THE INFLUENCE OF NONLOCAL NONLINEARITIES ON THE LONG TIME BEHAVIOR OF SOLUTIONS OF BURGERS'S EQUATION

By

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Abstract. We study the long time behavior of solutions of Burgers's equation with nonlocal nonlinearities: $u_t = u_{xx} + \varepsilon uu_x + \frac{1}{2}(a\|u(\cdot, t)\|^{p-1} + b)u$, $0 < x < 1$, $a, \varepsilon \in \mathbb{R}$, $b > 0$, $p > 1$, subject to $u(0, t) = u(1, t) = 0$. A stability-instability analysis is given in some detail, and some finite time blow up results are given.

1. Introduction. In this paper, we consider the following initial-boundary value problem:

$$\begin{aligned} u_t &= u_{xx} + \varepsilon uu_x + \frac{1}{2}(a\|u(\cdot, t)\|^{p-1} + b)u & 0 < x < 1, \quad t > 0; \\ u(0, t) &= u(1, t) = 0 & t > 0; \\ u(x, 0) &= u_0(x) & 0 \leq x \leq 1. \end{aligned} \tag{P}$$

Here a, b, ε, p are given constants with $\varepsilon \geq 0$ (without loss of generality), $b > 0$, $p > 1$, and $u_0(\cdot)$ is a continuous function with $u_0(0) = u_0(1) = 0$. Moreover, we take

$$\|u(\cdot, t)\| = \left(\int_0^1 |u(x, t)|^q dx \right)^{1/q}.$$

If, as we will do in the sequel, we require that $u_0 \geq 0$, then, by the maximum principle $u(\cdot, t) \geq 0$ for all t in the existence interval.

Our interest in (P) is twofold. First, it is closely related to a one-dimensional turbulence model proposed by Burgers [2, 3] and studied by Horgan and Olmstead [8]. (See also Drazin and Reid [5].) The major difference between (P) and the model considered earlier is the use of the L^q norm in (P) rather than the L^2 norm. A number of authors [1, 6, 8, 12] have investigated nonlocal problems as models for

Received October 5, 1990.

The research of Deng and Levine was supported by NSF Grant DMS-8822788, that of Kwong by Argonne National Laboratory.

local problems. They also restricted their attention to the case $q = 2$. To the best of our knowledge, no one has done this for problems in which a convective term (εuu_x) is present. However, convective terms have a remarkable effect on the dynamical behavior of solutions of equations. For example, consider

$$\begin{aligned} u_t &= u_{xx} + \varepsilon uu_x + \frac{1}{2}bu & 0 < x < 1, \quad t > 0; \\ u(0, t) &= u(1, t) = 0 & t > 0; \\ u(x, 0) &= u_0(x) \geq 0 & 0 \leq x \leq 1. \end{aligned}$$

When $\varepsilon = 0$ and $b > 2\pi^2$, this problem possesses the exponentially growing solution $e^{(b/2-\pi^2)t} \sin \pi x$, whereas for $\varepsilon > 0$, all solutions are bounded for any b (since a super solution of the form $M(1 - \alpha x)$ with large M and $\alpha \in (0, 1)$ exists). We shall see that this phenomenon persists, although to a somewhat less pronounced effect, for (P).

Secondly, (P) (with $b = 0$) is closely related to the same initial-boundary value problem for the equation

$$u_t = u_{xx} + \varepsilon uu_x + a|u|^{p-1}u,$$

where now the nonlocal nonlinearity is replaced by the more standard local term. This problem was studied extensively in [4, 11], where the stabilizing effect of convective terms was noted; however, the results there are somewhat incomplete. Therefore, the study of (P) was taken up in the hope that more complete results could be obtained for a closely related problem.

In [14] the authors consider, from a computational point of view, the same initial-boundary value problem for

$$Lu = u_t + 2uu_x - R^{-1}u_{xx},$$

where

$$Lu = \begin{cases} u - Ru\|u\|^2, & \text{or} \\ u, & \text{or} \\ u + Ru\|u\|, \end{cases}$$

and where $\|\cdot\|$ denotes the L^2 norm, and where $R > 0$ may be thought of as the Reynolds number. Under appropriate scaling, this problem is included in (P) when $q = 2$. In particular, we are able to verify theoretically for (P) all (except one) of the numerical observations made in [14].

The plan of the paper is as follows: In the next section we state the results in the absence of convection. In Sec. 3 we characterize the set of stationary solutions of (P) when $\varepsilon > 0$. In Sec. 4 we discuss the long time behavior of solutions of (P) (stability, asymptotic stability, global existence, and nonexistence, etc.). The necessary local existence theorems and comparison theorems are discussed in the appendix. The arguments are more or less standard, but we could not find any reference to such results for nonlocal problems of the type considered here.

2. Discussion for the case $\varepsilon = 0$. In this section, we consider the following problem:

$$\begin{aligned} u_t &= u_{xx} + \frac{1}{2}(a\|u(\cdot, t)\|^{p-1} + b)u & 0 < x < 1, \quad t > 0; \\ u(0, t) &= u(1, t) = 0 & t > 0; \\ u(x, 0) &= u_0(x) & 0 \leq x \leq 1. \end{aligned} \quad (\mathbf{P}^0)$$

Because $(\mathbf{P}^0)$ is an explicitly resolvable problem, we can easily examine the questions of stability, global existence, and nonexistence of nonnegative solutions.

First, for the stationary solutions of $(\mathbf{P}^0)$, we need to solve

$$\begin{aligned} v'' + \frac{1}{2}(a\|v\|^{p-1} + b)v &= 0 & 0 < x < 1, \\ v(0) &= v(1) = 0. \end{aligned} \quad (\mathbf{S}^0)$$

The nonnegative solutions of $(\mathbf{S}^0)$ must be in the form $v(x) = c_1 \varphi_1(x) = c_1 \sin \pi x$. Substituting such v in the equation, we find

$$-\pi^2 + \frac{1}{2}(ac_1^{p-1}\|\varphi_1\|^{p-1} + b) = 0. \quad (2.1)$$

Thus for $a > 0$, there is no positive solution for $b \geq 2\pi^2$ and one positive solution for $b < 2\pi^2$. For $a < 0$, there exists a unique positive solution for $b > 2\pi^2$ while there are none for $b \leq 2\pi^2$.

Next for $(\mathbf{P}^0)$, we seek solutions of the form

$$u(x, t) = \sum_{n=1}^{\infty} a_n(t) \varphi_n(x), \quad (2.2)$$

where $\varphi_n(x) = \sin n\pi x$. Let

$$f(t) = \|u(\cdot, t)\|^{p-1} = \left(\int_0^1 \left| \sum_{n=1}^{\infty} a_n(t) \varphi_n(x) \right|^q dx \right)^{(p-1)/q}.$$

Through a straightforward computation, we have

$$f(t) = (h(t))^{p-1} \exp \left(\frac{a(p-1)}{2} \int_0^t f(\eta) d\eta \right), \quad (2.3)$$

where

$$h(t) = e^{bt/2} \left(\int_0^1 \left| \sum_{n=1}^{\infty} a_n(0) e^{-n^2 \pi^2 t} \varphi_n(x) \right|^q dx \right)^{1/q}.$$

From (2.3), after a quadrature, we can rewrite $f(t)$,

$$f(t) = (h(t))^{p-1} \left/ \left[1 - \frac{a(p-1)}{2} \int_0^t (h(\eta))^{p-1} d\eta \right] \right. \quad (2.4)$$

Let

$$H(t) = \int_0^t h^{p-1}(\eta) d\eta. \quad (2.5)$$

If $b < 2\pi^2$, it follows that $H(t) \rightarrow H_{\infty} < \infty$ for some constant H_{∞} , which is proportional to $|a_N(0)|$ where N is the smallest integer such that $a_N(0) \neq 0$. Thus

if $a(p-1)H_\infty < 2$, which will be the case if $a_N(0)$ is sufficiently small, then $f(t) \rightarrow 0$ as $t \rightarrow \infty$ and the trivial solution is stable in L^q .

If $a(p-1)H_\infty > 2$ then clearly $f(t) \rightarrow \infty$ in finite time so some solutions of (P^0) blow up in finite time. If $a(p-1)H_\infty = 2$ then $f(t)$ can have a finite, nonzero limit as $t \rightarrow \infty$. This happens, for example, when u is a stationary solution of (P^0) .

In order to show that $v(x)$ is unstable, we look for solutions of (P^0) of the form

$$u(x, t) = a_1(t)\varphi_1(x) = \alpha(t)v(x),$$

where $\alpha(t) = a_1(t)/c_1$ and $\alpha(0) \neq 1$. Then, using (S^0) we see that

$$\alpha'(t) = c^2(\alpha^{p-1}(t) - 1)\alpha(t),$$

where $c^2 = \frac{1}{2}a\|v\|_q^{p-1}$. If $\alpha(0) < 1$, $\alpha'(t) < 0$ for small $t > 0$ and consequently for all t (since $p > 1$). Therefore $\alpha(t) < \alpha(0)$ and $\alpha'(t) < -c^2(1 - \alpha^{p-1}(0))\alpha(t)$ so that $\alpha(t) \rightarrow 0$ as $t \rightarrow \infty$. If $\alpha(0) > 1$, then it is easy to see that $\alpha' > 0$ for all t for which $\alpha(t)$ exists. It then follows that $\alpha(t)$ blows up in finite time. Thus for any $r > 1$

$$\liminf_{t \rightarrow \infty} \|u(\cdot, t) - v\|_r > 0;$$

and consequently v is not stable (in any norm).

If $b \geq 2\pi^2$ then zero is unstable. To see this, we note that if $a_1(0) \neq 0$, then, in view of the asymptotic behavior of $h(t)$ near $+\infty$, $H(t)$ grows either exponentially or linearly so that it passes $2/(a(p-1))$ in finite time. Consequently, whenever $(u_0, \varphi_1) \neq 0$, solutions of (P^0) blow up in finite time.

For the case $a < 0$, we again have subcases. If $b < 2\pi^2$ then zero is the only nonnegative stationary solution. From the form of $h(t)$ and the fact that $a < 0$, we see that $h(t) \rightarrow 0$ as $t \rightarrow \infty$ and consequently $\|u(\cdot, t)\| \rightarrow 0$ also.

If $b = 2\pi^2$ then $h(t) \rightarrow |a_1(0)|\|\varphi_1\|$. If $|a_1(0)| = 0$, then from (2.4) since $a < 0$, we have $f(t) \rightarrow 0$ as $t \rightarrow \infty$. If $|a_1(0)| \neq 0$, then $H(t)$ behaves like $(|a_1(0)|\|\varphi_1\|)^{p-1}t$ so that $f(t) \rightarrow 0$ in this case too. Therefore zero is globally asymptotically stable in this case also.

If $b > 2\pi^2$, we claim zero is unstable and the positive stationary solution, $v(x)$, is stable. For the first of these, let $u(x, t) = \varepsilon(t)v(x)$. Then, as before,

$$\varepsilon'(t) = c^2\varepsilon(t)(1 - \varepsilon^{p-1}(t)),$$

where $c^2 = |a|\|v\|^{p-1}$. Thus, if $0 < \varepsilon(0) < 1$, $\varepsilon' > 0$ for small t , and hence for all t for which $\varepsilon(0) \leq \varepsilon(t) \leq 1$. It follows that if $(0, T)$ is the largest interval on which $\varepsilon'(t) > 0$, then $\varepsilon(t) \rightarrow 1^-$ as $t \nearrow T$ and $T = +\infty$. This establishes the instability of the null solution.

In order to show that $v(x)$ is stable for $a < 0$ and $b > 2\pi^2$, we write

$$u(x, t) = a_1(t)\varphi_1(x) + \sum_{n=2}^{\infty} a_n(t)\varphi_n(x), \quad (2.6)$$

where

$$a_1(0) = (1 + \delta)c_1 \quad \text{and} \quad |\delta| \ll 1, \quad \delta \neq 0,$$

and where $v(x) = c_1 \varphi_1(x)$ solves (S^0) . Then a tedious, but routine, computation yields

$$a_n(t) = a_n(0) \exp((-n^2 \pi^2 + \frac{1}{2}b)t) \left[1 - \frac{a(p-1)}{2} H(t) \right]^{-1/(p-1)} \quad (2.7)$$

and

$$u(x, t) - v(x) = (a_1(t)/c_1 - 1)v(x) + \sum_{n=2}^{\infty} a_n(t) \varphi_n(x) \quad (2.8)$$

or

$$\begin{aligned} u(x, t) - v(x) &= \left[(1 + \delta) \exp((\frac{1}{2}b - \pi^2)t) \left[1 - \frac{a(p-1)}{2} H(t) \right]^{-1/(p-1)} - 1 \right] v(x) \\ &\quad + \exp((\frac{1}{2}b - \pi)t) \left[1 - \frac{a(p-1)}{2} H(t) \right]^{-1/(p-1)} \\ &\quad \times \sum_{n=2}^{\infty} a_n(0) \exp(-(n^2 - 1)\pi^2 t) \varphi_n(x). \end{aligned} \quad (2.9)$$

Since $a_1(0) \neq 0$, we see that $h(t) \sim (1 + \delta)\|v\| \exp((\frac{1}{2}b - \pi^2)t)$ as $t \rightarrow \infty$. Together with (2.1), this shows us that

$$\begin{aligned} \left[1 - \frac{1}{2}a(p-1)H(t) \right]^{1/(p-1)} &\sim \left(\frac{-a\|v\|^{p-1}}{b - 2\pi^2} \right)^{1/(p-1)} (1 + \delta) \exp((\frac{b}{2} - \pi^2)t) \\ &\sim (1 + \delta) \exp((\frac{b}{2} - \pi^2)t) \end{aligned}$$

as $t \rightarrow +\infty$. Using these estimates in (2.9) and the triangle inequality, we see that

$$\begin{aligned} \|u(\cdot, t) - v\|_q &\leq \varepsilon_1(t) \|v\|_q \\ &\quad + e^{-\pi^2 t} (1 - \varepsilon_2(t)) \left[\int_0^t \left| \sum_{n=2}^{\infty} a_n(0) e^{-(n^2-2)\pi^2 t} \varphi_n(x) \right|^q dx \right]^{1/q}, \end{aligned}$$

where $\varepsilon_1(t), \varepsilon_2(t) \rightarrow 0$ as $t \rightarrow \infty$. Consequently $\|u(\cdot, t) - v\|_q \rightarrow 0$ as $t \rightarrow +\infty$.

3. The stationary solutions ($\varepsilon > 0$). If we write $u = \sigma v$ with $\sigma > 0$ such that $\sigma^{p-1}|a| = 1$ and replace ε by $\sigma\varepsilon$, we can reduce (P) to an equivalent problem for v with $a = 1$ or $a = -1$. For this reason, we seek all positive solutions of

$$\begin{aligned} 0 &= w_{xx} + \varepsilon w w_x + \frac{1}{2}(\delta \|w\|^{p-1} + b)w \quad 0 < x < 1; \\ w(0) &= w(1) = 0, \end{aligned} \quad (S)$$

where $\delta = 1$ or $\delta = -1$. We let

$$\|w\| = \|w\|_q = \left(\int_0^1 w^q dx \right)^{1/q}$$

and set

$$y = \int_0^x w(\eta) d\eta, \quad Y = \int_0^1 w(\eta) d\eta. \quad (3.1)$$

Under this change of variable, we have

$$\frac{d}{dx} = w \frac{d}{dy}$$

so that in lieu of (S), with $v(y) = w^2(x(y))$, we find

$$0 = v_{yy} + \varepsilon v_y + \delta Y_q^{p-1} + b, \quad (3.2a)$$

$$v(0) = v(Y) = 0, \quad (3.2b)$$

where $Y_q = \|w\|_q$ and $Y_1 = Y$. The positive solutions of (3.2a,b) are given by

$$v(y, Y) = \varepsilon^{-1} Y (\delta Y_q^{p-1} + b) J(\varepsilon y, \varepsilon Y), \quad (3.3a)$$

where

$$J(y, z) = \frac{1 - e^{-y}}{1 - e^{-z}} - \frac{y}{z}, \quad (3.3b)$$

and it follows that $\delta Y_q^{p-1} + b > 0$, since $v = w^2$ and $J > 0$ if $y < z$.

The values of Y , Y_q are then determined by solving the nonlinear system

$$1 = \int_0^1 dx = \int_0^Y \frac{dy}{\sqrt{v(y)}} \quad (3.4a)$$

$$Y_q^q = \int_0^1 w^q(x) dx = \int_0^Y v^{(q-1)/2}(y) dy. \quad (3.4b)$$

We define, for $s \geq 0$, $z > 0$,

$$H_s(z) = \int_0^1 [J(\sigma z, z)]^{(s-1)/2} d\sigma.$$

A routine calculation shows that with

$$\begin{aligned} Z_q &= \varepsilon Y_q, \\ Z_1 &= Z = \varepsilon Y, \end{aligned} \quad (3.5)$$

we have from (3.4a,b) for $1 \leq q < \infty$,

$$Z_q \equiv Z_q(Z) = Z [H_q(Z) H_0^{q-1}(Z)]^{1/q} \quad (Z \geq 0) \quad (3.6)$$

while

$$b = Z H_0^2(Z) - \delta \varepsilon^{-(p-1)} Z_q^{p-1}(Z) = b(Z). \quad (3.7)$$

When $q = +\infty$,

$$Z_q(Z) = Z^{-1/2} \left[\frac{e^{-Z} - (1-Z)}{1 - e^{-Z}} + \ln \left(\frac{1 - e^{-Z}}{Z} \right) \right]^{1/2} H_0(Z).$$

Useful asymptotic formulas are, with $1 \leq q < \infty$

$$\frac{Z_q(Z)}{Z} = \begin{cases} 2/(1+q)^{1/q} & Z \rightarrow \infty, \\ [\pi^{q-1} \Gamma^2(\frac{1}{2}(1+q))/\Gamma(1+q)]^{1/q} & Z \rightarrow 0^+, \end{cases}$$

and for $q = +\infty$

$$\frac{Z_\infty(Z)}{Z} = \begin{cases} 2 & Z \rightarrow +\infty, \\ \pi/2 & Z \rightarrow 0^+. \end{cases}$$

We have plotted $Z_q(Z)$ and $Z_q(Z)/Z$ against Z for various q in Fig. 1 and Fig. 2 respectively.

FIG. 1. L_q norm of the stationary solution as a function of L_1 norm of the stationary solution ($\varepsilon = 1.0$) for various q .

FIG. 2. L_q norm/ L_1 norm of the stationary solution as a function of L_1 of the stationary solution norm ($\varepsilon = 1.0$) for various q .

For fixed b , the cardinality of the set of stationary solutions will turn out to be the same as the cardinality of the set of solutions of (3.7).

To see this, we will need some lemmas. We begin with the following

LEMMA 3.1. For any $q \geq 1$, $p > 1$, all the positive solutions of (S) are concave.

Proof. This follows immediately from (3.3a,b). We have

$$w_{xx} = w \frac{d}{dy} \left(w \frac{dw}{dy} \right) = \frac{1}{2} \sqrt{v(y)} v_{yy}(y).$$

Clearly, from (3.3a,b) $v_{yy} < 0$.

This establishes the concavity of the steady states found in [14] numerically.

LEMMA 3.2. The function $h(Z) := ZH_0^2(Z)$ ($Z > 0$) is strictly increasing.

The proof of this lemma is rather long and is therefore included in Appendix B.

LEMMA 3.3. The function

$$h_q(Z) := Z^q H_q(Z) H_0^{q-1}(Z)$$

is strictly increasing for $q \geq 1$.

FIG. 3. Plot of $ZH_0^2(Z)$ against Z . (See Lemma 3.3.) With $\epsilon = 1$, Z is the L^1 norm of a stationary solution.

Proof. We can write

$$h_q(Z) = Z^{(q+1)/2} H_q(Z) (h(Z))^{(q-1)/2}.$$

Therefore, by Lemma 3.2, it suffices to show that $H_q(Z)$ is strictly increasing. This in turn will hold if $\frac{d}{dz} J(\sigma z, z) = [\sigma e^{-\sigma z} (1 - e^{-z}) - e^{-z} (1 - e^{-\sigma z})] / (1 - e^{-z})^2 \geq 0$. If we write $x = e^{-z}$, then $x \in (0, 1]$, and

$$G(\sigma, x) \equiv \frac{d}{dz} J(\sigma z, z) = \left[\sigma x^\sigma - \frac{x(1-x^\sigma)}{1-x} \right] / (1-x).$$

In view of the convexity of $1 - x^\sigma$, we find

$$\sigma x^{\sigma-1} \geq \frac{1 - x^\sigma}{1 - x};$$

hence $G(\sigma, x) \geq 0$, and the lemma is true.

LEMMA 3.4. Let w_1, w_2 be two nonnegative stationary solutions of (P) with $\|w_1\|_1 \leq \|w_2\|_1$. Then either $w_1 < w_2$ on $(0, 1)$ or else $w_1 \equiv w_2$.

Proof. Any such solution can be written in the form

$$\begin{aligned} \varepsilon^2 v \left(z, \frac{Z}{\varepsilon} \right) &= Z(\delta \varepsilon^{-(p-1)} Z_q^{p-1}(Z) + b)J(z, Z) \\ &= Zh(Z)J(z, Z), \end{aligned}$$

where $z = \varepsilon y$, $Z = \varepsilon \|w\|_1$, and $v(y) = w^2(x)$, in view of (3.7).

A straightforward computation shows that

$$\begin{aligned} \varepsilon^2 \frac{\partial v}{\partial Z} \left(z, \frac{Z}{\varepsilon} \right) &= \varepsilon v_{,2} \left(z, \frac{Z}{\varepsilon} \right) \\ &= h'(Z)ZJ(z, Z) + h(Z)\frac{\partial}{\partial Z}(ZJ(z, Z)) \\ &= h'(Z)ZJ(z, Z) + h(Z)(1 - e^{-Z})(1 - e^{-Z} - Ze^{-Z})/(1 - e^{-Z})^2 \\ &> 0. \end{aligned}$$

Thus, if $Y_i = \|w_i\|_1$, we have $Y_1 \leq Y_2$ and

$$v(y, Y_1) \leq v(y, Y_2).$$

We need to show that if $w_1 \neq w_2$ then $w_1 < w_2$ on $(0, 1)$. Note that $w_i^2(x) = v(y_i, Y_i)$, and $dy_i/dx = w_i(x)$ for $i = 1, 2$. Thus, for any $x \in (0, 1)$,

$$x = \int_0^{y_1(x)} (v(\sigma, Y_1))^{-1/2} d\sigma = \int_0^{y_2(x)} (v(\sigma, Y_2))^{-1/2} d\sigma,$$

where

$$y_i(x) = \int_0^x w_i(\eta) d\eta.$$

We see that if $Y_1 = Y_2$ then $y_1(x) \equiv y_2(x)$, so that $w_1(x) \equiv w_2(x)$. Hence, we may assume $Y_1 < Y_2$.

Since $v(\sigma, Y_1) < v(\sigma, Y_2)$, if $\sigma > 0$ we must have $y_1(x) < y_2(x)$ on $(0, 1]$. Let $\bar{x}$ denote the unique x in $(0, 1)$ where $w_2'(\bar{x}) = 0$ (Lemma 3.1). Then for $x \in (0, 1)$, $y_2(x) \leq y_2(\bar{x})$. Now

$$\begin{aligned} &v(y_2(x), Y_2) - v(y_1(x), Y_1) \\ &= v(y_2(x), Y_2) - v(y_1(x), Y_2) + v(y_1(x), Y_2) - v(y_1(x), Y_1). \end{aligned}$$

Since $v(\cdot, Y_2)$ is increasing on $(0, y_2(\bar{x}))$ and $v_{,2} > 0$, the differences on the right are nonnegative on $(0, 1)$ with the second being strictly positive there. Thus $w_1 < w_2$ on $(0, 1)$.

These lemmas allow us to conclude the following.

THEOREM 3.5. Let $w(x)$ be a nonnegative solution of (S). If $Z = \varepsilon\|w\|_1$ then Z solves (3.7). Conversely, if $Z (> 0)$ is a solution of (3.7), then $w(x) = (v(y))^{1/2}$ solves (S) with $dx/dy = (v(y, Y))^{-1/2}$, $Y = Z/\varepsilon$, and v is given by (3.3a,b).

We next count the solutions of (3.7) for fixed $\varepsilon > 0$. Let $Z(b)$ be a branch of solutions of (2.7). By the implicit function theorem, as long as

$$Q(Z) := \frac{\partial}{\partial Z}[ZH_0^2(Z) - \delta\varepsilon^{-(p-1)}Z_q^{p-1}(Z)] \neq 0$$

on this branch, Z will be a C^1 function of b and

$$Z'(b)Q(Z(b)) = 1.$$

For fixed b , the number of solutions of (3.7) is then the same as the number of sign changes of $Q(Z)$ on $(0, \infty)$. However, the sign analysis of Q is complicated when $\delta > 0$, since, in principle, a function of bounded variation (the difference of two increasing functions) can have infinitely many zeros.

The case $\delta < 0$ is easy to treat. The function $b(Z)$ given by (3.7) is strictly increasing for $Z \geq 0$ and has range $[2\pi^2, \infty)$. Therefore, for each $b > 2\pi^2$, there is exactly one positive solution of (S), and no nontrivial solution for $b \leq 2\pi^2$.

Let us observe that $b(Z)$ has the following asymptotic properties:

$$b(Z) = \begin{cases} 4Z - \delta\varepsilon^{-(p-1)}\varepsilon_\infty^{p-1}Z^{p-1} & Z \rightarrow \infty, \quad 1 \leq q \leq \infty, \\ 2\pi^2 - \delta\varepsilon^{-(p-1)}\varepsilon_0^{p-1}Z^{p-1} & Z \rightarrow 0^+, \quad 1 \leq q \leq \infty, \end{cases}$$

where

$$\varepsilon_\infty = \begin{cases} 2(1+q)^{-1/q} & 1 \leq q < \infty, \\ 2 & q = \infty \end{cases}$$

and

$$\varepsilon_0 = \begin{cases} [\pi^{q-1}\Gamma^2(\frac{1}{2}(1+q))/\Gamma(1+q)]^{1/q} & 1 \leq q < \infty, \\ \pi/2 & q = \infty. \end{cases}$$

These asymptotic formulas, together with the numerical results of Figs. 4–11, allow us to assert that for $\delta > 0$

N-1. If $1 < p < 2$, there is $b(p, q, \varepsilon) < 2\pi^2$ such that

- (a) If $b < b(p, q, \varepsilon)$, there are no positive stationary solutions.
- (b) If $b = b(p, q, \varepsilon)$ or $b > 2\pi^2$, there is exactly one solution.
- (c) If $b(p, q, \varepsilon) < b < 2\pi^2$, there are exactly two solutions.

N-2. If $p = 2$, we have the following:

- (a) If $\varepsilon < \frac{1}{4}\varepsilon_\infty$, there is exactly one positive stationary solution for each $b < 2\pi^2$, none for $b \geq 2\pi^2$.
- (b) If $\varepsilon = \frac{1}{4}\varepsilon_\infty$, there is $b(\varepsilon_\infty) < 2\pi^2$ such that
 - (i) if $b \leq b(\varepsilon_\infty)$ or $b > 2\pi^2$, there are no positive stationary solutions;
 - (ii) if $b(\varepsilon_\infty) < b < 2\pi^2$, there is exactly one solution.
- (c) If $\varepsilon > \frac{1}{4}\varepsilon_\infty$, there is $b(\varepsilon_\infty) < 2\pi^2$ such that
 - (i) if $b < b(\varepsilon_\infty)$, there are no positive stationary solutions;

- (ii) if $b = b(\varepsilon_\infty)$ or $b > 2\pi^2$, there is exactly one solution;
- (iii) if $b(\varepsilon_\infty) < b < 2\pi^2$, there are exactly two solutions.

N-3. If $p > 2$, there is a critical number $\varepsilon_1(p, q)$ such that

- (a) If $\varepsilon \leq \varepsilon_1$, we have the following:
 - (i) if $b \geq 2\pi^2$, there are no positive stationary solutions;
 - (ii) if $b < 2\pi^2$, there is exactly one solution.
- (b) If $\varepsilon > \varepsilon_1$, there is $b(p, q, \varepsilon) > 2\pi^2$ such that
 - (i) if $b > b(p, q, \varepsilon)$, there are no positive stationary solutions;
 - (ii) if $b = b(p, q, \varepsilon)$ or $b < 2\pi^2$, there is exactly one solution;
 - (iii) if $2\pi^2 < b < b(p, q, \varepsilon)$, there are exactly two solutions.

A word of explanation about the figures is in order. In Figs. 4–6, we have set $q = 2$ and chosen $p = 1.5, 2, 3$ respectively, plotting the solution set of (2.7) for various ε . The above assertions are based on these figures, the asymptotic formulas following (3.7) and (H). In Figs. 7–11, we fixed ε and p and plotted the solution sets as functions of Q .

4. Stability and global nonexistence ($\varepsilon > 0$). Now we consider the following problem equivalent to (P):

$$\begin{aligned} u_t &= u_{xx} + \varepsilon u u_x + \frac{1}{2}(\delta \|u(\cdot, t)\|^{p-1} + b)u & 0 < x < 1, \quad 0 < t < T; \\ u(0, t) &= u(1, t) = 0 & 0 < t < T, \\ u(x, 0) &= u_0(x) & 0 \leq x \leq 1. \end{aligned} \quad (\text{P}')$$

For simplicity, we let $D_T = (0, 1) \times (0, T)$ and $D_T \cup \Gamma_T = [0, 1] \times [0, T)$.

Our primary interest is in the stability properties of the steady states, and the asymptotic behavior of solutions of (P') for a given initial datum u_0 . To pursue this interest further, we first establish a relationship between solutions of (P') and (S).

LEMMA 4.1. If u is a bounded monotone (in time) solution of (P'), then u tends to a solution of (S) as $t \rightarrow \infty$.

Proof. First, we note that such a solution must be global in time by the continuation statement in Appendix A. Suppose $\lim_{t \rightarrow \infty} u(x, t) = \varphi(x)$, and let

$$F(x, t) = \int_0^1 G(x, y) u(y, t) dy, \quad (4.1)$$

where

$$G(x, y) = \begin{cases} x(1-y) & 0 \leq x \leq y \leq 1, \\ y(1-x) & 0 \leq y \leq x \leq 1 \end{cases}$$

is the Green's function for $-d^2/dy^2$ with Dirichlet boundary conditions.

Under the assumptions for u , F is bounded in $[0, 1] \times [0, \infty)$ and

$$\begin{aligned}
 F_t(x, t) &= \int_0^1 G(x, y) u_t(y, t) dy \\
 &= -u(x, t) - \frac{\varepsilon}{2} \int_0^x u^2(y, t) dy + \frac{\varepsilon}{2} x \int_0^1 u^2(y, t) dy \\
 &\quad + \frac{1}{2} (\delta \|u(\cdot, t)\|^{p-1} + b) \int_0^1 G(x, y) u(y, t) dy \\
 &\rightarrow -\varphi(x) - \frac{\varepsilon}{2} \int_0^x \varphi^2(y) dy + \frac{\varepsilon}{2} x \int_0^1 \varphi^2(y) dy \\
 &\quad + \frac{1}{2} (\delta \|\varphi\|^{p-1} + b) \int_0^1 G(x, y) \varphi(y) dy
 \end{aligned} \tag{4.2}$$

as $t \rightarrow \infty$. This limit, which is of one sign depending on whether $u_t \geq 0$ or $u_t \leq 0$, is in fact zero for $x \in [0, 1]$; otherwise F would not have a finite limit as $t \rightarrow \infty$. Therefore,

$$\varphi(x) = -\frac{\varepsilon}{2} \int_0^x \varphi^2(y) dy + \frac{\varepsilon}{2} x \int_0^1 \varphi^2(y) dy + \frac{1}{2} (\delta \|\varphi\|^{p-1} + b) \int_0^1 G(x, y) \varphi(y) dy,$$

and hence φ is a solution of (S).

By means of this lemma, we can obtain a complete result for stability and instability of stationary solutions of (P') with $\delta = 1$. This time, we treat the solution of (S) as a function depending on the parameter b and denote it by $w(x, b)$.

THEOREM 4.2. Let $w(x, b)$ be a continuously differentiable positive solution of (S) with $\delta = 1$ on some b interval $[\alpha, \beta]$, and let $Z(b)$ be the corresponding solution of (3.7). Then if $Z'(b) > 0$ on $[\alpha, \beta]$, the solutions are stable, whereas they are unstable if $Z'(b) < 0$.

Proof. For the case $Z'(b) > 0$ we first show that $w(x, b_1) < w(x, b_2)$ on $(0, 1)$ for $\alpha \leq b_1 < b_2 \leq \beta$.

From $Z'(b) > 0$, it follows that $Z(b_1) < Z(b_2)$ if $\alpha \leq b_1 < b_2 \leq \beta$. In view of (3.7), $Z(b) = \varepsilon \|w(\cdot, b)\|_1$ and $v(\varepsilon y, Z/\varepsilon, b) = w^2(x, b)$. Using the form of $v(\varepsilon y, Z/\varepsilon, b)$ in Lemma 3.3, we see that

$$\begin{aligned}
 \varepsilon^2 v\left(\varepsilon y, \frac{Z(b_1)}{\varepsilon}, b_1\right) &= (\varepsilon^{-(p-1)} Z_q^{p-1}(Z(b_1)) + b_1) Z(b_1) J(\varepsilon y, Z(b_1)) \\
 &< (\varepsilon^{-(p-1)} Z_q^{p-1}(Z(b_2)) + b_1) Z(b_2) J(\varepsilon y, Z(b_2)) \\
 &< (\varepsilon^{-(p-1)} Z_q^{p-1}(Z(b_2)) + b_2) Z(b_2) J(\varepsilon y, Z(b_2)) \\
 &= \varepsilon^2 v\left(\varepsilon y, \frac{Z(b_2)}{\varepsilon}, b_2\right),
 \end{aligned}$$

since $\partial v / \partial Z > 0$ and $b_1 < b_2$.

Thus, letting $Y(b) = \|w(\cdot, b)\|_1$, we have

$$v(y, Y(b_1), b_1) < v(y, Y(b_2), b_2) \quad \text{for } \alpha \leq b_1 < b_2 \leq \beta.$$

Then following the same reasoning as in Lemma 3.3, we find that

$$w(x, b_1) < w(x, b_2) \quad \text{on } (0, 1) \text{ for } \alpha \leq b_1 < b_2 \leq \beta.$$

Let $u(x, t, b_1)$ be a solution of (P') with $u_0(x, b_1) = w(x, b_2)$. Then, on $(0, 1)$, we have

$$\begin{aligned} u_0'' + \varepsilon u_0' + \frac{1}{2}(\|u_0\|^{p-1} + b_1)u_0 \\ = w_{xx}(x, b_2) + \varepsilon w(x, b_2)w_x(x, b_2) + \frac{1}{2}(\|w(\cdot, b_2)\|^{p-1} + b_1)w(x, b_2) \\ < w_{xx}(x, b_2) + \varepsilon w(x, b_2)w_x(x, b_2) + \frac{1}{2}(\|w(\cdot, b_2)\|^{p-1} + b_2)w(x, b_2) \\ = 0. \end{aligned}$$

Hence, recalling the corollary in Appendix A, we have $u_t < 0$ in D_T . From the comparison theorem in Appendix A and monotonicity of u , we also have, on $(0, 1)$

$$w(x, b_1) < u(x, t, b_1) \leq w(x, b_2).$$

By Lemma 4.1, $\varphi(x, b_1) = \lim_{t \rightarrow \infty} u(x, t, b_1)$ exists, and $w(x, b_1) \leq \varphi(x, b_1) \leq w(x, b_2)$. Letting $b_2 \rightarrow b_1^+$ yields $\varphi(x, b_1) \equiv w(x, b_1)$, which shows that $w(x, b_1)$ is stable from above. We can also prove similarly that $w(x, b_1)$ is stable from below.

If $Z'(b) < 0$, then $Z(b_2) < Z(b_1)$, and consequently, $\|w(\cdot, b_2)\|_1 < \|w(\cdot, b_1)\|_1$ for $\alpha \leq b_1 < b_2 \leq \beta$. Thus there is a subinterval $[x_0, x_1]$ contained in $[0, 1]$ such that $w(x, b_2) < w(x, b_1)$ on $[x_0, x_1]$. Now let $u(x, t, b_2)$ be a solution of (P') with $u_0(x, b_2) = w(x, b_1)$. Then, on $(0, 1)$, we find that

$$\begin{aligned} u_0'' + \varepsilon u_0' + \frac{1}{2}(\|u_0\|^{p-1} + b_2)u_0 \\ = w_{xx}(x, b_1) + \varepsilon w(x, b_1)w_x(x, b_1) + \frac{1}{2}(\|w(\cdot, b_1)\|^{p-1} + b_2)w(x, b_1) \\ > w_{xx}(x, b_1) + \varepsilon w(x, b_1)w_x(x, b_1) + \frac{1}{2}(\|w(\cdot, b_1)\|^{p-1} + b_1)w(x, b_1) \\ = 0. \end{aligned}$$

Therefore, $u_t > 0$ in D_T . Hence, $u(x, t, b_2)$ is increasing in t , and it follows that $w(x, b_2)$ is unstable from above. Similarly, $w(x, b_2)$ is unstable from below when $b_1 > b_2$.

Using this theorem combined with the characterization of the stationary solutions in Sec. 3, we obtain the following stability and instability results for the case $\delta = 1$.

C-1. For $1 < p < 2$, if $b(p, q, \varepsilon) < b < 2\pi^2$, there are two branches of solutions of (S)—one stable but the other unstable; if $b > 2\pi^2$, the unique solution is stable.

C-2. For $p = 2$,

- (a) if $\varepsilon < \frac{1}{4}\varepsilon_\infty$ and $b < 2\pi^2$, $w(x, b)$ is unstable;
- (b) if $\varepsilon = \frac{1}{4}\varepsilon_\infty$ and $b(\varepsilon_\infty) < b < 2\pi^2$, $w(x, b)$ is unstable;
- (c) if $\varepsilon > \frac{1}{4}\varepsilon_\infty$, and if
 - (i) $b(\varepsilon_\infty) < b < 2\pi^2$, there are two branches—one stable, the other unstable;
 - (ii) $b > 2\pi^2$, $w(x, b)$ is stable.

C-3. For $p < 2$,

- (a) if $\varepsilon \leq \varepsilon_1$ and $b < 2\pi^2$, $w(x, b)$ is unstable;
- (b) if $\varepsilon > \varepsilon_1$ and if
 - (i) $b < 2\pi^2$, $w(x, b)$ is unstable;
 - (ii) $2\pi^2 < b < b(p, q, \varepsilon)$, there are two branches—one unstable, the other stable.

In particular, we see from Figs. 4–6, as ε increases, the portion of the bifurcation curve along which $Z'(b) > 0$ becomes more pronounced. This illustrates the stabilizing effect of the convection term in the dynamical equation.

From Figs. 7–11 (see pp. 188–190), we see that, for fixed ε , the choice $q = 1$ leads to the “most stability” while the choice $q = +\infty$ leads to the “least stability” in the sense of the preceding paragraph. Increasing q has the effect of decreasing the set $\{b | Z'(b) > 0\}$.

FIG. 4. Solution set of the bifurcation equation (3.7) for $p = 1.50$, $q = 2$, and various ε .

FIG. 5. Solution set of the bifurcation equation (3.7) for $p = 2.0$, $q = 2$, and various ϵ .

FIG. 6. Solution set of the bifurcation equation (3.7) for $p = 3$, $q = 2$, and various ϵ .

FIG. 7. Solution set of the bifurcation equation (3.7) for $p = 1.5$, $\varepsilon = 0.004$, and various q .

FIG. 8. Solution set of the bifurcation equation (3.7) for $p = 2$, $\varepsilon = 0.350$, and various q .

FIG. 9. Solution of the bifurcation equation (3.7) for $p = 3$, $\varepsilon = 1.5$, and various q .

FIG. 10. Solution of the bifurcation equation (3.7) for $p = 3$, $\varepsilon = 1.5$, and various q .

FIG. 11. Solution of the bifurcation equation (3.7) for $p = 3$, $\varepsilon = 4.0$, and various q .

The case $\delta = -1$, due to the lack of a comparison principle, is not so amenable to analysis, but bearing numerical evidence in mind (see [14]), we may conjecture as follows: the positive stationary solution branch is stable, where it exists.

Next we discuss the asymptotic stability of the trivial stationary solution of (P') . Based on semigroup theory, Henry [7] used the linearization method to analyze the asymptotic stability. His principle has wide application, but it does not appear to extend to the current nonlocal problem. Therefore, we adopt another approach.

Letting $F_2(t) = \int_0^1 u^2 dx$ we find

$$\begin{aligned} F_2'(t) &= 2 \int_0^1 uu_{xx} dx + 2\varepsilon \int_0^1 uu_x dx + bF_2 + \delta \|u\|^{p-1} F_2 \\ &= -2 \int_0^1 u_x^2 dx + (b + \delta \|u\|^{p-1}) F_2 \\ &\leq (-2\pi^2 + b + \delta \|u\|^{p-1}) F_2, \end{aligned}$$

since $\pi^2 \int_0^1 u^2 dx \leq \int_0^1 u_x^2 dx$.

For $\delta = 1$, for any $b < 2\pi^2$ and sufficiently small initial values u_0 , we have $F_2(t) \leq F_2(0)e^{-kt}$, which means that $v \equiv 0$ is an asymptotically stable solution of (S) in $L_2(0, 1)$. Similarly, $v \equiv 0$ is a globally asymptotically stable solution of (S) in $L_2(0, 1)$ when $b \leq 2\pi^2$ for $\delta = -1$.

Finally, we investigate the global existence and nonexistence of solutions of (P') . We first show the following for the case $\delta = 1$.

THEOREM 4.3. For $1 < p < 2$ and any $\varepsilon > 0$, or $p = 2$ and $\varepsilon > \frac{1}{2}(1+q)^{-1/q}$, the solutions of (P') are uniformly bounded on $[0, 1] \times [0, \infty)$.

Proof. We look for a supersolution v in the form $v(x) = M(1 - \sigma x)$ with $M > 0$ and $0 < \sigma < 1$ to be chosen. We will be done if M and σ satisfy

$$-\varepsilon \sigma M^2(1 - \sigma x) + \frac{1}{2}[b + M^{p-1}(\alpha(1 + q))^{-(p-1)/q}]M(1 - \sigma x) \leq 0 \quad (4.3)$$

and

$$M(1 - \sigma) \geq \max_{0 \leq x \leq 1} u_0(x). \quad (4.4)$$

(4.3) is equivalent to

$$M^{p-1}[\varepsilon \sigma M^{2-p} - \frac{1}{2}(\alpha(1 + q))^{-(p-1)/q}] \geq \frac{1}{2}b. \quad (4.5)$$

For $1 < p < 2$, (4.4) and (4.5) will hold if M is sufficiently large; while if $p = 2$, letting $\varepsilon = \frac{1}{2}(1 + \gamma)(1 + q)^{-1/q}$ ($\gamma > 0$), we have that $\sigma^{(1+q)/q} \leq (1 + \gamma)^{-1}$, i.e., $\sigma \leq (1 + \gamma)^{-q/(1+q)} < 1$. For such σ , we can choose M so large that (4.4) holds.

For the case $\delta = -1$, since the solution of (P') is a subsolution of (P') with $\delta = 1$ and $1 < p < 2$, all solutions are bounded on $[0, 1] \times [0, \infty)$.

Next for $p > 2$ or $p = 2$ with small ε , we prove that with sufficiently large initial data, problem (P') with $\delta = 1$ does not have global solutions. To show this, we employ two different arguments.

THEOREM 4.4A. Let $p > 2$. Then there exists $c_0 = c_0(\varepsilon, p, b) < \infty$ such that if $u_0 \geq c_0 \sin \pi x$, the solution of (P') blows up in finite time. The same is true for $p = 2$ if ε is small enough.

Proof. We seek a subsolution $w(x, t)$ in the form $w(x, t) = h(t) \sin \pi x$ with $h(t)$ becoming unbounded in finite time. To see this, we need

$$h'(t) \leq -\pi^2 h(t) + \pi \varepsilon h^2(t) \cos \pi x + \frac{1}{2}(A_0 h^p(t) + b h(t)), \quad (4.6)$$

where $A_0 = [\int_0^1 (\sin \pi x)^q dx]^{(p-1)/q}$.

Set

$$Q(s) = \frac{A_0}{2} s^{p-2} - \pi \varepsilon \quad (4.7a)$$

if $b < 2\pi^2$ or

$$Q(s) = \frac{A_0}{2} s^{p-2} - \pi \varepsilon - \left(\pi^2 - \frac{b}{2}\right) s^{-1} \quad (4.7b)$$

for $b \geq 2\pi^2$.

For $p > 2$, if s_0 is the largest positive root of Q , then $Q(s), Q'(s) > 0$ for $s > s_0$.

Choosing $c_0 = s_0$ and letting $h(t)$ be the solution of the following problem:

$$\begin{aligned} h'(t) &= Q(h)h^2(t) \quad t > 0, \\ h(0) &= c_0, \end{aligned} \quad (4.8)$$

one easily sees that (3.6) is satisfied and $h(t)$ blows up in finite time.

For $p = 2$ and sufficiently small $\varepsilon > 0$, a similar argument also holds.

REMARK. The result for $p = 2$ is in contrast to that of [4] where it is shown that for any $\varepsilon > 0$, the solution of a local problem with our nonlocal term replaced by $|u|^{p-1}u$ remains bounded on $[0, 1] \times [0, \infty)$ when $p = 2$.

For any large initial value, with more restrictions on p and q , we also have

THEOREM 4.4B. Let $p > 3$, $q \geq 2$. Then there exists $c_1 = c_1(\varepsilon, p, b)$ such that if $\int_0^1 u_0(x) dx > c_1$, the solution of (P') blows up in finite time.

Proof. Now we use a variation of the eigenfunction method [9].

Set

$$J(t) = \int_0^1 u(x, t) \psi(x) dx, \quad (4.9)$$

where $\psi(x) = \frac{\pi}{2} \sin \pi x$.

A routine calculation shows that

$$\begin{aligned} J'(t) \geq & -\pi^2 \int_0^1 u(x, t) \psi(x) dx - \varepsilon \int_0^1 u^2(x, t) \psi'(x) dx \\ & + \frac{1}{2} \left[\int_0^1 u^q(x, t) dx \right]^{(p-1)/q} \int_0^1 u(x, t) \psi(x) dx \\ & + \frac{b}{2} \int_0^1 u(x, t) \psi(x) dx. \end{aligned} \quad (4.10)$$

Using Hölder's inequality, we see that

$$\varepsilon \int_0^1 u^2(x, t) \psi'(x) dx \leq A_1 \left[\int_0^1 u^q(x, t) dx \right]^{2/q}, \quad (4.11)$$

$$\int_0^1 u(x, t) \psi(x) dx \leq A_2 \left[\int_0^1 u^q(x, t) dx \right]^{1/q}, \quad (4.12)$$

with $A_1 = \varepsilon \|\psi'\|_{L^\infty(0,1)}$ and $A_2 = \pi/2$.

Let

$$R(s) = A_3 s^{(p-1)/q} - A_1 s^{2/q} \quad (4.13a)$$

if $b > 2\pi^2$ or

$$R(s) = A_3 s^{(p-1)/q} - A_1 s^{2/q} - \left(\pi^2 - \frac{b}{2} \right) s^{1/q}, \quad (4.13b)$$

for $b \leq 2\pi^2$, where $A_3 = \frac{1}{2} \int_0^1 u_0(x) \psi(x) dx$.

If s_1 is the largest positive root of R , we find that $R(s), R'(s) > 0$ for $s > s_1$. Letting $c_1 = s_1$ and $\int_0^1 u_0(x) dx > c_1$ implies that $\int_0^1 u^q(x, t) dx > s_1$ for all t and so

$$J'(t) \geq R \left(\int_0^1 u^q(x, t) dx \right) \geq R \left(\frac{2}{\pi} J^q(t) \right). \quad (4.14)$$

Since $p > 3$, $J(t)$ must blow up in finite time.

Appendix A. Here we will establish the comparison principle and local existence of solutions for the following general problem:

$$\begin{aligned} u_t &= u_{xx} + (f(u))_x + g(u, \|u\|) & 0 < x < 1, \quad 0 < t < T; \\ u(0, t) &= u(1, t) = 0 & 0 < t < T; \\ u(x, 0) &= u_0(x) & 0 \leq x \leq 1; \end{aligned} \quad (G)$$

here $\|u\| = [\int_0^1 |u|^q dx]^{1/q}$.

First, we proceed to the definitions of subsolution and supersolution of (G). As in Sec. 4, we let $D_T = (0, 1) \times (0, T)$ and $D_T \cup \Gamma_T = [0, 1] \times [0, T]$.

DEFINITION. A function $u(x, t)$ is called a *subsolution* of (G) on D_T if $u \in C^{2,1}(D_T) \cap C(D_T \cup \Gamma_T)$ satisfies

$$\begin{aligned} u_t &\leq u_{xx} + (f(u))_x + g(u, \|u\|) & 0 < x < 1, \quad 0 < t < T; \\ u(0, t) &\leq 0, \quad u(1, t) \leq 0 & 0 < t < T; \\ u(x, 0) &\leq u_0(x) & 0 \leq x \leq 1. \end{aligned} \quad (G')$$

A *supersolution* is defined by (G') with each " $\leq$ " replaced by " $\geq$ ".

COMPARISON THEOREM. Suppose that f and g are continuously differentiable and $g_{,2} \geq 0$. Let u and v be a nonnegative supersolution and a nonnegative subsolution of (G), respectively, with $u(x, 0) \geq v(x, 0)$ for $x \in (0, 1)$. Then $u \geq v$ in $D_T \cup \Gamma_T$.

Proof. For every $t \in (0, T)$ and every nonnegative $\varphi(x, t) \in C^{2,1}(\overline{D}_T)$ with $\varphi(0, t) = \varphi(1, t) = 0$, the subsolution v satisfies the following integral inequality

$$\begin{aligned} \int_0^1 v(x, t) \varphi(x, t) dx &\leq \int_0^1 v_0(x) \varphi(x, 0) dx \\ &\quad + \int_0^1 \int_0^t [v \varphi_\tau - (v_x + f(u)) \varphi_x + g(v, \|v\|) \varphi] dx d\tau. \end{aligned}$$

The supersolution u satisfies the above with reversed inequality.

We integrate by parts in both the above inequality and that satisfied by u , and subtract the two resultant expressions; then

$$\begin{aligned} &\int_0^1 (v(x, t) - u(x, t)) \varphi(x, t) dx \\ &\leq \int_0^1 (v(x, 0) - u(x, 0)) \varphi(x, 0) dx \\ &\quad + \int_0^t \int_0^1 (v - u) (\varphi_\tau + \varphi_{xx} - A(x, \tau) \varphi_x + B(x, \tau) \varphi) dx d\tau \\ &\quad + \int_0^t \int_0^1 \varphi(x, \tau) \int_0^1 C(s, \tau) (v - u) ds dx d\tau, \end{aligned} \quad (A1)$$

where

$$\begin{aligned} A(x, t) &= f'(\theta_1(x, t)), \\ B(x, t) &= g_{,1}(\theta_2(x, t), \|v\|), \\ C(x, t) &= g_{,2}(u, \theta_3(t)) \theta_4^{(1/q)-1}(t) \theta_5^{q-1}(x, t), \end{aligned}$$

with $\theta_1, \theta_2, \theta_5$ between u and v , and θ_3, θ_4 between $\|u\|$ and $\|v\|$.

Note that by the hypotheses for f and g , A, B, C are bounded on $\overline{D}_T$ in the uniform norm. We denote the bound by M_0 .

Now we define sequences $\{A_n\}$ and $\{B_n\}$ in such a way that

- (i) $A_n, B_n \in C^\infty(\overline{D}_T)$;
- (ii) $A_n \rightarrow A, B_n \rightarrow B$ in (D_T) as $n \rightarrow \infty$,

and set up a backward problem on D_t ,

$$\begin{aligned} \varphi_{n\tau} + \varphi_{nxx} - A_n \varphi_{nx} + B_n \varphi &= 0 & 0 < x < 1, \quad 0 < \tau < t; \\ \varphi_n(0, \tau) = \varphi_n(1, \tau) &= 0 & 0 < \tau < t; \\ \varphi_n(x, t) &= \chi(x) & 0 \leq x \leq 1. \end{aligned} \quad (G^*)$$

Here, $\chi(x) \in C_0^\infty(0, 1)$, $0 \leq \chi \leq 1$.

Recalling some standard theory (in [10] for example), we find that $\varphi = \lim_{n \rightarrow \infty} \varphi_n$ is a solution of (G^*) with A_n, B_n replaced by A, B , and $\varphi \in C^{2,1}(\overline{D}_T)$. The initial and boundary values for φ_n imply that $\varphi \geq 0$ in D_T .

Substituting φ in (A1) yields

$$\begin{aligned} \int_0^1 (v(x, t) - u(x, t)) \chi(x) dx &\leq M_1 \int_0^1 (v(x, 0) - u(x, 0))^+ dx \\ &\quad + \int_0^t \int_0^1 \varphi(x, \tau) \int_0^1 C(s, \tau) (v - u) ds dx d\tau \end{aligned}$$

where $M_1 = \sup_{\overline{D}_T} |\varphi|$.

Since the inequality holds for every χ , we can choose a sequence $\{\chi_n\}$ on $(0, 1)$ converging to

$$\chi = \begin{cases} 1 & \text{if } v(x, t) - u(x, t) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Meanwhile, noting that $C \geq 0$, $v - u = (v - u)^+ - (v - u)^-$, we find that

$$\begin{aligned} \int_0^1 (v(x, t) - u(x, t))^+ dx &\leq M_1 \int_0^1 (v(x, 0) - u(x, 0))^+ dx \\ &\quad + M_0 M_1 \int_0^t \int_0^1 (v(x, \tau) - u(x, \tau))^+ dx d\tau, \end{aligned}$$

which by Gronwall's inequality leads to

$$\int_0^1 (v(x, t) - u(x, t))^+ dx \leq M_1 (1 + e^{M_1 t}) \int_0^1 (v(x, 0) - u(x, 0))^+ dx.$$

Thus, the conclusion follows from the condition on initial data.

COROLLARY. If $u_0'' + f'(u_0)u_0' + g(u_0, \|u_0\|) \geq 0$ (≤ 0) on $(0, 1)$, then $u_t(x, t) \geq 0$ (≤ 0) in D_T .

Proof. The condition on u_0 implies that u_0 is a subsolution (supersolution) of (G). Thus $u(x, t) \geq u_0(x)$ ($\leq u_0(x)$) in $D_T \cup \Gamma_T$. Let $v(x, t) = u(x, t+h)$ ($h > 0$). Then v is a supersolution (subsolution) of (G), and therefore $u(x, t+h) \geq u(x, t)$

$(\leq u(x, t))$. Since h is arbitrary, u is increasing (decreasing) in t for fixed x , and hence $u_t \geq 0$ (≤ 0).

Next we establish the existence of solutions of (G) on $D_T \cup \Gamma_T$ for sufficiently small T and certain initial values. This time we only assume that f and g are continuously differentiable. We shall also define $f_M \equiv \sup_{|u| \leq M} |f(u)|$ and $g_M \equiv \sup_{|u| \leq M} |g(u, \|u\|)|$.

Let $G(x, y; t)$ denote Green's function for

$$Lu = u_t - u_{xx} \quad 0 < x < 1, \quad t > 0,$$

with boundary conditions

$$u(0, t) = u(1, t) = 0 \quad t > 0,$$

i.e., $G(x, y; t) = 2 \sum_{n=1}^{\infty} \sin(n\pi x) \sin(n\pi y) e^{-n^2 \pi^2 t}$.

Then $G(x, 0; t) = G(x, 1; t) = 0$. Also u is a solution of (G) on $D_T \cup \Gamma_T$ if and only if for $(x, t) \in D_T \cup \Gamma_T$

$$\begin{aligned} u(x, t) &= \int_0^1 G(x, y; t) u_0(y) dy \\ &+ \int_0^t \int_0^1 [G(x, y; t - \eta) g(u(y, \eta), \|u(\cdot, \eta)\|) \\ &\quad - G_y(x, y; t - \eta) f(u(y, \eta))] dy d\eta \\ &\equiv Tu(x, t). \end{aligned} \tag{A2}$$

To show that (A2) is solvable for sufficiently small T , we use a contraction mapping argument. To this end, we define

$$u_1(x, t) = 0 \tag{A3.1}$$

and

$$u_{n+1}(x, t) = Tu_n(x, t). \tag{A3.2}$$

LOCAL EXISTENCE THEOREM. Let the initial datum for problem (G) be continuous on $[0, 1]$ and satisfy

$$0 < m < \int_0^1 \int_0^1 G(x, y; t) u_0(y) dy dx \tag{A4}$$

for $0 \leq t \leq 1$, say. Then for sufficiently small T , (G) has a unique solution that satisfies

$$\|u\| \geq \frac{m}{2} \tag{A5}$$

on $[0, T]$. The solution is C^1 in t and C^2 in x on D_T and continuous on $\bar{D}_T$.

Proof. First, we define

$$\begin{aligned} M_0 &= \sup_{0 \leq x \leq 1} |u_0(x)|, \\ \mu(t) &= \sup_{\substack{0 \leq x \leq 1 \\ 0 \leq \tau \leq t}} \int_0^\tau \int_0^1 G(x, y; \tau - \eta) dy d\eta, \\ \nu(t) &= \sup_{\substack{0 \leq x \leq 1 \\ 0 \leq \tau \leq t}} \int_0^\tau \int_0^1 |G_y(x, y; \tau - \eta)| dy d\eta. \end{aligned}$$

Clearly, $\mu(t)$ and $\nu(t)$ tend to zero as $t \rightarrow 0^+$. For fixed $M > M_0$, choose T so small that $T \leq 1$ and

$$\nu(T)f_M + \mu(T)g_M < \max(M - M_0, \tfrac{1}{2}m), \quad (\text{A6})$$

$$\begin{aligned} \alpha &\equiv \nu(T) \sup_{|\xi| \leq M} |f'(\xi)| \\ &+ \mu(T) \left(\sup_{\substack{|\xi| \leq M \\ |\eta| \leq M}} |g_\xi(\xi, \eta)| + \sup_{\substack{|\xi| \leq M \\ |\eta| \leq M}} |g_\eta(\xi, \eta)| \left(\frac{m}{2} \right)^{(1/q)-1} M^{q-1} \right) \\ &< 1. \end{aligned} \quad (\text{A7})$$

It then follows from (A3.1), (A3.2), (A6), and induction that on $\overline{D}_T$

$$\|u_n\|_{L^\infty(\overline{D}_T)} \leq M$$

for all $n = 1, 2, \dots$. Moreover, we have from (A3.2) and (A6),

$$\|u_n\| \geq \|u_n\|_{L^1(\overline{D}_T)} > \frac{m}{2} \quad (\text{A8})$$

for all $n = 2, 3, \dots$. By making use of (A7) and letting $\beta_n \equiv \|u_{n+1} - u_n\|_{L^\infty(\overline{D}_T)}$, we can see that

$$\beta_{n+1} \leq \alpha \beta_n \leq \dots \leq \alpha^n \beta_1.$$

Therefore, $\{u_n\}$ is uniformly convergent on $\overline{D}_T$ and

$$u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t) \quad (\text{A9})$$

solves (A2) with $\|u\| \geq m/2$.

The asserted interior regularity follows from the properties of G and the continuity of u in $\overline{D}_T$. We omit the standard arguments.

The above result also allows us to make the following continuation statement: If u is a classical solution on D_T , bounded on $\overline{D}_T$, then u may be extended to $\overline{D}_{T+\delta}$ for some $\delta > 0$.

Appendix B. Here we establish Lemma 3.2.

LEMMA 3.2. The integral

$$\int_0^1 \frac{z^{1/2}}{((1 - e^{-\sigma z})/(1 - e^{-z}) - \sigma)^{1/2}} d\sigma$$

is increasing in z for $z \geq 0$.

We see that the statement is equivalent to showing that

$$\begin{aligned} & \int_0^{1/2} \left[\frac{z^{1/2}}{((1 - e^{-\sigma z})/(1 - e^{-z}) - \sigma)^{1/2}} + \frac{z^{1/2}}{((1 - e^{-(1-\sigma)z})/(1 - e^{-z}) - (1 - \sigma))^{1/2}} \right] d\sigma \\ &= \int_0^{1/2} \left[\left(\frac{z}{f(z, \sigma)} \right)^{1/2} + \left(\frac{z}{g(z, \sigma)} \right)^{1/2} \right] d\sigma \end{aligned}$$

is increasing in z for $z \geq 0$.

To establish this assertion, we will show that

$$\left(\frac{z}{f} \right)^{1/2} + \left(\frac{z}{g} \right)^{1/2}$$

is increasing in z , for $\sigma \in [0, 1/2]$.

REDUCTION 1. The presence of square roots in the integrand causes difficulties. By taking the square, we see that the lemma is a consequence of showing that

$$\frac{z}{f} + \frac{z}{g} \quad \text{and} \quad \frac{z^2}{fg}$$

are increasing.

REDUCTION 2. Let us assume that we already know that a^2/fg is increasing. We have

$$\frac{z}{f} + \frac{z}{g} = \frac{z}{(fg)^{1/2}} \cdot \left(\left(\frac{f}{g} \right)^{1/2} + \left(\frac{g}{f} \right)^{1/2} \right). \quad (\text{B1})$$

The first factor is increasing by assumption. We therefore have (B1) once we can show that the second factor is also increasing. We claim that this is equivalent to showing that f/g is decreasing in z , for $\sigma \in [0, 1/2]$. Let

$$F^2(z) = \frac{f}{g}. \quad (\text{B2})$$

Using L'Hopital's rule, we obtain

$$F(0) = \lim_{z \rightarrow 0} \frac{f}{g} = 1. \quad (\text{B3})$$

The derivative of the second factor in (B1) is

$$\frac{F'(z)[F^2(z) - 1]}{F^2(z)}. \quad (\text{B4})$$

If we can show that $F(z)$ is decreasing in z , then, in view of (B3), both factors in the numerator are negative (and hence the second factor in (B1) is increasing).

We have thus reduced the problem to proving that $z^2/(fg)$ is increasing and f/g is decreasing in z for all $\sigma \in [0, 1/2]$.

Proof that f/g is decreasing. After simplification and the substitution $y = e^{-z}$,

$$\frac{f}{g} = \frac{1 - y^\sigma - \sigma(1 - y)}{1 - y^{1-\sigma} - (1 - \sigma)(1 - y)},$$

$y \in [0, 1]$, $\sigma \in [0, 1/2]$. The numerator of its derivative, namely, $f'g - fg'$, is

$$X(y) = (4\sigma - 2) + (\sigma^2 - 2\sigma + 1)y^\sigma - \sigma^2 y^{\sigma-1} + (\sigma^2 - 2\sigma + 1)y^{-\sigma} - \sigma^2 y^{1-\sigma}.$$

We need to show that it is nonpositive. Note that $X(1) = 0$; hence, it suffices to show that $X'(y) \geq 0$ in $[0, 1]$, or equivalently that

$$Y(y) = \frac{yX'}{\sigma(1 - \sigma)} = -\sigma y^{1-\sigma} - (1 - \sigma)y^{-\sigma} + (1 - \sigma)y^\sigma + \sigma y^{\sigma-1} \geq 0.$$

Again we see that $Y(1) = 0$; hence, it remains to show that $Y'(y) \leq 0$ in $[0, 1]$. But

$$Y'(y) = -\frac{\sigma(1 - \sigma)(1 - y)(1 - y^{1-2\sigma})}{y^{2-\sigma}} \leq 0,$$

as desired (recall that $0 < \sigma < 1/2$).

Now it remains to prove $z^2/(fg)$ is increasing, or rather its reciprocal

$$K(z, \sigma) = \frac{(1 - e^{-\sigma z} - \sigma(1 - e^{-z}))(1 - e^{-(1-\sigma)z} - (1 - \sigma)(1 - e^{-z}))}{z^2(1 - e^{-z})^2} \quad (\text{B5})$$

is decreasing in $z \geq 0$, for $\sigma \in [0, \frac{1}{2}]$.

REDUCTION 3. Note that $K(z, 0) \equiv 0$. Hence, trivially, $K_z(z, 0) = 0$. If we can show that

$$K_{z\sigma}(z, \sigma) \leq 0, \quad (\text{B6})$$

for $z \geq 0$, $\sigma \in [0, \frac{1}{2}]$, then we know that for a fixed z , K_z is a decreasing function in σ . Since K_z starts out at 0 when $\sigma = 0$, K_z must be nonpositive for all $\sigma > 0$ up to $\sigma = \frac{1}{2}$, and (B5) is then proved. Inequality (B6) is equivalent to showing that K_σ is decreasing in $z \geq 0$, for $\sigma \in [0, \frac{1}{2}]$. After differentiating K with respect to σ , we can separate out those terms in the numerator that have the extra factor z ; $K_\sigma = H + L$, where

$$H = \frac{(1 - 2\sigma)(1 - e^{-z})^2 + e^{-(1-\sigma)z} + e^{-(1+\sigma)z} - e^{-\sigma z} - e^{-(2-\sigma)z}}{z^2(1 - e^{-z})^2} \quad (\text{B7})$$

and

$$L = \frac{\sigma e^{\sigma z} - \sigma e^{-(2-\sigma)z} + (1 - \sigma)e^{-(1+\sigma)z} - (1 - \sigma)e^{-(1-\sigma)z}}{z(1 - e^{-z})^2}. \quad (\text{B8})$$

We have thus made the final reduction to proving that H and L are decreasing in z for $z \geq 0$ and all $\sigma \in [0, 1/2]$.

Proof that H is decreasing. The crucial observation is that the numerator of H has the factor $(1 - e^{-z})$. After cancellation, we have

$$\begin{aligned} H &= \frac{(1 - 2\sigma)(1 - e^{-z}) + e^{-(1-\sigma)z} - e^{-\sigma z}}{z^2(1 - e^{-z})} \\ &= \frac{(1 - 2\sigma)(e^{z/2} - e^{-z/2}) + e^{-(1-\sigma)z/2} - e^{(1-\sigma)z/2}}{z^2(e^{z/2} - e^{-z/2})}. \end{aligned} \quad (\text{B9})$$

The next step involves an obvious change of variables

$$t = 1 - \sigma, \quad y = z/2,$$

and the use of hyperbolic functions. We have

$$H = \frac{t \sinh(y) - \sinh(ty)}{4y^2 \sinh^2(y)}, \quad (\text{B10})$$

$y \geq 0$, $t \in [0, 1]$. Note that

$$H(y, 0) = H(y, 1) = 0.$$

Hence,

$$H_y(y, 0) = H_y(y, 1) = 0. \quad (\text{B11})$$

Let us fix y and see how $R(t) = H_y(\cdot, t)$ changes with t . We need to show that $R(t) \leq 0$ in order to establish that H is decreasing. By (B11),

$$R(0) = R(1) = 0. \quad (\text{B12})$$

By differentiating (B10) twice with respect to t and once with respect to y , we see that

$$R''(t) = \left(-\frac{\sinh(ty)}{4 \sinh(y)} \right)_y = \frac{\cosh(y) \sinh(ty) - t \sinh(y) \cosh(ty)}{4 \sinh^2(y)}, \quad (\text{B13})$$

where the subscript y outside the parentheses denotes partial differentiation in y . If we can show that $R''(t) \geq 0$, then $R(t)$ is a convex function of t and by virtue of (B12), $R(t) \leq 0$ as required. The nonnegativity of R'' is implied by

$$\cosh(y) \sinh(ty) \geq t \sinh(y) \cosh(ty),$$

or equivalently,

$$\frac{y \cosh(y)}{\sinh(y)} \geq \frac{(ty) \cosh(ty)}{\sinh(ty)} \quad t \in [0, 1]. \quad (\text{B14})$$

This is a consequence of the elementary fact that $y \cosh(y)/\sinh(y)$ is an increasing function in y .

Proof that L is decreasing in z . We follow the same line of attack. Denote $S(\sigma) = L_z(z, \sigma)$ for a fixed z . Note that

$$L(z, 0) = L(z, 1/2) \equiv 0 \quad (\text{B15})$$

implies that

$$S(0) = S(1/2) = 0. \quad (\text{B16})$$

Unlike the function $R(t)$, the second derivative of $S(\sigma)$ does not have a simple form from which we can deduce $S''(\sigma) \geq 0$. We first rewrite L in the form

$$L(z, \sigma) = \frac{\sigma \sinh(1 - \sigma)z - (1 - \sigma) \sinh(\sigma z)}{z \sinh^2(z/2)}. \quad (\text{B17})$$

Taking the derivative twice with respect to σ , we have

$$L_{\sigma\sigma} = -2 \frac{\cosh(1 - \sigma)z - \cosh \sigma z}{\sinh^2(z/2)} + z^2 L. \quad (\text{B18})$$

Using the sum and product formula on the first term, we obtain the simplified equation

$$L_{\sigma\sigma} = -4 \frac{\sinh((1/2 - \sigma)z)}{\sinh(z/2)} + z^2 L. \quad (\text{B19})$$

The first term on the right-hand side is an increasing function of z , since it is really the same function occurring in R'' in (B13) (take $y = z/2$, and $t = (1 - 2\sigma) < 1$). Now take the z derivative of (B19) to obtain

$$S''(\sigma) = \left[\left(-\frac{\sinh((1/2 - \sigma)z)}{\sinh(z/2)} \right)_z + 2zL \right] + z^2 S(\sigma). \quad (\text{B20})$$

The terms inside the square brackets are positive. Now suppose that L is not decreasing; then for some z , $S(\sigma)$ must have a strictly positive maximum in $[0, \frac{1}{2}]$. At this maximum, the left-hand side of (B20) is nonpositive while the right-hand side is strictly positive, giving a contradiction.

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