i(t)}{t} \right) = u_0(y_i(t)), \quad b \left(\frac{x_i(t) - y_{i+1}(t)}{t} \right) = u_0(y_{i+1}(t)). \quad (4.1)$$

Then $x_i(t)$ ($t < t_c$) is a shock curve and

$$u(x_i(t) - 0, t) = u_0(y_i(t)), \quad u(x_i(t) + 0, t) = u_0(y_{i+1}(t)).$$

Now we seek the asymptotic expansions of $x_i(t)$, $y_i(t)$, and $y_{i+1}(t)$ in the following forms:

$$x_i(t) \sim x_c + \sum_{k \geq 1} c_k \varepsilon^k,$$

$$y_i(t) \sim p_i + \sum_{k \geq 1} a_k \varepsilon^k, \quad y_{i+1}(t) \sim p_{i+1} + \sum_{k \geq 1} b_k \varepsilon^k, \quad \varepsilon = t - t_c.$$

Then we can apply a similar procedure as in the proof of Theorem 3.1 to determine the coefficients a_k , b_k , and c_k successively. Indeed, from (4.1) we obtain a system (3.5), (3.6), and (3.7) with t^* replaced by t_c . Note that this time the Jacobian of this system does not vanish at $\varepsilon = 0$, so the proof is simpler than that of Theorem 3.1. For $t > t_c$ the same argument can also be applied to obtain the asymptotic formulas for $x(t)$ and $u(x(t) \pm 0, t)$. Q.E.D.

FIG. 8

B. Finally, we shall give the asymptotic formula of the shock wave for large time in the case where the initial data u_0 has compact support. In this case the asymptotic behavior of the shock wave for large time has been studied by several authors (see Dafermos [2] and [4, 6, 7]). Here, we shall apply our asymptotic method to the case where u_0 is a function such that $\text{Supp } u_0(y) = [c, d]$ and $u_0(y) > 0$ for $y \in (c, d)$ as in Fig. 8. Put

$$\int_c^d u_0(y) dy = M. \quad (4.2)$$

THEOREM 4.2. Let $u_0(y) \in C^{N+1}([c, d])$, $f(u) \in C^{N+2}(\mathbf{R})$, and $u'_0(c+0) > 0$ ($N \geq 1$). Then for large time the solution contains precisely one shock curve $x(t)$ and admits the following asymptotic formulas as $t \rightarrow +\infty$:

$$\begin{aligned} x(t) &= \sum_{k=1}^N c_k t^{1-k/2} + O(t^{-(N-1)/2}), \\ u(x(t) - 0, t) &= \sum_{k=1}^N A_k t^{-k/2} + O(t^{-(N+1)/2}), \\ u(x(t) + 0, t) &= 0. \end{aligned}$$

The coefficients c_k , A_k ($k \geq 1$) can be determined successively. In particular,

$$\begin{aligned} c_1 &= \{2f^{(2)}M\}^{1/2}, & c_2 &= c + \frac{f^{(3)}M}{3f^{(2)}}, \\ A_1 &= \left(\frac{2M}{f^{(2)}}\right)^{1/2}, & A_2 &= \frac{-2f^{(3)}}{3(f^{(2)})^2}, \end{aligned}$$

where $f^{(i)} = f^{(i)}(0)$.

Proof. Let t be sufficiently large and let the two graphs $z = u_0(y)$ and $z = b((x-y)/t)$ intersect at the three points $y = y_1(x, t)$, $\bar{y}(x, t)$, x ($y_1(x, t) < \bar{y}(x, t) < x$) (see Fig. 9). Let D_1 and D_2 be the left and right regions surrounded by the two graphs, respectively. According to Theorem 2.1, the shock curve $x(t)$ satisfies $|D_1| = |D_2|$. Combining with (4.2) we have

$$\int_c^{y_1(t)} u_0(y) dy + \int_{y_1(t)}^{x(t)} b\left(\frac{x(t)-y}{t}\right) dy = M,$$

FIG. 9

where $y_1(t) = y_1(x(t), t)$.

Since

$$u_0(y_1(t)) = b \left(\frac{x(t) - y_1(t)}{t} \right), \quad (4.3)$$

we have

$$\int_c^{y_1(t)} u_0(y) dy + t \{ a \circ u_0(y_1(t)) u_0(y_1(t)) - f \circ u_0(y_1(t)) \} = M.$$

Then

$$\varepsilon \int_c^{y_1(\varepsilon)} u_0(y) dy + a \circ u_0(y_1(\varepsilon)) u_0(y_1(\varepsilon)) - f \circ u_0(y_1(\varepsilon)) - \varepsilon M = 0, \quad \varepsilon = \frac{1}{t}. \quad (4.4)$$

Now we seek the asymptotic expansion of $y_1(t)$ in the following form:

$$y_1(t) = y_1(\varepsilon) \sim c + \sum_{k \geq 1} a_k \varepsilon^{k/2}. \quad (4.5)$$

Expanding $u_0(y)$, $a \circ u_0(y)$, and $f \circ u_0(y)$ in (4.4) at $y = c$ we substitute (4.5) into (4.4). Equating all the coefficients in each term of the same order of ε to zero we can determine the coefficients a_k successively. Note that although the first derivative of the left-hand side of (4.4) with respect to y_1 vanishes at $\varepsilon = 0$ we can justify the above expansion by applying the contraction mapping theorem as in the proof of Theorem 3.1. Finally, from (4.3) and

$$u(x(t) - 0, t) = u_0(y_1(t)), \quad u(x(t) + 0, t) = u_0(x(t)) = 0$$

we obtain the result. Q.E.D.

REMARK. For more general initial data, [2, 6] showed that $y(x(t), t)$, the minimizing value for $G(x(t), \cdot, t)$ (see (1.4)), tends to $x(t)$ or c as $t \rightarrow +\infty$. Here, c is assumed to be the unique point where $\int_{-\infty}^y u_0(z) dz$ takes its minimum and $u'_0(c) > 0$. Then combining with the proof of Theorem 4.2 we can easily obtain a similar result as Theorem 4.2.

A. Appendix.

Proof of Lemma 3.2. The differentiability of $y_1(x, t)$, $y_2(x, t)$, and $x(t)$ is easily proved by applying the implicit function theorem to (2.2) and (3.1). Now we shall

prove (3.8). Differentiating (3.4) gives

$$\begin{aligned} & u_0(y_1(t)) \frac{dy_1(t)}{dt} + a \circ u_0(y_1(t)) u_0(y_1(t)) - f(u_0(y_1(t))) \\ & + t u_0(y_1(t)) (a \circ u_0)'(y_1(t)) \frac{dy_1(t)}{dt} \\ & - \left(u_0(y_2(t)) \frac{dy_2(t)}{dt} + a \circ u_0(y_2(t)) u_0(y_2(t)) - f(u_0(y_2(t))) \right. \\ & \left. + t u_0(y_2(t)) (a \circ u_0)'(y_2(t)) \frac{dy_2(t)}{dt} \right) = 0. \end{aligned}$$

Here we have used the equality $g \circ a(u) = a(u)u - f(u)$. Then combining with (3.3) and

$$\begin{aligned} (a \circ u_0)'(y_i(t)) \frac{dy_i(t)}{dt} &= \frac{-(x(t) - y_i(t))}{t^2} \\ &+ \frac{1}{t} \left(\frac{dx(t)}{dt} - \frac{dy_i(t)}{dt} \right), \quad i = 1, 2, \end{aligned}$$

we obtain (3.8).

Conversely, (3.8) implies

$$\int_{y_2(t)}^{y_1(t)} u_0(y) dy + t g \circ a \circ u_0(y_1(t)) - t g \circ a \circ u_0(y_2(t)) = C,$$

where C is a constant independent of t . By letting $t \rightarrow t^* + 0$ we have $y_1(t) \rightarrow \tau$, $y_2(t) \rightarrow \tau$. Therefore, we obtain $C = 0$. Q.E.D.

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