EXISTENCE AND NONEXISTENCE RESULTS ON THE RADIALLY SYMMETRIC CAVITATION PROBLEM

By

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Abstract. We investigate the problem of radially symmetric cavitation for a hyperelastic ball in $\mathbb{R}^n$, $n \geq 2$. The radial equilibrium equation is analyzed by a shooting argument. The basic formulation of the problem is the same as C. A. Stuart's formulation in [10], but an asymptotic study of the solutions of the radial equilibrium equation allows us to enlarge the discussion of cavitation to cases that are excluded from the context of [10]. Finally, criteria for nonexistence to the problem of cavitation are briefly discussed. They have a physical interpretation through relations between the total energy and the radial stress.

1. Introduction.

1.1. *Boundary value problems and initial value problem (IVP).* We consider radial deformations of the unit ball $B(0, 1)$ in $\mathbb{R}^n$, $n \geq 2$. In particular, we are interested in the question of existence and nonexistence of a solution to the problem of radially symmetric cavitation of the ball. In other words, under what conditions on the stored energy function does the equilibrium state of the ball present a radial hole at the center? A radially symmetric equilibrium is a solution of the form $u(x) = U(r)x/r$ to the general equilibrium equation

$$\operatorname{div}(S(\nabla u(x))) = 0, \quad (1.1)$$

where $S(\nabla u)$ is the first Piola-Kirchhoff stress tensor (see [4]). The medium is supposed to be homogeneous and hyperelastic. Then there exists a stored energy function $W : M_+ \rightarrow \mathbb{R}$ where $M_+ = \{F, n \times n \text{ matrix: } \det(F) > 0\}$ such that

$$S(F) = \partial W(F) / \partial F, \quad \forall F \in M_+. \quad (1.2)$$

Furthermore we suppose that the medium is isotropic and frame indifferent. In this case it is well known (see [4]) that there exists a symmetric function $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$W(F) = \Phi(\lambda_1, \dots, \lambda_n),$$

where λ_i is the i th proper value of $(F^t F)^{1/2}$. The λ_i are the principal stretches associated with the deformation u and the λ_i are subjected to the usual restriction, $\lambda_i > 0$, $\forall i = 1, \dots, n$. For the radial case, a direct calculation gives $\lambda_1 =$

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$U'(r)$, $\lambda_2 = \lambda_3 = \dots = \lambda_n = U(r)/r$. For a stored energy function given by $\Phi(U(r), U(r)/r, \dots, U(r)/r)$ the radial equilibrium equation has the form

$$r\Phi_{11}U''(r) = (n-1)\left(\Phi_2 - \Phi_1 - \left(U'(r) - \frac{U(r)}{r}\right)\Phi_{12}\right), \quad (1.4)$$

where Φ_i is the i th partial derivative of $\Phi(\lambda_1, \lambda_2, \dots, \lambda_n)$ at the point $(U(r), U(r)/r, \dots, U(r)/r)$.

The question of cavitation was first studied by J. Ball in [1] in the context of the direct methods of the calculus of variations. His approach has been extended by Sivaloganathan in [8, 9]. In preliminary results J. Ball makes precise the sense in which solutions of (1.4), which can be singular at $r = 0$, are radial solutions of the equilibrium equation (1.1) in the ball $B(0, 1)$ by introducing a definition of weak solution of (1.1).

DEFINITION 1.1.1. Let Ω be a bounded domain in $\mathbb{R}^n$. u is a weak solution of (1.1) in $\Omega \Leftrightarrow u \in W^{1,1}(\Omega)$, $\nabla u \in M_+$ a.e. Ω , $S(\nabla u) \in L^1(\Omega)$, and

$$\int_{\Omega} \text{tr}(S(\nabla u), \nabla V^t) dx = 0, \quad \forall V \in C_0^\infty(\Omega).$$

For radial deformation, the following theorem holds under mild assumptions on the stored energy function [1].

THEOREM 1.1.2. Let Φ satisfy (A1), (A2), and (A8) of section 1.2, and let $u(x) = U(r)x/r$, $u(x) \in W^{1,1}(B(0, 1))$. Then u is a weak solution of (1.1) if and only if

$$\begin{cases} \text{(i)} & U(r) \in C^2((0, 1]), \quad U(r) > 0, \quad U'(r) > 0, \quad 0 < r \leq 1; \\ \text{(ii)} & r^{n-1}\Phi_i\left(U'(r), \frac{U(r)}{r}, \dots, \frac{U(r)}{r}\right) \in L^1(0, 1), \quad i = 1, 2; \\ \text{(iii)} & r \frac{d}{dr}\Phi_1 = (n-1)(\Phi_2 - \Phi_1), \quad 0 < r \leq 1. \end{cases} \quad (\text{REE})$$

The equation 1.1.2(iii) is an equivalent form for (1.4). This theorem gives a relation between solutions $U(r)$ of (1.4) and weak solutions $u(x)$ of (1.1) on $B(0, 1)$. The boundary value problems that we consider will give a precise sense to $u(x) = U(r)x/r$ on $B(0, 1)$ if we require that $U(r)$ satisfies the conditions (REE). It is shown in [10] that it is a direct consequence of the sufficient conditions for existence of solutions to the problem of cavitation.

The question of cavitation gives rise to the following two boundary value problems. Find $U \in C^2((0, 1])$ such that $U(r)$ satisfies (REE), $U(1) = \lambda$, and either

$$\lim_{r \rightarrow 0} U(r) = U(0) = 0 \quad (1.5)$$

or

$$\lim_{r \rightarrow 0} U(r) > 0 \quad \text{and} \quad \lim_{r \rightarrow 0} T(r) = 0. \quad (1.6)$$

$U(1) = \lambda$ characterises a displacement problem. The condition $U(0) = 0$ will give continuous solutions since the deformation $u(x)$ is continuous at $x = 0$. On the contrary, any solution satisfying (1.6) corresponds to a discontinuous deformation at the origin of the ball. $T(r)$ is the normal component of the Cauchy stress tensor

and the last boundary condition in (1.6) means that if the deformation opens a hole at the center of the ball, the pressure in the cavity is supposed to be zero. In this paper we study the question of existence and nonexistence to these boundary value problems with a shooting argument (proposed by Stuart in [10]). We then consider the following initial value problem:

$$\begin{cases} U''(r) = \frac{n-1}{r\Phi_{11}} \left\{ \Phi_2 - \Phi_1 - \left(U'(r) - \frac{U(r)}{r} \right) \Phi_{12} \right\}, \\ U(1) = \lambda, \quad U'(1) = \alpha, \\ U(r) > 0, \quad U'(r) > 0, \quad 0 < r \leq 1. \end{cases} \quad (\text{IVP})$$

All solutions of (REE) are solutions of (IVP). In particular, for a fixed $\lambda > 0$ solutions of (1.5) or (1.6) must occur as solutions of (IVP) denoted $U(\lambda, \alpha)$ for some particular values α (depending on λ) such that the maximal interval $J(\lambda, \alpha) = (0, 1]$ and such that $U(\lambda, \alpha)$ satisfies one of the second boundary conditions at $r = 0$.

1.2. *Description of results.* We first present the main result obtained in [10], which will make clear the motivation of our approach of problem (1.6). In particular, we explain why we are led to the study of the behavior of the solutions of (IVP) near $r = 0$, which occupies the main part of this paper. This study is based on an appropriate method and gives a generalisation of [10], which is illustrated for a wide class of Ogden materials.

Our study of (IVP) begins with similar basic hypotheses to those of [10].

- (A1) Φ is symmetric and $\Phi \in C^3((0, \infty)^n; \mathbb{R}^+)$.
- (A2) $\Phi_{11}(q, t, \dots, t) > 0, \forall q, t > 0$.
- (A3) $\Phi_2(q, t, \dots, t) - \Phi_1(q, t, \dots, t) - (q - t)\Phi_{12}(q, t, \dots, t) > 0, \forall q, t > 0$.
- (A4)

$$R(q, t) = \frac{q\Phi_1(q, t, \dots, t) - t\Phi_2(q, t, \dots, t)}{q - t} \geq 0, \quad \forall q, t > 0, \quad q \neq t.$$

These hypotheses are common to different articles devoted to the problem of cavitation [1, 8–11]. Basic consequences of (A1)–(A4) show that the unique solution of (1.5) is $U(\lambda, \lambda)(r) = \lambda r$ and that solutions of (1.6) have to be searched for in $D = \{(\lambda, \alpha) : 0 < \alpha < \lambda\}$. For this purpose we introduce the following change of variable in (IVP) [10, 1],

$$t = \frac{U(\lambda, \alpha)(r)}{r}, \quad q(t) = U'(\lambda, \alpha)(r), \quad (\lambda, \alpha) \in D. \quad (1.7)$$

One can verify that $q(t)$ satisfies the initial value problem

$$\begin{cases} q'(t) = \frac{(n-1)}{\Phi_{11}} \left\{ \frac{\Phi_2 - \Phi_1}{q - t} - \Phi_{12} \right\}, \quad \text{where } \Phi_i = \Phi_i(q(t), t, \dots, t), \\ q(t) > 0, \quad \forall t \geq \lambda, \\ q(\lambda) = \alpha, \quad 0 < \alpha < \lambda. \end{cases} \quad (\text{P})$$

The basic lemmas obtained by (A1)–(A3) show that a solution of (P) defined on $[\lambda, \infty)$ is a solution of (IVP) defined on $(0, 1]$, i.e., $\inf J(\lambda, \alpha) = 0$ and

$U(\lambda, \alpha)(0) > 0$. Moreover, the asymptotic behaviour of $q(t)$ at infinity corresponds to the asymptotic behaviour of $U'(\lambda, \alpha)$ near $r = 0$. In what follows the motivation for this study of (IVP) is explained.

First we present the main results of [10]. Let

$$E(\lambda, \alpha) = \omega_n \int_0^1 r^n \Phi \left(U'(\lambda, \alpha)(r), \frac{U(\lambda, \alpha)(R)}{r}, \dots, \frac{U(\lambda, \alpha)(r)}{r} \right) dr, \quad (1.8)$$

where ω_n is the $(n-1)$ -dimensional surface of the unit sphere,

be the total energy associated to a solution of (IVP) defined on $(0, 1]$, and let

$$T(\lambda, \alpha)(r) = \left(\frac{U(r)}{r} \right)^{1-n} \Phi_1 \left(U'(r), \frac{U(r)}{r}, \dots, \frac{U(r)}{r} \right), \quad \text{where } U = U(\lambda, \alpha), \quad (1.9)$$

be the normal component of Cauchy stress tensor.

Now for $0 < \alpha < \lambda$, if $U(\lambda, \alpha)$ is defined on $(0, 1]$, (A1)–(A4) shows that one can define

$$\tau(\lambda, \alpha) = \lim_{r \rightarrow 0} T(\lambda, \alpha)(r). \quad (1.10)$$

$\tau(\lambda, \alpha)$ is the value of the Cauchy stress tensor on the surface of the cavity of radius $U(\lambda, \alpha)(0)$. The conditions (1.6) require that the cavity be empty and define the boundary condition $\tau(\lambda, \alpha) = 0$. The approach to problem (1.6) consists of showing the existence of a critical value $\lambda^* > 0$ such that for $\lambda > \lambda^*$ the equation $\tau(\lambda, \alpha) = 0$ admits a unique solution in $D = \{(\lambda, \alpha) : 0 < \alpha < \lambda\}$, noted $(\lambda, w(\lambda))$, and has no solution for $\lambda < \lambda^*$. To get these results about existence, Stuart completes the assumptions on the stored energy function in the following manner.

(A'5) $\exists A, B > 0, \beta > 0$ such that $R(q, t) < A + Bt^\beta, \forall 0 < q < t$.

(A'6) $\exists K > 0, T > 0, \gamma < 2(n-1)$ such that $|\frac{\partial R(q, t)}{\partial q}| < Kt^\gamma, \forall 0 < q < t, \forall t > T$.

(A'7) $\exists C > 0$ such that $\Phi_{11}(q, t, \dots, t) \geq Ct^{2(n-1)}, \forall 0 < q < t$.

(A'8) $\forall b > 0,$

$$\lim_{(q, t) \rightarrow (0, b)} t^{1-n} \Phi_1(q, t, \dots, t) = -\infty,$$

$$\lim_{(q, t) \rightarrow (b, \infty)} t^{1-n} \Phi_1(q, t, \dots, t) = +\infty,$$

(A'9) $\lim_{t \rightarrow 0} t^{1-n} \Phi_1(t, \dots, t) = -\infty, \lim_{t \rightarrow \infty} t^{1-n} \Phi_1(t, \dots, t) = +\infty$.

The following theorem summarises the main results of [10].

THEOREM 1.2.1. Let $\Phi(q, t, \dots, t)$ satisfy (A1)–(A4), (A'5)–(A'9).

(i) If in (A'5) $\beta < n$, then all the $U(\lambda, \alpha)$ are defined on $(0, 1]$ and $u(x) = U(\lambda, \alpha)(r)x/r$ is a weak radial equilibrium solution in $B(0, 1)$ for $0 < \alpha < \lambda$.

(ii) If in (A'5) $\beta < n-1$, all these solutions have a finite energy. Moreover $\exists \lambda^* > 0$ and $w \in C^1((\lambda^*, \infty), \mathbb{R})$, strictly decreasing such that $0 < w(\lambda) < \lambda, \forall \lambda > \lambda^*, \lim_{\lambda \rightarrow \lambda^*} w(\lambda) = \lambda^*, \lim_{\lambda \rightarrow \infty} w(\lambda) = 0$, and $\{(\lambda, \alpha) \in D : \tau(\lambda, \alpha) = 0\} = \{(\lambda, w(\lambda)) : \lambda > \lambda^*\}$.

The solution of (IVP) $U(\lambda, w(\lambda))$ defined by (ii) is then the unique solution of (1.6) and Theorem 1.1.2(i) shows that it is a weak radial equilibrium solution in the sense given by Definition 1.1.1.

The set of assumptions (A'5)–(A'7) gives rise to a remark that is at the origin of our approach to the problem of cavitation. The assumptions (A'5)–(A'7) are uniform estimations in the variable q and are quite restrictive. In fact, (A'5)–(A'7) exclude stored energy functions like that of [5], which exhibits the phenomenon of radial cavitation. The example of [5] is known as a Blatz and Ko Material and is defined by

$$\Phi(q, t) = \frac{\mu}{2} \left\{ \frac{1}{q^2} + \frac{1}{t^2} + 2qt - 4 \right\}, \quad \text{where } \mu > 0. \quad (1.11)$$

In that simple case one verifies that

$$\forall T > 0, \quad \lim_{(q, t) \rightarrow (0, T)} R(q, t) = \infty, \quad (1.12)$$

which is clearly incompatible with (A'5). Moreover, one has the estimation

$$\exists A, B, C \geq 0, \quad \beta, \nu, \xi \geq 0 \text{ such that } R(q, t) < A + Bt^\beta + Ct^\xi q^{-\nu}, \quad \forall 0 < q < t. \quad (1.13)$$

The main purpose of this paper is to include situations for which estimations like (1.13) hold in a new existence theorem.

A solution to this problem is based on the following remark. Since we consider only the solutions of (IVP), the proofs leading to Theorem 1.2.1 make use of (A'5)–(A'7) only for $q = q(t)$ ($q(t)$ is a solution of (P) defined on $[\lambda, \infty)$), and with (A1)–(A4), what we know about $q(t)$ is only that

$$0 < q(t) < t, \quad \forall t \geq \lambda.$$

That explains why the domain of validity of (A'5)–(A'7) is of the form

$$M = \{(q, t) : 0 < q < t, \quad t > T\}, \quad \text{with } T > 0.$$

In this paper we show that an asymptotic study of $q(t)$, for large t , leads to replacing M by the appropriate smaller set,

$$S = \{(q, t) : t^{-\nu} < q < t^{-\delta}, \quad \forall t > T\}, \quad \text{with } T > 0.$$

Thus if $R(q, t)$ satisfies the estimation (1.13) in M , then a uniform estimation, independent of q , holds for $R(q, t)$ in S . It is consequently possible to follow the ideas of [10] for getting a new existence theorem by using hypotheses similar to (A'5)–(A'7) but formulated on this smaller set S .

This existence theorem provides a generalisation of Theorem 1.2.1 in the sense that for any stored energy function of the form

$$\begin{aligned}\Phi(\lambda_1, \lambda_2, \lambda_3) &= \sum_{i=1}^3 \phi(\lambda_i) + \sum_{i=1}^3 \kappa(\lambda_i \lambda_{i+1}) + h(\lambda_1 \lambda_2 \lambda_3) \\ \text{for } \phi(s) &= \sum_{i=1}^K (A_i s^{\alpha_i} + C_i s^{-\gamma_i}), \quad K \geq 1; \\ \kappa(s) &= \sum_{j=1}^L (B_j s^{\beta_j} + D_j s^{-\delta_j}), \quad L \geq 1; \\ h(p) &= \sum_{m=1}^M (H_m p^{\sigma_m} + K_m p^{-\tau_m}), \quad M \geq 1;\end{aligned}$$

where $\alpha_i, \beta_j > 1$, $A_i, B_j, C_i, D_j \geq 0$, $\gamma_i, \delta_j \geq 0$, $\forall i = 1, 2, \dots, K$, $\forall j = 1, 2, \dots, L$, and $H_m, K_m > 0$, $\sigma_m > 1$, $\tau_m > 0$, $\forall m = 1, 2, \dots, M$, our approach applies whereas the one of [10] fails if $C_i > 0$, $\gamma_i > 0$ or $D_j > 0$, $\delta_j > 0$ for some i, j . In fact, a direct calculation shows that $R(q, t)$ satisfies (1.12).

The set of assumptions under which one gets our existence theorem is then the following:

- (A5) $\exists A, B > 0$, $\beta > 0$ such that $R(q, t) < A + Bt^\beta$, $\forall (q, t) \in S$.
- (A6) $\exists K > 0$, $\gamma_1 > 0$ such that $|\frac{\partial R(q, t)}{\partial q}| < Kt^{\gamma_1}$, $\forall (q, t) \in S$.
- (A7) $\exists C > 0$, $\gamma_2 > \gamma_1$ such that $\Phi_{11}(q, t, \dots, t) \geq Ct^{\gamma_2}$, $\forall (q, t) \in S$.
- (A8) $\forall b > 0$,

$$\begin{aligned}\lim_{(q, t) \rightarrow (0, b)} t^{1-n} \Phi_1(q, t, \dots, t) &= -\infty, \\ \lim_{(q, t) \rightarrow (b, \infty)} t^{1-n} \Phi_1(q, t, \dots, t) &> 0.\end{aligned}$$

- (A9) $\lim_{t \rightarrow 0} t^{1-n} \Phi_1(t, \dots, t) = -\infty$, $\lim_{t \rightarrow \infty} t^{1-n} \Phi_1(t, \dots, t) = \infty$.

The introduction of constants $\gamma_2 > \gamma_1 > 0$ in (A6) and (A7) represents only a secondary and straightforward generalisation of [10]. Nevertheless the example [5] satisfies (A5)–(A7), but not with $\gamma_2 \geq 2(n-1)$ and $\gamma_1 < 2(n-1)$ as we shall see later. (A8) is only slightly weakened and (A9) is not changed. The analysis of the asymptotic behaviour of the solutions of (P) requires new assumptions on the stored energy function. These are expressed on a new function $G(q, t)$, defined by writing (P) in the following way:

$$\begin{cases} q'(t) = -\frac{q}{t}(G(q(t), t)), & \forall t \geq \lambda, \\ q(\lambda) = \alpha, & 0 < \alpha < \lambda, \\ q(t) > 0. \end{cases} \quad (P')$$

In this study we suppose that $\Phi(q, t, \dots, t)$ satisfies (A1)–(A3) and that $G(q, t)$ satisfies

$$\forall T > 0, \quad \limsup_{(q, t) \rightarrow (0, T)} G(q, t) < \infty. \quad (P1)$$

FIG. 1. Illustration of the sets A^ν and A_b^δ .

$$\begin{aligned} \exists \nu > 0, \quad T(\nu) > 0 \text{ such that } G(q, t) < \nu, \\ \forall (q, t) \in A^\nu = \{(q, t) : 0 < q < t^{-\nu}, \quad t > T(\nu)\}. \end{aligned} \quad (\text{P2})$$

$$\begin{aligned} \exists \delta > 0 \text{ such that } \forall b > 0 \exists T(\delta, b) > 0 \text{ such that } G(q, t) > \delta, \\ \forall (q, t) \in A_b^\delta = \{(q, t) : t^{-\delta} < q < b, \quad t > T(\delta, b)\}. \end{aligned} \quad (\text{P3})$$

The assumptions (P1)–(P3) with (A1)–(A3) give the main theorem of this paper.

THEOREM 1.2.2. Let $\Phi(q, t, \dots, t)$ satisfy (A1)–(A3) and (P1)–(P3). Let $\lambda > 0$, X be a compact subset of $(0, \lambda]$, and let $Q = \{q(\lambda, \alpha) : q(\lambda, \alpha) \text{ solution of (P) with } \alpha \in X\}$. Then there exist $\nu_0, \delta_0, T > \max\{1, \lambda\}$ such that

- (i) $0 < \delta_0 \leq \nu_0$,
- (ii) $\forall \varepsilon > 0, \quad t^{-\nu_0-\varepsilon} < q(\lambda, \alpha)(t) < t^{-\delta_0+\varepsilon}, \quad \forall t > T, \quad \forall q(\lambda, \alpha) \in Q$.

This theorem is proved in the next section and will be illustrated for a wide class of Ogden materials. Finally, these preliminary considerations on the radial equilibrium equation give the following generalisation of Theorem 1.2.1.

THEOREM 1.2.3. Let $\Phi(q, t, \dots, t)$ satisfy (A1)–(A9) and (P1)–(P3). Then the conclusions (i) and (ii) of Theorem 1.2.1 hold.

The proof of Theorem 1.2.3 differs from that of Theorem 1.2.2 only because of the introduction of the constants γ_2, γ_1 in (A6) and (A7). These details are given in [7] and will not be discussed here.

It is important to remark that (P1)–(P3) have no justification coming from nonlinear elasticity. The justification for (P1)–(P3) given here is that they hold for a quite general empirical model of stored energy function, the Ogden material, which is able to describe phenomena related to large elastic deformations. This form corresponds

to many empirical models of the stored energy function. From that point of view we have considered that the assumptions (P1)–(P3) are reasonable.

This paper is concluded by two criteria for nonexistence of solution to the problem of cavitation. They have a physical interpretation through two relations involving the radial stress and the total energy. The last relation gives a further important property of the energy of the solution of (IVP) in $D = \{(\lambda, \alpha) : 0 < \alpha < \lambda\}$ and also generalises the results of [10].

2. Asymptotic behaviour of the solutions of (IVP). We first give the lemmas that show the solutions of (1.6) must be solutions of (P) on $[\lambda, \infty)$ and the unique solution to (1.5) is $U(\lambda, \lambda)(r) = \lambda r$.

LEMMA 2.1.1. Let $\Phi(q, t, \dots, t)$ satisfy (A1)–(A4), and let $U(\lambda, \alpha)(r) : J(\lambda, \alpha) \rightarrow \mathbb{R}$ be a solution of (IVP). Then the $U(\lambda, \alpha)(r)$ has the following properties:

- (i) $0 < \alpha < \lambda : (\frac{U(\lambda, \alpha)(r)}{r})' < 0, T'(\lambda, \alpha)(r) \geq 0, U''(\lambda, \alpha)(r) > 0$, for $r \in J(\lambda, \alpha)$;
- (ii) $0 < \alpha = \lambda : U(\lambda, \alpha)(r) = \lambda r, 0 \leq r \leq 1$;
- (iii) $0 < \lambda < \alpha : (\frac{U(\lambda, \alpha)(r)}{r})' > 0, T'(\lambda, \alpha)(r) \leq 0, U''(\lambda, \alpha)(r) < 0$, for $r \in J(\lambda, \alpha)$.

Proof. cf. [10, Lemmas 2 and 3].

LEMMA 2.1.2. Let $\Phi(q, t, \dots, t)$ satisfy (A1)–(A3), and let $U(\lambda, \alpha)(r) : J(\lambda, \alpha) \rightarrow \mathbb{R}$ be a solution of (IVP). Then

- (i) for $0 < \lambda < \alpha : \inf J(\lambda, \alpha) > 0$;
- (ii) for $0 < \alpha < \lambda : \lim_{r \rightarrow 0} U(\lambda, \alpha)(r) > 0$, whenever $\inf J(\lambda, \alpha) = 0$.

Proof. cf. [10, Lemmas 2 and 3].

The discussion is now devoted to problem (1.6). We focus our attention on the solution of (IVP) in $D = \{(\lambda, \alpha) : 0 < \alpha < \lambda\}$. Let $U(\lambda, \alpha)$ be a solution of (IVP) in D . By Lemmas 2.1.1 and 2.1.2 we can define

$$t = \frac{U(\lambda, \alpha)(r)}{r}, \quad q(t) = U'(\lambda, \alpha)(r). \quad (2.1)$$

A direct calculus shows that $q(t)$ satisfies the initial value problem (P). As noted before we first have to show that each solution of (P) is defined on $[\lambda, \infty)$. Then we will establish the desired asymptotic properties of $q(t)$.

First we remark that if $G(q, t)$ satisfies (P2) and (P3) for ν_1, δ_1 then $G(q, t)$ satisfies (P2) and (P3) for $\nu > \nu_1$ and $\delta < \delta_1$. Moreover we must have $\nu_1 \geq \delta_1$. Thus by introducing

$$\nu_0 = \inf\{\nu : \nu \text{ satisfying (P2)}\}, \quad \delta_0 = \sup\{\delta : \delta \text{ satisfying (P3)}\}, \quad (2.2)$$

we must have $0 \leq \delta_0 \leq \nu_0$. Note that (P3) ensures that δ_0 is finite. In all of what follows, $q(t) = q(\lambda, \alpha)(t)$ for a fixed α . Now from (P') we obtain by a simple integration a representation of a solution $q(t)$:

$$q(t) = \alpha \exp\left(-\int_{\lambda}^t \frac{G(q(s), s)}{s} ds\right). \quad (2.3)$$

This immediately gives the following lemma.

LEMMA 2.1.3. Let $G(q, t)$ satisfy (P1). Then $q(t)$ is defined on $[\lambda, \infty)$.

Proof. As $G(q, t)$ satisfies (P1) we have $q'(t) < 0$, $\forall t \geq \lambda$. With (P2) and the formula (2.3) for $q(t)$, it follows straightforwardly that if $\lim_{t \rightarrow T} q(t) = 0$ then $T = \infty$. $\square$

PROPOSITION 2.1.4. Let $G(q, t)$ satisfy (P1), (P2) for $\nu > \nu_0$. Then $\exists T \geq \max\{1, \lambda\}$ such that $q(t) \geq t^{-\nu}$, $\forall t \geq T$.

Proof. Let $\nu > \nu_0$. Then $\exists \nu_0 < \nu_1 < \nu$ such that (P_{ν}) holds for ν_1 with $1 \leq T(\nu) \leq T(\nu_1)$. Let $B^\nu = \{t > T(\nu_1) : q(t) < t^{-\nu}\}$. Clearly B^ν is open and it is sufficient to show that B^ν is bounded or empty.

If B^ν is nonempty, let $J = (t_1, t_2)$ be a nonempty connected component of B^ν . For $t \in JCB^\nu$, $(q(t), t) \in A^{\nu_1}$. Consequently,

$$q'(t) > -\nu_1 \frac{q(t)}{t}, \quad \text{for } t \in J. \quad (2.4)$$

Hence by integration of (2.4) we find that,

$$\frac{q(s)}{q(t)} > \left(\frac{s}{t}\right)^{-\nu_1}, \quad \text{for } t_1 < t < s < t_2.$$

Since $(q(t), t) \in A^\nu$, $\forall t \in J$,

$$\left(\frac{s}{t}\right)^{-\nu_1} < \frac{q(s)}{q(t)} < \left(\frac{s}{q(t)}\right)^{-\nu}, \quad \text{for } t_1 < t < s < t_2. \quad (2.5)$$

Now $\nu_1 < \nu$ implies that $t_2 < \infty$. Furthermore $q(t_1) = t_1^{-\nu}$ and $q(t_2) = t_2^{-\nu}$. Then, returning to (2.5), we deduce that

$$\left(\frac{t_2}{t_1}\right)^{-\nu} = \frac{q(t_2)}{q(t_1)} \geq \left(\frac{t_1}{t_2}\right)^{-\nu_1}, \quad (2.6)$$

contradicting $t_1 > t_2$. Hence if B^ν is nonempty, then $B^\nu = (T(\nu_1), t_2)$ where $t_2 < \infty$. $\square$

PROPOSITION 2.15. Let $G(q, t)$ satisfy (P1)–(P3) for $\delta > \delta_0$. Then $\exists T \geq \max\{1, \lambda\}$ such that $q(t) \leq t^{-\delta}$, $\forall t \geq T$.

Proof. Let $\delta < \delta_0$; then $\exists \delta_1 > \delta$ such that (P3) holds for δ_1 . By (P1), $0 < q(t) < \lambda$. Consequently, we can set $b = \lambda$ and (P3) holds for δ with $1 \leq T(\delta, \lambda) \leq T(\delta_1, \lambda)$. Let $C^\delta = \{t > T(\delta, \lambda) : q(t) > t^{-\delta}\}$. Clearly C^δ is open. It is sufficient to show that C^δ is bounded or empty.

If C^δ is nonempty, let $J = (t_1, t_2)$ be a nonempty connected component of C^δ . For $t \in JCC^\delta$, we have that $(q(t), t) \in A_\lambda^\delta CA_\lambda^{\delta_1}$ and so,

$$q'(t) < -\delta_1 \frac{q(t)}{t}, \quad \text{for } t \in J. \quad (2.7)$$

By integration of (2.7),

$$\frac{q(s)}{q(t)} < \left(\frac{s}{t}\right)^{-\delta_1}, \quad \text{for } t_1 < t < s < t_2. \quad (2.8)$$

Since $(q(t), t) \in A_\lambda^\delta$, $\forall t \in J$,

$$\left(\frac{s}{q(t)}\right)^{-\delta} < \frac{q(s)}{q(t)} < \left(\frac{s}{t}\right)^{-\delta_1}, \quad \text{for } t_1 < t < s < t_2. \quad (2.9)$$

Now, $\delta_1 > \delta$ implies that $t_2 < \infty$. As in the proof of Proposition 2.1.4 it follows that if C^δ is nonempty then $C^\delta = (T(\nu_1), t_2)$, where $t_2 < \infty$. $\square$

To complete the proof of Theorem 1.2.2 we must prove a certain uniformity of these estimations on compact subsets of $(0, \lambda]$.

This uniformity is a direct consequence of basic theorems on ordinary differential equations. The continuous dependence theorem and the existence and uniqueness theorem of Picard immediately give the continuity and the monotonicity of $q(\lambda, \cdot)(t)$ with respect to α . In fact,

$$\text{for } \mu < \alpha, \quad q(\lambda, \mu)(t) < q(\lambda, \alpha)(t), \quad \forall t \geq \lambda.$$

Now by applying Proposition 2.1.5 for $\alpha = \lambda$, the $T(\delta, \lambda)$ we get holds for all $\mu \leq \lambda$. Then the uniformity of this estimation in α on X , a compact subset of $(0, \lambda]$, follows by choosing $T(X) = T(\delta, \lambda)$.

On the other hand, $\sigma = \inf X > 0$. Then by applying Proposition 2.1.4 for $\alpha = \sigma$, the $T(\nu, \sigma)$ we get holds for all $\mu \in X$, since the monotonicity of $q(\lambda, \cdot)(t)$ implies that

$$t^{-\nu} < q(\lambda, \sigma)(t) \leq q(\lambda, \mu)(t), \quad \forall t > T(\nu, \sigma), \quad \forall \mu \in X.$$

This closes the demonstration of Theorem 1.2.2.

3. Application to a class of Ogden materials. In this section we apply the preceding results to a particular three-dimensional model of a stored energy function, called Ogden materials. For $n = 3$, $\lambda_1 = U'(r)$, $\lambda_2 = \lambda_3 = U(r)/r$. The model of Ogden materials supposes that a stored energy function can be written in the form

$$\Phi(\lambda_1, \lambda_2, \lambda_3) = \sum_{i=0}^3 \phi(\lambda_i) + \sum_{i=0}^3 \kappa(\lambda_i \lambda_{i+1}) + h(\lambda_1 \lambda_2 \lambda_3), \quad \lambda_4 = \lambda_1.$$

The hypotheses on the function ϕ , κ , h are

- (H1) $\phi, \kappa, h \in C^3((0, \infty); \mathbb{R}^+)$.
- (H2) $\phi''(s) \geq 0$, $\kappa''(s) \geq 0$, $\forall s > 0$, $h''(p) > 0$, $\forall p > 0$.
- (H3) $\exists \sigma_1 \geq 2$, $H_1 > 0$ such that $\lim_{p \rightarrow 0} p^{\sigma_1} h''(p) = H_1$.
- (H4) $\exists \sigma_2 \geq 0$, $H_2 > 0$ such that $\lim_{p \rightarrow \infty} p^{-\sigma_2} h''(p) = H_2$.
- (H5) $\exists \nu_1, \mu_1 \geq 0$ and $A_1, B_1 \geq 0$ such that

$$\lim_{s \rightarrow 0} s^{\nu_1} \phi''(s) = A_1, \quad \lim_{s \rightarrow 0} s^{\mu_1} \kappa''(s) = B_1,$$

- (H6) $\exists \nu_2, \mu_2 \geq 0$ and $A_2, B_2 \geq 0$ such that

$$\lim_{s \rightarrow \infty} s^{-\nu_2} \phi''(s) = A_2, \quad \lim_{s \rightarrow \infty} s^{-\mu_2} \kappa''(s) = B_2.$$

The function h is related to the energy associated with changes of volume. For the three-dimensional case, we have made some reinforced convexity and growth hypotheses on $h(p)$. A discussion about necessary and sufficient conditions for the existence of a solution to the problem of cavitation gives some justification of this choice (see [7, Chapter 3]). In this section these assumptions on $h(p)$ play an important role. In (H5) $\nu_1, \mu_1 \geq 0$ means that we treat both the singular and regular cases for ϕ, κ . The case where ϕ, κ are regular at the origin corresponds to the situation in [10].

Let us first remark that (H1)–(H2) give (A1)–(A4) and consequently all the results related to them. In fact,

$$R(q, t) = \frac{q\phi'(q) - t\phi'(t)}{q - t} + \frac{t(q\kappa'(qt) - t\kappa'(t^2))}{q - t}, \quad (3.1)$$

$$\frac{\Phi_2 - \Phi_1}{q - t} - \Phi_{12} = \frac{\phi'(t) - \phi'(q)}{q - t} + \frac{t(\kappa'(t^2) - \kappa'(qt))}{q - t} - qt\kappa'(qt) - qt^3h''(qt^2); \quad (3.2)$$

$$\Phi_{11} = \phi''(q) + 2t^2\kappa''(qt) + t^4h''(qt^2). \quad (3.3)$$

We shall now examine how (H1)–(H6) give (P1)–(P3). Lemma 3.1.1 gives two basic estimations on the right-hand side $F(q, t)$ of the equation for $q(t)$ that depend only on the hypotheses of convexity on ϕ, κ, h .

LEMMA 3.1.1. Let $\Phi(q, t, t)$ satisfy (H1), (H2), and let $F(q, t)$ be the right-hand side of the equation (P). Then for $0 < q < t$,

$$\begin{aligned} \text{(i)} \quad \frac{1}{2}F(q, t) &\geq -\frac{q}{t} \left\{ \frac{t}{t - q} + \frac{t(\phi'(t) + t\kappa'(t^2))}{q\Phi_{11}(t - q)} \right. \\ &\quad \left. - \frac{t(q\phi''(q) + \phi'(q))}{q\Phi_{11}(t - q)} - \frac{t^2(qt\kappa''(qt) + \kappa'(qt))}{q\Phi_{11}(t - q)} \right\}. \\ \text{(ii)} \quad \frac{1}{2}F(q, t) &\leq -\frac{q}{t} \left\{ 1 + \frac{t(\phi'(t) + t\kappa'(t^2))}{q\Phi_{11}(t - q)} \right. \\ &\quad \left. - \left| \frac{t(q\phi''(q) + \phi'(q))}{q\Phi_{11}(t - q)} \right| - \left| \frac{t^2(qt\kappa'(qt) + \kappa'(qt))}{q\Phi_{11}(t - q)} \right| \right\}. \end{aligned}$$

Proof. We know that

$$\begin{aligned} \frac{\Phi_2 - \Phi_1}{q - t} - \Phi_{12} &= \frac{\phi'(t) - \phi'(q)}{q - t} + \frac{t(\kappa'(t^2) - \kappa'(qt))}{q - t} - qt\kappa''(qt) - qt^3h''(qt^2), \\ \Phi_{11} &= \phi''(q) + 2t^2\kappa''(qt) + t^4h''(qt^2). \end{aligned}$$

Consequently,

$$\frac{1}{2}F(q, t) = \frac{1}{\Phi_{11}} \left[\frac{\phi'(t) - \phi'(q)}{q - t} + \frac{t(\kappa'(t^2) - \kappa'(qt))}{q - t} - qt\kappa'(qt) - qt^3h''(qt^2) \right]. \quad (3.4)$$

By adding and subtracting

$$\frac{1}{\Phi_{11}} \frac{q\phi''(q)}{t - q} \quad \text{and} \quad \frac{1}{\Phi_{11}} \frac{qt^2\kappa''(qt)}{t - q}, \quad (3.5)$$

(3.4) becomes

$$\frac{1}{2}F(q, t) = \frac{1}{\Phi_{11}} \left\{ \frac{\phi'(t) - \phi'(q)}{q - t} + \frac{q\phi''(q)}{t - q} + \frac{t(\kappa'(t^2) - \kappa'(qt))}{q - t} + \frac{qt^2\kappa'(qt)}{t - q} - \frac{q}{t - q}(\phi''(q) + t(t + (t - q))\kappa''(qt) + t^3(t - q)h''(qt^2)) \right\}. \quad (3.6)$$

Using (H2) for $0 < q < t$, we get

$$\Phi_{11} \geq \phi''(q) + t(t + (t - q))\kappa''(qt) + t^3(t - q)h''(qt^2). \quad (3.7)$$

Then

$$\frac{1}{2}F(q, t) \geq \frac{1}{\Phi_{11}} \left\{ \frac{\phi'(t) - \phi'(q)}{q - t} + \frac{q\phi''(q)}{t - q} + \frac{t(\kappa'(t^2) - \kappa'(qt))}{q - t} + \frac{qt^2\kappa''(qt)}{t - q} \right\} - \frac{q}{t - q}. \quad (3.8)$$

Regrouping the terms with the same argument we have,

$$\frac{1}{2}F(q, t) \geq \frac{1}{\Phi_{11}} \left\{ \frac{\phi'(t) + t\kappa'(t^2)}{q - t} + \frac{q\phi''(q) + \phi'(q)}{t - q} + \frac{t(qt\kappa''(qt) + \kappa'(qt))}{t - q} \right\} - \frac{q}{t - q}. \quad (3.9)$$

Finally by writing

$$\begin{aligned} \frac{1}{2}F(q, t) \geq & -\frac{q}{t} \left\{ \frac{t}{t - q} + \frac{t(\phi'(t) + t\kappa'(t^2))}{q\Phi_{11}(t - q)} \right. \\ & \left. - \frac{t(q\phi''(q) + \phi'(q))}{q\Phi_{11}(t - q)} - \frac{t^2(qt\kappa''(qt) + \kappa'(qt))}{q\Phi_{11}(t - q)} \right\}, \end{aligned}$$

we get the lower estimate for $F(q, t)$. For (ii) the procedure is similar. By adding and subtracting

$$\frac{qt\kappa''(qt)}{\Phi_{11}} \quad \text{and} \quad \frac{q\phi''(q)}{t\Phi_{11}}, \quad (3.10)$$

in (3.4) we get

$$\begin{aligned} \frac{1}{2}F(q, t) = & \frac{1}{\Phi_{11}} \left\{ \frac{\phi'(t) - \phi'(q)}{q - t} + \frac{q\phi''(q)}{t} + \frac{t(\kappa'(t^2) - \kappa'(qt))}{q - t} + qt\phi''(qt) \right\} \\ & - \frac{1}{\Phi_{11}} \frac{q}{t} \left\{ \phi''(q) + 2t^2\kappa''(qt) + t^4h''(qt^2) \right\}. \end{aligned} \quad (3.11)$$

Since

$$\Phi_{11} = \phi''(q) + 2t^2\kappa''(qt) + t^4h''(qt^2)$$

it follows that for $0 < q < t$,

$$\frac{1}{2}F(q, t) \leq \frac{1}{\Phi_{11}} \left(\frac{\phi'(t) + t\kappa'(t^2)}{q - t} + \frac{q\phi''(q) + \phi'(q)}{t - q} + \frac{t(qt\kappa''(qt) + \kappa'(qt))}{t - q} \right) - \frac{q}{t}. \quad (3.12)$$

Finally,

$$\begin{aligned} \frac{1}{2}F(q, t) &\leq -\frac{q}{t} \left(1 + \frac{t(\phi'(t) + t\kappa'(t^2))}{q\Phi_{11}(t-q)} - \frac{t(q\phi''(q) + \phi'(q))}{q\Phi_{11}(t-q)} - \frac{t^2(qt\kappa''(qt) + \kappa'(qt))}{q\Phi_{11}(t-q)} \right) \\ &\leq -\frac{q}{t} \left\{ 1 + \frac{t(\phi'(t) + t\kappa'(t^2))}{q\Phi_{11}(t-q)} \right. \\ &\quad \left. - \left| \frac{t(q\phi''(q) + \phi'(q))}{q\Phi_{11}(t-q)} \right| - \left| \frac{t^2(qt\kappa''(qt) + \kappa'(qt))}{q\Phi_{11}(t-q)} \right| \right\}. \end{aligned}$$

That is the second inequality for $F(q, t)$. $\square$

The following lemma is useful for establishing the properties (P2) and (P3). It gives a simple property of functions satisfying (H5).

LEMMA 3.1.2. Let $f \in C^2(0, \infty)$, $A, \alpha > 0$, and $\lim_{s \rightarrow 0} s^\alpha f''(s) = A$. Then $\forall \varepsilon > 0$ $\exists s_0(\varepsilon) \geq 0$ such that

$$\text{for } \xi > 2: \begin{cases} s(sf'(s))' \geq 0, \\ s(sf'(s))' \geq s(C_0(s_0)) + \left(\frac{A(\alpha-2) - \varepsilon\alpha}{\alpha-1} \right) s^{-\alpha+2}, \\ s(sf'(s))' \leq s(C_1(s_0)) + \left(\frac{A(\alpha-2) + \varepsilon\alpha}{\alpha-1} \right) s^{-\alpha+2}, \end{cases} \quad \forall s < s_0(\varepsilon);$$

for $0 \leq \xi \leq 2$: $\lim_{s \rightarrow 0} s(sf'(s))' = 0$.

Proof. By assumption, $\forall \varepsilon > 0$ $\exists s_0(\varepsilon)$ such that $(A - \varepsilon)s^{-\alpha} \leq f''(s) \leq (A + \varepsilon)s^{-\alpha}$, $\forall s < s_0(\varepsilon)$. By integrating this inequality from s to s_0 , we get for $\xi \neq 1$

$$f'(s_0) + \frac{A + \varepsilon}{\alpha - 1} (s_0^{-\alpha+1} - s^{-\alpha+1}) \leq f'(s) \leq f'(s_0) + \frac{A - \varepsilon}{\alpha - 1} (s_0^{-\alpha+1} - s^{-\alpha+1}).$$

This leads to the estimations,

$$\begin{aligned} s(sf'(s))' &\geq s \left\{ f'(s_0) + \left(\frac{A + \varepsilon}{\alpha - 1} \right) s_0^{-\alpha+1} \right\} + \left(\frac{A(\alpha-2) - \varepsilon\alpha}{\alpha-1} \right) s^{-\alpha+2}, \\ s(sf'(s))' &\leq s \left\{ f'(s_0) + \left(\frac{A - \varepsilon}{\alpha - 1} \right) s_0^{-\alpha+1} \right\} + \left(\frac{A(\alpha-2) + \varepsilon\alpha}{\alpha-1} \right) s^{-\alpha+2}, \quad \forall s < s_0(\varepsilon). \end{aligned}$$

For $\xi = 1$,

$$f'(s_0) + (A - \varepsilon) + (A + \varepsilon) \ln \left(\frac{s}{s_0} \right) \leq (sf'(s))' \leq f'(s_0) + (A + \varepsilon) + (A - \varepsilon) \ln \left(\frac{s}{s_0} \right).$$

Then,

$$s \left\{ f'(s_0) + (F + \varepsilon) \ln \left(\frac{s}{s_0} \right) \right\} \leq s(sf'(s))' \leq s \left\{ f'(s_0) + (F - \varepsilon) \ln \left(\frac{s}{s_0} \right) \right\}.$$

For $\alpha > 2$ and sufficiently small s , $s\{(sf'(s))'\} \geq 0$ and for $0 < \alpha \leq 2$, both estimations converge to zero when $s \rightarrow 0$. $\square$

As in the previous section, $q(t) = q(\lambda, \alpha)(t)$ denotes a solution of (P) for a fixed α in $(0, \lambda]$.

PROPOSITION 3.1.3. Let $\Phi(q, t, t)$ verify (H1)–(H3), (H5). Then $q(t)$ is defined on $[\lambda, \infty)$.

Proof. $G(q, t) = -\frac{t}{q}F(q, t)$, where $F(q, t)$ is the right-hand side of the equation for $q(t)$. Once we show that $G(q, t)$ has the property (P1), the conclusion follows from Lemma 2.2.1. One has to distinguish the cases where $\nu_1 > 2$, $\nu_1 \leq 2$ (and $\mu_1 > 2$, $\mu_1 \leq 2$).

Suppose that $\nu_1 > 2$ and $\mu_1 > 2$. Then for $t \rightarrow T$ and $q \rightarrow 0$, (H6) and Lemma 3.1.2 imply that

$$\exists \varepsilon > 0 \text{ such that } \frac{t(q\phi''(q) + \phi'(q))}{q\Phi_{11}(t-q)} \geq 0 \text{ and } \frac{t^2(qt\kappa''(qt) + \kappa'(qt))}{q\Phi_{11}(t-q)} \geq 0, \quad \forall 0 < q < \varepsilon.$$

Lemma 3.1.1(i) then gives the following estimation,

$$\frac{1}{2}F(q, t) \geq -\frac{q}{t} \left\{ \frac{t}{t-q} + \frac{t(\phi'(t) + t\kappa'(t^2))}{q\Phi_{11}(t-q)} \right\}. \quad (3.13)$$

Moreover since $p = qt^2 \rightarrow 0$, (H2) and (H4) imply that

$$q\Phi_{11} \geq qt^4 h''(qt^2) = q^{-1} p^2 h''(p) \geq q^{-1} H, \quad H > 0. \quad (3.14)$$

Consequently for $t \rightarrow T$ and $q \rightarrow 0$,

$$\left| \frac{t(\phi'(t) + t\kappa'(t^2))}{q\Phi_{11}(t-q)} \right| \leq \frac{q}{H_1} \frac{(t\phi'(t) + t^2\kappa'(t^2))}{(t-q)} \rightarrow 0. \quad (3.15)$$

On the other hand, $t/(t-q) \rightarrow 1$ as $q \rightarrow 0$. Then we get for $\nu_1 > 2$ and $\mu_1 > 2$,

$$\begin{aligned} \forall c > 0 \exists \varepsilon > 0, \quad T', \lambda \leq T' < T \text{ such that } \frac{1}{2}F(q, t) \geq -\frac{q}{t}(1+c), \\ \forall t \in (T', T), \quad \forall q \in (0, \varepsilon). \end{aligned} \quad (3.16)$$

For the case where $0 \leq \nu_1, \mu_1 \leq 2$, it follows from Lemma 3.1.1(i) that

$$\begin{aligned} \frac{1}{2}F(q, t) \geq -\frac{q}{t} \left\{ \frac{t}{t-q} + \left| \frac{t(\phi'(t) + t\kappa'(t^2))}{q\Phi_{11}(t-q)} \right| \right. \\ \left. + \left| \frac{t(q\phi''(q) + \phi'(q))}{q\Phi_{11}(t-q)} \right| + \left| \frac{t^2(qt\kappa''(qt) + \kappa'(qt))}{q\Phi_{11}(t-q)} \right| \right\}. \end{aligned}$$

Then from (3.14) and Lemma 3.1.2 we see that for $q \rightarrow 0$ and $t \rightarrow T$,

$$\begin{cases} \left| \frac{t(q\phi''(q) + \phi'(q))}{q\Phi_{11}(t-q)} \right| \leq \frac{t}{t-q} \left| \frac{q\{(q\phi'(q))'\}}{H_1} \right| \rightarrow 0, \\ \left| \frac{t^2(qt\kappa''(qt) + \kappa'(qt))}{q\Phi_{11}(t-q)} \right| \leq \frac{t}{t-q} \left| \frac{qt\{(qt\kappa'(qt))'\}}{H_1} \right| \rightarrow 0. \end{cases} \quad (3.17)$$

It is then easy to see that one gets an estimation like (3.16) in this case and that this estimation also holds for the cases $0 \leq \nu_1 \leq 2 < \mu_1$ or $0 \leq \mu_1 \leq 2 < \nu_1$. To conclude, it suffices to remark that (P1) follows directly from (3.16). $\square$

PROPOSITION 3.1.4. Let $\Phi(q, t, t)$ satisfy (H1)–(H3), (H5), (H6). Then $\forall \nu > \max\{\nu_2 + 1; 2\mu_2 + 3; 2\}$ $\exists T(\nu) \geq \max\{1, \lambda\}$ such that $q(t) \geq t^{-\nu}$, $\forall t > T(\nu)$.

Proof. We show that $G(q, t)$ satisfies (P2). Let $\nu > \nu_0 = \max\{\nu_2 + 1; 2\mu_2 + 3; 2\}$ and define $A^\nu = \{(q, t) : 0 < q < t^{-\nu}, t \geq 1\}$. We show that for a fixed $\nu > \nu_0$, $F(q, t)$ verifies

$$\forall \varepsilon > 0 \quad \exists T(\varepsilon) \geq 1 \quad \text{such that} \quad F(q, t) > -(2 + \varepsilon)q/t, \quad \forall t > T(\varepsilon), \quad (3.18)$$

$$\forall (q, t) \in A^\nu.$$

Then (P3) follows directly from (3.18).

In A^ν , $\lim_{t \rightarrow \infty} q = 0$ and $\lim_{t \rightarrow \infty} p = qt^2 = 0$. The calculations leading to (3.15) hold and for our proof it suffices to revise the estimations of

$$\frac{-t\phi'(t)}{q\Phi_{11}(t-q)}, \quad \frac{-t^2\kappa'(t^2)}{q\Phi_{11}(t-q)} \quad (3.19)$$

for $(q, t) \in A^\nu$ and $t \rightarrow \infty$. With (3.14) we deduce that $q\Phi_{11} \geq H_1 t^\nu$, $\forall (q, t) \in A^\nu$. Consequently

$$\begin{cases} \frac{-t\phi'(t)}{q\Phi_{11}(t-q)} \geq -\frac{t^{1-\nu}|\phi'(t)|}{K(t-q)}, \\ \frac{-t^2\kappa'(t^2)}{q\Phi_{11}(t-q)} \geq -\frac{t^{2-\nu}|\kappa'(t^2)|}{K(t-q)}. \end{cases} \quad (3.20)$$

With the choice of ν_0 and (H6) for $(q, t) \in A^\nu$ it follows that

$$\lim_{t \rightarrow \infty} \frac{t^{1-\nu}|\phi'(t)|}{K(t-q)} = 0, \quad \lim_{t \rightarrow \infty} \frac{t^{2-\nu}|\kappa'(t^2)|}{K(t-q)} = 0. \quad (3.21)$$

In A^ν we remark that $\forall \delta > 0 \quad \exists T(\varepsilon)$ such that $q < \delta$ if $(q, t) \in A^\nu$ and $t > T$. Consequently (3.21) and (3.13) allow us to write that $\forall \varepsilon > 0$, $\exists T(\varepsilon)$ such that

$$F(q, t) > -(2 + \varepsilon)\frac{q}{t}, \quad \forall t > T(\varepsilon), \quad (q, t) \in A^\nu. \quad (3.22)$$

$\nu_0 = \max\{\nu_2 + 1; 2\mu_2 + 3; 2\} \geq 2$. Thus by choosing $0 < \varepsilon < \nu - \nu_0$ one can define $T(\nu) \geq \lambda$ such that $G(q, t) < \nu$, $\forall (q, t) \in A^\nu$, $\forall t > T(\nu)$. The conclusion then follows from Proposition 2.1.4. $\square$

PROPOSITION 3.1.5. Let $\Phi(q, t, t)$ satisfy (H1)–(H2), (H4)–(H6), and let

$$\delta \in \left(0, \min \left\{ 2; \frac{2(\sigma_2 + 2)}{\nu_1 + \sigma_2}; 1 + \frac{\sigma_2 + 2}{\mu_1 + \sigma_2} \right\} \right).$$

Then $\forall b > 0 \quad \exists T(\delta, b) \geq \max\{1, \lambda\}$ such that $q(t) \leq t^{-\delta}$, $\forall t > T(\delta, b)$.

Proof. We show that $G(q, t)$ satisfies (P3). Let

$$0 < \delta < \delta_0 = \min\{2; (2(\sigma_2 + 2)/\nu_1 + \sigma_2); 1 + (\sigma_2 + 2/\mu_1 + \sigma_2)\}.$$

For $b > 0$ we set $A_b^\delta = \{(q, t) : t^{-\delta} < q < b, t \geq 1\}$. In A_b^δ , $p = qt^2$ is not bounded as $t \rightarrow \infty$. We show that in A_b^δ , $\forall \varepsilon > 0 \quad \exists T(\varepsilon)$ such that

$$F(q, t) < -(2 - \varepsilon)\frac{q}{t}, \quad \forall (q, t) \in A_b^\delta, \quad \forall t > T(\varepsilon). \quad (3.23)$$

Then the property (P3) follows directly. As in the proof of Proposition 3.1.4 the hypotheses on $h(p)$ play an important role. In fact, (H4) gives

$$q\Phi_{11} \geq H_2 q^{-1} p^{2+\sigma_2}, \quad (3.24)$$

$$q\Phi_{11} \geq H_2 t^2 p^{1+\sigma_2}, \quad \forall (q, t) \in A_b^\delta. \quad (3.25)$$

With (3.24) and (3.25) we get an estimation like (3.23) from Lemma 3.1.1(ii), which is

$$\begin{aligned} \frac{1}{2}F(q, t) \leq & -\frac{q}{t} \left(1 + \frac{t(\phi'(t) + t\kappa'(t^2))}{q\Phi_{11}(t-q)} - \left| \frac{t(q\phi''(q) + \phi'(q))}{q\Phi_{11}(t-q)} \right| \right. \\ & \left. - \left| \frac{t^2(qt\kappa''(qt) + \kappa'(qt))}{q\Phi_{11}(t-q)} \right| \right). \end{aligned}$$

Now we consider that $(q, t) \in A_b^\delta$ and the expressions to estimate are

$$\frac{t(\phi'(t) + t\kappa'(t^2))}{q\Phi_{11}(t-q)}; \quad (3.26)$$

$$\left| \frac{t(q\phi''(q) + \phi'(q))}{q\Phi_{11}(t-q)} \right|; \quad (3.27)$$

$$\left| \frac{t^2(qt\kappa''(qt) + \kappa'(qt))}{q\Phi_{11}(t-q)} \right|. \quad (3.28)$$

If (3.26) is nonnegative, an upper bound of $\frac{1}{2}F(q, t)$ is given by simply omitting this term in Lemma 3.1.1(ii). Else if $\phi' \leq 0$ (H2) implies that $\phi'(t)$ is bounded from below. The same remark holds if $\kappa' \leq 0$. Consequently for constants K_1, K_2 we get with (3.25),

$$0 \geq \frac{t(\phi'(t) + t\kappa'(t^2))}{q\Phi_{11}(t-q)} \geq \frac{t}{t-q} \frac{(K_1 + tK_2)}{q\Phi_{11}} \geq \frac{t}{t-q} \frac{t^{-2}(K_1 + tK_2)}{H_2 p^{1+\sigma_2}}.$$

Then if (3.26) is negative,

$$\lim_{t \rightarrow \infty} \frac{t(\phi'(t) + t\kappa'(t^2))}{q\Phi_{11}(t-q)} = 0.$$

Let $\varepsilon > 0$. For (3.27) one has to distinguish the two cases: $0 < q < \varepsilon$ and $q > \varepsilon$.

• If $0 < q < \varepsilon$, with (H6), Lemma 3.1.2, and (3.24) $\exists K, Q_1, Q_2 > 0$ such that

$$\left| \frac{t(q\phi''(q) + \phi'(q))}{q\Phi_{11}(t-q)} \right| \leq \frac{t}{t-q} \frac{K q^{-\nu_1+2}}{p^{2+\sigma_2}} \leq \begin{cases} Q_1 t^{\delta(\sigma_2+\nu_1)-2(\sigma_2+2)}, & \text{if } \nu_1 > 2, \\ \frac{Q_2}{p^{2+\sigma_2}}, & \text{if } 0 \leq \nu_1 \leq 2. \end{cases}$$

In both cases the choice of δ and δ_0 leads to

$$\lim_{t \rightarrow \infty} \frac{t(q\phi''(q) + \phi'(q))}{q\Phi_{11}(t-q)} = 0, \quad \forall \nu_1 \geq 0.$$

• If $q > \varepsilon$, the same conclusion follows immediately. The procedure is similar for (3.28). If $0 < qt < \varepsilon$, one also gets for some constants L_1, L_2 ,

$$\left| \frac{t^2(qt\kappa''(qt) + \kappa'(qt))}{q\Phi_{11}(t-q)} \right| \leq \begin{cases} L_1 t^{\delta(\mu_1 + \sigma_2) - (\mu_1 + 2 + 2\sigma_2)}, & \text{if } \mu_1 > 2, \\ \frac{L_2}{p^{1+s_2}}, & \text{if } 0 \leq \mu_1 \leq 2. \end{cases}$$

In both cases the choice of δ and δ_0 shows that

$$\lim_{t \rightarrow \infty} \frac{t^2(qt\kappa''(qt) + \kappa'(qt))}{q\Phi_{11}(t-q)} = 0, \quad \forall \mu_1 \geq 0.$$

• If $qt > \varepsilon$, one compares (3.28) and (3.26),

$$\left(\frac{t^2(qt\kappa''(qt) + \kappa'(qt))}{q\Phi_{11}(t-q)} - \frac{t^2\kappa'(t^2)}{q\Phi_{11}(t-q)} \right) = \frac{t^2\kappa'(t^2)}{q\Phi_{11}(t-q)} \left(\frac{(qt\kappa''(qt) + \kappa'(qt))}{\kappa'(t^2)} - 1 \right).$$

With (H7) we get

$$\lim_{t \rightarrow \infty} \left| \frac{(qt\kappa''(qt) + \kappa'(qt))}{\kappa'(t^2)} \right| \leq Cte \lim_{t \rightarrow \infty} (qt)^{\mu_2+1} t^{-2(\mu_2+1)} = 0.$$

Then $((qt\kappa''(qt) + \kappa'(qt))/\kappa'(t^2)) - 1$ is negative for sufficiently large values of t . Consequently zero is an upper asymptotic bound.

By substituting all these estimations it follows that $F(q, t)$ has the desired property,

$$\begin{aligned} \forall \varepsilon > 0 \quad \exists T(\varepsilon) > 0, \text{ such that } F(q, t) < -(2 - \varepsilon) \frac{q}{t}, \quad \forall t > T(\varepsilon), \\ \forall (q, t) \in A_b^\delta. \end{aligned} \quad (3.29)$$

Since

$$\delta_0 = \min \left\{ 2; \frac{2(\sigma_2 + 2)}{\nu_1 + \sigma_2}; 1 + \frac{\sigma_2 + 2}{\mu_1 + \sigma_2} \right\} \leq 2,$$

by choosing $\varepsilon \in (0, \delta_0 - \delta)$ one can define $T(\delta) > 0$ such that $G(q, t) > \delta$, $\forall (q, t) \in A_b^\delta$, $\forall t > T(\delta)$. Then $G(q, t)$ satisfies (P3) and the conclusion follows from Proposition 2.1.5. $\square$

REMARK. (a) bidimensional case. For the bidimensional case the stored energy function is of the form

$$\Phi(\lambda_1, \lambda_2) = \sum_{i=0}^2 \phi(\lambda_i) + \sum_{i=0}^2 \kappa(\lambda_i \lambda_{i+1}) \lambda_3 = \lambda_1.$$

One can remove the hypotheses of the case $n = 3$ in the case $n = 2$ by giving the “dominant role” to the function $\kappa(s)$, related to energy associated with surface area deformations. The treatment of the three-dimensional case remains valid if we suppose

- (B1) $\phi, \kappa \in C^2((0, \infty); \mathbb{R}^+)$.
- (B2) $\phi''(s) \geq 0$, $\forall s > 0$, $\kappa''(s) > 0$, $\forall p > 0$.
- (B3) $\exists \gamma_1 \geq 2$, $D_1 > 0$ such that $\lim_{s \rightarrow 0} s^{\gamma_1} \kappa''(s) = D_1$.
- (B4) $\exists \gamma_2 \geq 0$, $D_2 > 0$ such that $\lim_{s \rightarrow \infty} s^{-\gamma_2} \kappa''(s) = D_2$.

(B5) $\exists \nu_1 \geq 0, A_1 \geq 0$ such that $\lim_{s \rightarrow 0} s^{\nu_1} \phi''(s) = A_1$.

(B6) $\exists \nu_2 \geq 0, A_2 \geq 0$ such that $\lim_{s \rightarrow \infty} s^{-\nu_2} \phi''(s) = A_2$.

In [7] it is shown that $G(q, t)$ satisfies (P1)–(P3) for

$$\nu_0 = \max\{1, \nu_2 + 1\}, \quad \delta_0 = \min\left\{1, \frac{\gamma_2 + 2}{\nu_1 + \gamma_2}\right\}.$$

(b) limit case in the estimations. For the three-dimensional case, we have seen that

$$\delta_0 = \min\left\{2; \frac{2(\sigma_2 + 2)}{\nu_1 + \sigma_2}; 1 + \frac{\sigma_2 + 2}{\mu_1 + \sigma_2}\right\} \leq 2 \leq \nu_0 = \max\{\nu_2 + 1; 2\mu_2 + 3; 2\}.$$

For some values of $\nu_1, \nu_2, \mu_1, \mu_2, \delta_0 = 2 = \nu_0$, the estimation of $q(t)$ is

$$\forall \varepsilon > 0 \quad \exists T(\varepsilon) \geq \max\{1, \lambda\} : t^{-2-\varepsilon} \leq q(t) \leq t^{-2+\varepsilon}, \quad \forall t > T(\varepsilon). \quad (3.30)$$

This estimation suggests studying the behaviour of $q(t)t^2$ and seeking conditions such that there exists P_0, P_1 such that $P_0 \leq q(t)t^2 \leq P_1, \forall t \geq \lambda$. This is done in [7]. We show that

- (i) If $\nu_1 < 2, \mu_1 < 2, \nu_2 < 1$, and $\mu_2 = 0$ then $\exists P_1 \geq P_0 > 0$ such that $P_0 \leq p(t) \leq P_1, \forall t \geq \lambda$.
- (ii) If $\nu_1 \geq 2$ or $\mu_1 \geq 2$, then $\lim_{t \rightarrow \infty} p(t) = \infty$.

4. Nonexistence of solutions to the problem of radial cavitation. The approach of existence consists of showing that there is a unique value $\alpha(\lambda) \in (0, \lambda)$ such that

$$\tau(\lambda, \alpha(\lambda)) = 0. \quad (4.1)$$

Nonexistence criteria can be deduced from that equation by seeking conditions on the stored energy function such that $\tau(\lambda, \alpha) = 0$ admits no solution for $\alpha(\lambda) \in (0, \lambda)$. We shall give two criteria. Each of them has a physical interpretation based on relations between the total energy $E(\lambda, \alpha)$ and $\tau(\lambda, \alpha)$, λ fixed and $\alpha(\lambda) \in (0, \lambda)$. These relations are also a generalisation of the considerations given in [10] about the energy of the solutions of the boundary value problems defined by (1.5) and (1.6). All these results are illustrated for an Ogden material satisfying (H1)–(H7) in [7].

For the first criterion we need a representation formula for $T(\lambda, \alpha)(r)$, where $T(\lambda, \alpha)(r)$ is the normal component of the Cauchy stress tensor defined at (1.9).

LEMMA 4.1.1. Let (A1)–(A4) hold, $U = U(\lambda, \alpha)$ be a solution of (IVP) defined on $(0, 1]$, and $T(\lambda, \alpha)(r)$ be the normal component of the Cauchy stress tensor associated to U . Then

$$T(\lambda, \alpha)(r) = T(\lambda, \alpha)(1) + (n-1) \int_r^1 \left(\frac{U(s)}{s}\right)' \left(\frac{U(s)}{s}\right)^{-n} R\left(U'(s), \frac{U(s)}{s}\right) ds.$$

Proof. Set $U = U(\lambda, \alpha)(r)$. The normal component of the Cauchy stress tensor is given by

$$T(\lambda, \alpha)(r) = \left(\frac{U(r)}{r}\right)^{1-n} \Phi_1\left(U'(r), \frac{U(r)}{r}, \dots, \frac{U(r)}{r}\right). \quad (4.2)$$

By deriving $T(\lambda, \alpha)(r)$ and using the equilibrium equation, we get

$$T'(\lambda, \alpha)(r) = -(n-1) \left(\frac{U(r)}{r} \right)' \left(\frac{U(r)}{r} \right)^{-n} R \left(U'(r), \frac{U(r)}{r} \right), \quad 0 < r \leq 1. \quad (4.3)$$

The conclusion follows by integrating (4.3). $\square$

With the change of variable (2.1), $T(\lambda, \alpha)(r)$ can be written as

$$S(\lambda, \alpha)(t) = \lambda^{1-n} \Phi_1(\alpha, \lambda, \dots, \lambda) - (n-1) \int_{\lambda}^t S^{-n} R(q(\lambda, \alpha)(s), s) ds, \quad \forall t \geq \lambda. \quad (4.4)$$

Now if for any compact subset X of $(0, \lambda]$ there exists $A, B > 0$, $\tau \geq n-1$, and $T(X) \geq \max\{1, \lambda\}$ such that

$$R(q(\lambda, \alpha)(t), t) \geq A + Bt^{\tau}, \quad \forall t > T, \quad \forall \alpha \in X. \quad (4.5)$$

Then $\lim_{t \rightarrow \infty} S(t) = \tau(\lambda, \alpha) = -\infty$. Clearly the second boundary condition is never satisfied and problem (1.6) has no solution. The physical interpretation of that criterion arises from the following relation. Set $\Phi = \Phi(U'(r), U(r)/r, \dots, U(r)/r)$.

PROPOSITION 4.1.2. Let $u(x) = U(\lambda, \alpha)(r)x/r$, a weak equilibrium solution. Then

- (i) If $E(\lambda, \alpha) < \infty$, $E(\lambda, \alpha) = (\omega_n/n)[\Phi(\alpha, \lambda, \dots, \lambda) - (\lambda - \alpha)\Phi_1(\alpha, \lambda, \dots, \lambda) - (U(\lambda, \alpha)(0))^n \tau(\lambda, \alpha)]$.
- (ii) $E(\lambda, \alpha) < \infty \Leftrightarrow \tau(\lambda, \alpha) > -\infty$.

Proof. (i) As in [10] one can define for $U = U(\lambda, \alpha)$,

$$\Psi(r) = \Phi - \left(U'(r) - \frac{U(r)}{r} \right) \Phi_1. \quad (4.6)$$

Since U is an equilibrium solution we see that

$$\frac{d}{dr}(r^n \Psi(r)) = nr^{n-1} \Phi. \quad (4.7)$$

Then if $E(\lambda, \alpha) < \infty$, one can write

$$E(\lambda, \alpha) = \omega_n \int_0^1 r^{n-1} \Phi dr = \frac{\omega_n}{n} [r^n \Psi(r)]_0^1.$$

This means

$$E(\lambda, \alpha) = \frac{\omega_n}{n} \left\{ \Phi(\alpha, \lambda, \dots, \lambda) - (\lambda - \alpha)\Phi_1(\alpha, \lambda, \dots, \lambda) - \lim_{r \rightarrow 0} r^n \left(\Phi - \left(U'(r) - \frac{U(r)}{r} \right) \Phi_1 \right) \right\}. \quad (4.8)$$

Let us now show that

$$\lim_{r \rightarrow 0} r^n U'(\lambda, \alpha)(r) \Phi_1 = 0.$$

As $u(x)$ is a weak equilibrium solution, Theorem 1.1.2 gives

$$r^{n-1} \Phi_i \in L^1((0, 1]), \quad i = 1, 2.$$

Then using the radial equilibrium equation we get for $0 < \varepsilon < 1$,

$$\begin{aligned} \int_{\varepsilon}^1 r^{n-1} \Phi_1 dr &= \frac{1}{n} \left\{ r^n \Phi_1|_{\varepsilon}^1 - \int_{\varepsilon}^1 r^n \frac{d\Phi_1}{dr} dr \right\}, \\ &= \frac{1}{n} \left\{ r^n \Phi_1|_{\varepsilon}^1 - (n-1) \int_{\varepsilon}^1 r^{n-1} (\Phi_2 - \Phi_1) dr \right\}, \end{aligned}$$

and so,

$$\int_{\varepsilon}^1 r^{n-1} \Phi_1 dr = r^n \Phi_1|_{\varepsilon}^1 - (n-1) \int_{\varepsilon}^1 r^{n-1} \Phi_2 dr.$$

Consequently $\lim_{r \rightarrow 0} r^n \Phi_1$ exists and is zero since $r^{n-1} \Phi_i \in L^1((0, 1])$. Moreover, by Lemma 2.1.1(i) $U''(\lambda, \alpha)(r) > 0$. Then $0 < U'(\lambda, \alpha)(r) < \alpha$ and finally,

$$\lim_{r \rightarrow 0} r^n \Phi_1 U'(\lambda, \alpha)(r) = 0. \quad (4.9)$$

Now, since

$$(U(r))^{n-1} T(\lambda, \alpha)(r) = r^{n-1} \Phi_1,$$

it follows from (4.9) that

$$\begin{aligned} E(\lambda, \alpha) &= \Phi(\alpha, \lambda, \dots, \lambda) + (\lambda - \alpha) \Phi_1(\alpha, \lambda, \dots, \lambda) - \lim_{r \rightarrow 0} r^n \left(\Phi + \left(\frac{U(r)}{r} \right) \Phi_1 \right) \\ &= \Phi(\alpha, \lambda, \dots, \lambda) + (\lambda - \alpha) \Phi_1(\alpha, \lambda, \dots, \lambda) - \lim_{r \rightarrow 0} (r^n \Phi + (U(r))^n T(\lambda, \alpha)(r)). \end{aligned} \quad (4.10)$$

By assumption, $0 \leq E(\lambda, \alpha) < \infty$. Then $\lim_{r \rightarrow 0} r^n \Phi$ exists and is zero. Passing to the limit we get (i).

For (ii), suppose that $E(\lambda, \alpha) < \infty$. By (i) it is implied that $\lim_{r \rightarrow 0} (U(r))^n T(r) > -\infty$. Then $\tau(\lambda, \alpha) > -\infty$ since $\lim_{r \rightarrow 0} U(r) > 0$. Conversely, let $\tau(\lambda, \alpha) > -\infty$. Since $u(x)$ is a weak equilibrium solution we always have $\lim_{r \rightarrow 0} r^n \Phi_1 U'(r) = 0$ and (4.10). Now with (A1), $E(\lambda, \alpha) \geq 0$. Then we must have that $E(\lambda, \alpha) < \infty$. $\square$

This criteria means that all the solutions of (IVP) presenting a hole at $r = 0$ have an infinite energy.

By considering the formula (4.4) it is easily verified that (A5) with $\beta < n - 1$ ensures that $\tau(\lambda, \alpha) > -\infty$ and $E(\lambda, \alpha) < \infty$ (see Theorem 1.2.3(ii)), but the definition of weak radial solutions does not guarantee this fact (see Theorem 1.2.3(i)).

In the second approach we suppose that $E(\lambda, \alpha) < \infty \forall \alpha \in (0, \lambda)$ (i.e., $\tau(\lambda, \alpha) > -\infty$) and give conditions such that $\tau(\lambda, \alpha) < 0, \forall \alpha \in (0, \lambda)$. The second boundary condition is also never satisfied.

PROPOSITION 4.1.3. Suppose that $\Phi(q, t, \dots, t)$ satisfies (A1)–(A4) and that $\Phi_{ii}(q, t, \dots, t) > 0$, for $i = 1, 2$ and $\Phi_{12}(q, t, \dots, t) \leq 0, \forall q, t \geq 0$. Let $q = q(\lambda, \alpha)(t)$ be a solution of (P) on $[\lambda, \infty)$. Then $\tau(\lambda, \alpha) < 0, \forall \alpha \in (0, \lambda)$.

Proof. Set $\Phi_j = \Phi_j(q, t, \dots, t)$. $\tau(\lambda, \alpha)$ can be written as

$$\begin{aligned} \tau(\lambda, \alpha) &= \lambda^{1-n} \Phi_1(\alpha, \lambda, \dots, \lambda) - (n-1) \int_{\lambda}^{\infty} t^{-n} R(q, t) dt \\ &= -(n-1) \int_{\lambda}^{\infty} t^{-n} \left(\frac{(q \Phi_1 - t \Phi_2)(q, t, \dots, t) - (q-t) \Phi_1(\alpha, \lambda, \dots, \lambda)}{q-t} \right) dt. \end{aligned}$$

Thus

$$\tau(\lambda, \alpha) = -(n-1) \int_{\lambda}^{\infty} t^{-n} \left(\frac{q(\Phi_1 - \Phi_1(\alpha, \lambda, \dots, \lambda)) - t(\Phi_2 - \Phi_1(\alpha, \lambda, \dots, \lambda))}{q-t} \right) dt.$$

Now, $\Phi_{11} > 0 \Rightarrow \Phi_1(\alpha, \lambda, \dots, \lambda) < \Phi_1(\lambda, \dots, \lambda)$, and since $\Phi_1(\lambda, \dots, \lambda) = \Phi_2(\lambda, \dots, \lambda)$, we have that

$$\tau(\lambda, \alpha) < -(n-1) \int_{\lambda}^{\infty} t^{-n} \left(\frac{q(\Phi_1 - \Phi_1(\alpha, \lambda, \dots, \lambda)) - t(\Phi_2 - \Phi_2(\lambda, \lambda, \dots, \lambda))}{q-t} \right) dt.$$

On the other side,

$$\begin{aligned} \Phi_1(q, t, \dots, t) - \Phi_1(\alpha, \lambda, \dots, \lambda) &= \Phi_1(q, t, \dots, t) - \Phi_1(\alpha, t, \dots, t) \\ &\quad + \Phi_1(\alpha, t, \dots, t) - \Phi_1(\alpha, \lambda, \dots, \lambda). \end{aligned}$$

Then if $\Phi_{11} > 0$ and $\Phi_{12} \leq 0$,

$$\Phi_1(q, t, \dots, t) - \Phi_1(\alpha, t, \dots, t) < 0, \quad \Phi_1(\alpha, t, \dots, t) - \Phi_1(\alpha, \lambda, \dots, \lambda) < 0.$$

Consequently $\Phi_1(q, t, \dots, t) - \Phi_1(\alpha, \lambda, \dots, \lambda) < 0$.

By repeating this argument for $\Phi_2 - \Phi_2(\lambda, \lambda, \dots, \lambda)$, we find that $\tau(\lambda, \alpha) < 0$, $\forall \alpha \in (0, \lambda)$. $\square$

For an n -dimensional stored energy function of the form

$$\Phi(q, t, \dots, t) = \phi(q) + (n-1)\phi(t) + h(p), \quad \text{where } h''(p) \geq 0, \quad \phi''(s) \geq 0, \quad (4.11)$$

the condition that $h'(p) < 0$ is necessary for having $\Phi_{12} \leq 0$. In fact,

$$\Phi_{12} = t^{n-2}(h'(p) + ph''(p)) \leq 0 \Rightarrow h'(p) \leq 0, \quad \forall p > 0. \quad (4.12)$$

For an Ogden material for which κ is nonidentically zero, this hypothesis on $h'(p)$ can also lead to a conclusion similar to that of Proposition 4.1.3.

PROPOSITION 4.1.4. Let $\Phi(q, t, t)$ satisfy (H1)–(H3). If $h'(p) < 0$, $\forall p > 0$, and if $q(t)t$ is decreasing on $[\lambda, \infty)$, then $\tau(\lambda, \alpha) < 0$, $\forall \alpha \in (0, \lambda)$.

Proof. As before,

$$\begin{aligned} \tau(\lambda, \alpha) &= \lambda^{-2} \Phi_1(\alpha, \lambda, \lambda) - 2 \int_{\lambda}^{\infty} t^{-3} R(q, t) dt \\ &= -2 \int_{\lambda}^{\infty} t^{-3} (R(q, t) - \Phi_1(\alpha, \lambda, \lambda)) dt \\ &= -2 \int_{\lambda}^{\infty} t^{-3} \left(\frac{q\Phi_1 - t\Phi_2 - (q-t)\Phi_1(\alpha, \lambda, \lambda)}{q-t} \right) dt \\ &= -2 \int_{\lambda}^{\infty} t^{-3} \left(\frac{q(\phi'(q) - \phi'(\alpha)) - t(\phi'(t) - \phi'(\alpha))}{q-t} \right) dt \\ &\quad - 2 \int_{\lambda}^{\infty} t^{-2} \left(\frac{q(\kappa'(qt) - \kappa'(\alpha\lambda)) - t(\kappa'(t^2) - \kappa'(\alpha\lambda))}{q-t} \right) dt + h'(\alpha\lambda^2). \end{aligned}$$

The three expressions of the right-hand side are negative by (H2) and because $q(t)t$ is decreasing on $[\lambda, \infty)$ and $h'(p) < 0$, $\forall p > 0$. Thus $\tau(\lambda, \alpha) < 0$, $\forall \alpha \in (0, \lambda)$. $\square$

Note that examples of stored energy functions such that $q(t)t$ is decreasing on $[\lambda, \infty)$ are easily deduced from section 3. These criteria of nonexistence have a physical interpretation through a second relation between the normal stress and the total energy. It describes a property of $E(\lambda, \cdot): (0, \lambda) \rightarrow \mathbb{R}$ and is based on estimations of

$$\frac{\partial U'(\lambda, \alpha)(r)}{\partial \alpha} \quad \text{and} \quad \frac{\partial U(\lambda, \alpha)(r)}{\partial \alpha}, \quad \text{for } (r, \alpha) \in (0, 1] \times (0, \lambda]. \quad (4.13)$$

These preliminary estimations are given in the following lemma.

LEMMA 4.1.5. (i) Let Φ satisfy (A1)–(A4), (A7) with $\gamma_2 > n - 2$ and (P1)–(P3). Then $\sup_{0 < r \leq 1} |\partial U(\lambda, \alpha)(r)/\partial \alpha|$ is uniformly bounded in α on the compacts of $(0, \lambda]$.

(ii) Let Φ satisfy (A1)–(A4), (A7) with $\gamma_2 > n - 1$ and (P1)–(P3). If

$$\sup_{t \geq \lambda} tq'(\lambda, \alpha)(t)$$

is uniformly bounded in α on any compact of $(0, \lambda]$, then $\sup_{0 < r \leq 1} |\partial U'(\lambda, \alpha)(r)/\partial \alpha|$ is uniformly bounded in α on any compact of $(0, \lambda]$.

Proof. Let $\alpha \in (0, \lambda]$. For $t = U(\lambda, \alpha)(r)/r$, $q(t) = U'(\lambda, \alpha)(r)$, one can consider that $\partial U(\lambda, \alpha)(r)/\partial \alpha$ is a solution of

$$\begin{cases} \frac{\partial U'(\lambda, \alpha)(r)}{\partial \alpha} = \frac{q'(t)}{r} \frac{\partial U(\lambda, \alpha)(r)}{\partial \alpha} + \frac{\partial q(t)}{\partial \alpha}, \\ \frac{\partial U(\lambda, \alpha)(1)}{\partial \alpha} = 0. \end{cases} \quad (4.14)$$

We have to estimate the solution of (4.14) for $r \rightarrow 0$ and $\alpha \in X$ a compact subset of $(0, \lambda]$. By introducing the usual change of variable, the integrating factor of (4.14) becomes

$$w(r) = \exp \left(\int_r^1 \frac{q'}{s} ds \right) = \exp \left(- \int_\lambda^t \frac{q'}{q-y} dy \right), \quad \text{where } q(t) = q. \quad (4.15)$$

We know that $0 < q(\lambda, \alpha)(t) < \alpha \quad \forall t > \lambda$ and that $\lim_{t \rightarrow \infty} q(\lambda, \alpha)(t) = 0$ with (P1)–(P3). Then for $\rho \in (0, \lambda]$, there exists $T' = T(\alpha, \rho)$ such that $q(\lambda, \alpha)(t) < \rho \quad \forall t > T'$, and

$$t - \rho < t - q(\lambda, \alpha)(t) < t, \quad \forall t > T'. \quad (4.16)$$

Note that on a compact X of $(0, \lambda]$, there exists $T = T(X, \rho) \geq \lambda$ such that (4.16) holds $\forall \alpha \in X$, by the continuous dependence theorem applied to (P). Then, the constants involved in (4.16) can be chosen uniformly on X . Now set $q = q(\lambda, \alpha)$ and write (4.15) as follows:

$$w(r) = C(\rho, \alpha) \exp \left(- \int_T^t \frac{q' - 1 + 1}{q - y} dy \right), \quad \text{where } C(\rho, \alpha) = \exp \left(- \int_\lambda^T \frac{q'}{q - y} dy \right).$$

Then by integration

$$w(r) = C(\rho, \alpha) \frac{T - q(T)}{t - q} \exp \left(\int_T^t \frac{1}{y - q} dy \right), \quad \text{where } q(t) = q, \quad (4.17)$$

and we get the following estimation with (4.16):

$$C(\rho, \alpha) \left(\frac{T - q(T)}{t - q} \frac{t}{T} \right) < w(r) < C(\rho, \alpha) \left(\frac{T - q(T)}{t - q} \frac{t - \rho}{T - \rho} \right) < C(\rho, \alpha). \quad (4.18)$$

On the other side, the solution of (4.14) can be written as

$$-\frac{\partial U(\lambda, \alpha)(r)}{\partial \alpha} = \frac{1}{w(r)} \int_r^1 w(s) \frac{\partial q}{\partial \alpha} ds.$$

An estimation follows from (4.18):

$$0 \leq -\frac{\partial U(\lambda, \alpha)(r)}{\partial \alpha} < \left(\frac{T - q(T)}{t - q} \frac{t}{T} \right)^{-1} \int_r^1 \frac{\partial q}{\partial \alpha} ds. \quad (4.19)$$

It remains to consider $\partial q / \partial \alpha$. Let us derive (P) with respect to α .

$$\frac{\partial q'}{\partial \alpha} = \frac{\partial}{\partial q}(F(q, t)) \frac{\partial q}{\partial \alpha}, \quad \forall t \geq \lambda \text{ and } \frac{\partial q(\lambda)}{\partial \alpha} = 1. \quad (P_\alpha)$$

By (A1), $\partial q / \partial \alpha$ is continuous on $[\lambda, \infty)$ and $\partial q / \partial \alpha = \exp\{\int_\lambda^t (\partial / \partial q) F(q, s) ds\}$, where

$$\frac{\partial F(q, t)}{\partial q} = \frac{\partial}{\partial q} \left\{ \frac{n-1}{\Phi_{11}} \left(\frac{\Phi_2 - \Phi_1}{q-t} - \Phi_{12} \right) \right\}. \quad (4.20)$$

By developing (4.20) we get for $q = q(t)$,

$$\begin{aligned} \frac{\partial F(q, t)}{\partial q} &= \frac{n-1}{\Phi_{11}} \left(\frac{\Phi_{21} - \Phi_{11}}{q-t} - \frac{\Phi_2 - \Phi_1}{(q-t)^2} - \Phi_{121} \right) - \frac{(n-1)\Phi_{111}}{\Phi_{11}^2} \left(\frac{\Phi_2 - \Phi_1}{q-t} - \Phi_{12} \right) \\ &= \frac{n-1}{\Phi_{11}} \left\{ \left(-\frac{\Phi_2 - \Phi_1}{q-t} + \Phi_{12} \right) \frac{\Phi_{111}}{\Phi_{11}} - \Phi_{121} \right. \\ &\quad \left. - \frac{1}{q-t} \left(\frac{\Phi_2 - \Phi_1}{q-t} - \Phi_{12} \right) + \frac{\Phi_{111}}{t-q} \right\}. \end{aligned}$$

On the other side,

$$\begin{aligned} \frac{d}{dt} \Phi_{11}(q, t, \dots, t) &= \Phi_{111} q' + (n-1) \Phi_{121} \\ &= (n-1) \left\{ \left(\frac{\Phi_2 - \Phi_1}{q-t} - \Phi_{12} \right) \frac{\Phi_{111}}{\Phi_{11}} + \Phi_{121} \right\}. \end{aligned}$$

Thus by substituting

$$\begin{aligned} \frac{\partial F(q, t)}{\partial q} &= \frac{1}{\Phi_{11}} \left\{ -\frac{d}{dt} (\Phi_{11}(q, t, \dots, t)) \right. \\ &\quad \left. + \frac{n-1}{t-q} \left(\frac{\Phi_2 - \Phi_1}{q-t} - \Phi_{12} \right) + (n-1) \frac{\Phi_{111}}{t-q} \right\}. \end{aligned}$$

Finally,

$$\frac{\partial F(q, t)}{\partial q} = -\frac{1}{\Phi_{11}} \frac{d}{dt} (\Phi_{11}(q, t, \dots, t)) + \frac{n-1}{t-q} \left(\frac{q'}{n-1} + 1 \right). \quad (4.21)$$

Since $q'(t) < 0$ by (A1)–(A3), (4.21) gives with (4.16),

$$\frac{\partial F(q, t)}{\partial q} < -\frac{1}{\Phi_{11}} \frac{d}{dt}(\Phi_{11}(q, t, \dots, t)) + \frac{n-1}{t-\rho}, \quad \forall t > T. \quad (4.22)$$

Now for $K = \exp\{\int_{\lambda}^T \frac{\partial F(q, s)}{\partial q} ds\}$, we can write

$$\begin{aligned} \frac{\partial q(t)}{\partial \mu} &= K \exp\left(\int_T^t \frac{\partial F(q, s)}{\partial q} ds\right) \\ &< K \left(\frac{\Phi_{11}(q(T), T, \dots, T)}{\Phi_{11}(q, t, \dots, t)}\right) \left(\frac{t-\rho}{T-\rho}\right)^{n-1}, \quad \forall t > T. \end{aligned}$$

Then (A7) gives the following estimation for $M > 0$:

$$0 \leq \frac{\partial q}{\partial \alpha} < Mt^{n-1-\gamma_2}, \quad \forall t > T. \quad (4.23)$$

By recalling that $t \leq \lambda/r$, the condition $\gamma_2 > n-2$ in (i) implies that $\partial q/\partial \alpha$ is integrable on $(0, 1)$. Thus $|\partial U(\lambda, \alpha)(r)/\partial \alpha|$ is bounded from above on $(0, 1]$. The estimation (4.23) is uniform on the compacts of $(0, \lambda]$, since $K = K(\lambda, T)$, $M = M(\lambda, T)$ in (4.23), and since we can choose T uniform in α on X in (4.16).

For (ii) one can write with (4.14), for $q = q(t)$,

$$\sup_{0 < r \leq 1} \left| \frac{\partial U'(\lambda, \alpha)(r)}{\partial \alpha} \right| = \sup_{0 < r \leq 1} \left| \left\{ \frac{q'}{r} \frac{\partial U(\lambda, \alpha)(r)}{\partial \alpha} + \frac{\partial q}{\partial \alpha} \right\} \right|.$$

Then by assumption on tq' and with (i),

$$\sup_{0 < r \leq 1} \left| \frac{q'}{r} \frac{\partial U(\lambda, \alpha)(r)}{\partial \alpha} \right| = \sup_{0 < r \leq 1} \left| \frac{tq'}{U(\lambda, \alpha)(r)} \frac{\partial U(\lambda, \alpha)(r)}{\partial \alpha} \right| < \infty,$$

uniformly in α on any compact of $(0, \lambda]$. On the other hand, since $\gamma_2 > n-1$, $\partial q/\partial \alpha$ is bounded from above by (4.23) and as in (i), this estimation is uniform on the compacts of $(0, \lambda]$. $\square$

PROPOSITION 4.1.6. Let Φ satisfy (A1)–(A4), (A5) with $\beta < n-1$ and the hypotheses of Lemma 4.1.5. Then $\partial_{\alpha} E(\lambda, \alpha)$ is continuous in α on $(0, \lambda]$ and

$$\frac{\partial E(\lambda, \alpha)}{\partial \alpha} = -(U(\lambda, \alpha)(0))^{n-1} \tau(\lambda, \alpha) \frac{\partial U(\lambda, \alpha)(0)}{\partial \alpha}.$$

Proof. If $\beta < n-1$, $E(\lambda, \alpha) < \infty$. For deriving $E(\lambda, \cdot)$ we need to show that $r^{n-1} \partial \Phi / \partial \alpha$ has an upper bound uniformly integrable in α on the compacts of $(0, \lambda]$. Let us consider

$$\begin{aligned} r^{n-1} \left| \frac{\partial \Phi}{\partial \alpha} \right| &= r^{n-1} \left| \Phi_1 \frac{\partial U'}{\partial \alpha} + \frac{n-1}{r} \Phi_2 \frac{\partial U}{\partial \alpha} \right| \\ &\leq \left| U^{n-1} T(r) \frac{\partial U'}{\partial \alpha} \right| + (n-1) \left| r^{n-2} \Phi_2 \frac{\partial U}{\partial \alpha} \right|. \end{aligned}$$

By using the form of the equilibrium equation given in Theorem 1.1.2 we get

$$r^{n-1} \left| \frac{\partial \Phi}{\partial \alpha} \right| \leq \left| U^{n-1} T(r) \frac{\partial U'}{\partial \alpha} \right| + \frac{\partial U}{\partial \alpha} \left| (n-1) U' U^{n-2} T(r) \right| + \left| U^{n-1} T'(r) \right| \frac{\partial U}{\partial \alpha}.$$

With (4.1.1) and knowing that $\lambda - \alpha < U(\lambda, \alpha)(r) < \lambda$ and $0 < U'(\lambda, \alpha)(r) < \alpha$ we get

$$r^{n-1} \left| \frac{\partial \Phi}{\partial \alpha} \right| \leq \left| \lambda^{n-1} T(r) \frac{\partial U'}{\partial \alpha} \right| + (n-1) \frac{\partial U}{\partial \alpha} \left\{ \alpha \lambda^{n-2} |T(r)| + \lambda^{n-1} \left| \left(\frac{U(r)}{r} \right)^{-n} \left(\frac{U(r)}{r} \right)' R \left(U'(r), \frac{U(r)}{r} \right) \right| \right\}.$$

Let X be a compact of $(0, \lambda]$. Since $E(\lambda, \alpha) < \infty$, $\tau(\lambda, \alpha) > -\infty$ by Proposition 4.1.2. On the other hand, $T(\lambda, \alpha)(r)$ is uniformly bounded for (r, α) in $(0, 1] \times X$. The same remark holds for $\partial_\alpha U'(\lambda, \cdot)(r)$, $\partial_\alpha U(\lambda, \cdot)(r)$ on $(0, 1] \times X$ by Lemma 4.1.5. Finally (A5) shows that $T'(r)$ is uniformly bounded from above on $(0, 1] \times X$ by an integrable function on $(0, 1)$. Therefore there exists $M(r, X) \in L^1(0, 1)$ such that

$$r^{n-1} \left| \frac{\partial \Phi}{\partial \alpha} \right| < M(r, X), \quad \forall \alpha \in X, \quad \forall r \in (0, 1]. \quad (4.24)$$

One can derive $E(\lambda, \alpha)$ with respect to α ,

$$\frac{\partial E(\lambda, \alpha)}{\partial \alpha} = \int_0^1 r^{n-1} \frac{\partial \Phi}{\partial \alpha} dr = \int_0^1 r^{n-1} \left(\Phi_1 \frac{\partial U'}{\partial \alpha} + \frac{n-1}{r} \Phi_2 \frac{\partial U}{\partial \alpha} \right) dr.$$

Since $\partial U'/\partial \alpha = (d/dr) \cdot (\partial U/\partial \alpha)$, one can write

$$r^{n-1} \Phi_1 \frac{\partial U'}{\partial \alpha} = \frac{d}{dr} \left(r^{n-1} \Phi_1 \frac{\partial U}{\partial \alpha} \right) - \frac{d}{dr} (r^{n-1} \Phi_1) \frac{\partial U}{\partial \alpha},$$

and with the equilibrium equation given in Theorem 1.1.2,

$$\begin{aligned} \frac{\partial E(\lambda, \alpha)}{\partial \alpha} &= \int_0^1 \left\{ \frac{d}{dr} \left(r^{n-1} \Phi_1 \frac{\partial U}{\partial \alpha} \right) - \frac{d}{dr} (r^{n-1} \Phi_1) \frac{\partial U}{\partial \alpha} + (n-1) r^{n-2} \Phi_2 \frac{\partial U}{\partial \alpha} \right\} dr \\ &= r^{n-1} \Phi_1 \frac{\partial U}{\partial \alpha} \Big|_0^1. \end{aligned}$$

Now $(\partial U/\partial \alpha)(1) = 0$ since $U(\lambda, \alpha)(1) = \lambda$, and then

$$\frac{\partial E(\lambda, \alpha)}{\partial \alpha} = -\lim_{r \rightarrow 0} r^{n-1} \Phi_1 \frac{\partial U}{\partial \alpha} = -(U(\lambda, \alpha)(0))^{n-1} \tau(\lambda, \alpha) \frac{\partial U(\lambda, \alpha)(0)}{\partial \alpha}. \quad \square$$

The physical interpretation of the second criterion of nonexistence follows straightforwardly from that proposition. Under the hypotheses of Proposition 4.1.6, $\tau(\lambda, \alpha) < 0$ implies that $\partial E(\lambda, \alpha)/\partial \alpha < 0$, $\forall \alpha \in (0, \lambda]$. Thus all the solutions of (IVP) presenting a hole at $r = 0$ are instable.

On the other hand, this proposition also shows that if there exists a solution $U(\lambda, w(\lambda))$ to the problem of cavitation, then $U(\lambda, w(\lambda))$ minimizes $E(\lambda, \cdot)$ on $(0, \lambda]$ since $\tau(\lambda, \cdot)$ is strictly monotone on $(0, \lambda]$. This result generalises the considerations of [10] about the energy of the solution of the boundary problems. A

property of convexity of the total energy is also used in [8] in another context for deducing uniqueness of solutions to problem (1.6).

The results about nonexistence and existence have to be considered with care in view of comparison with experimental data, since it has been shown that nonradial deformations, involving filamentary voids (see [6]) (observed experimentally in [3]) or multiple cavities (see [2]) have less than or the same energy as the pure radial deformation, presenting a single radial hole. Thus the results about nonexistence only mean that we have nonexistence of a solution to the problem of radially symmetric cavitation.

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