SCALAR WAVE SCATTERING OF A PROLATE SPHEROID AS A PARAMETER EXPANSION OF THAT OF A SPHERE

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Abstract. In this paper the scattering problem for the prolate spheroid,

$$\Delta U_p + k^2 U_p = 0,$$

is solved by way of an asymptotic parameter expansion, where the spheroid is considered a perturbation of a sphere (which has an exact solution). The error of the asymptotic approximation is then estimated.

Introduction. Generally, the problem of scattering is extremely difficult and few problems can be solved exactly. For most problems, one has to be content with some form of approximate solution. One such approximation type is that which is valid for some parameters. The sphere is one of the very few bodies for which an exact solution of the scattering problem is available, and, combined with the extreme simplicity of the shape, has necessitated the study of other convex bodies (one of which is the prolate spheroid) as a deviation from or a perturbation of the sphere.

There have been a number of studies on spheroids (oblate and prolate) by Van Buren and King [18], 1972, Senior and Uslenghi [6], 1969; Sidman and Handelman [13], 1972; and many others. However, the solution has always been in terms of spheroidal wave functions (radial and angular functions). Senior and Uslenghi (cf. [6]) expressed the scattering potential for the prolate spheroid in terms of spheroidal wave functions and later (under shape approximation) represented the scatter potential in terms of spherical functions. However, their expansion is not a parameter expansion.

In Sec. 1 we transform the spheroidal coordinates to spherical polar coordinates, in the limit, as the prolate spheroid tends to a sphere. Then we obtain the reduced wave equation for the prolate spheroid in the form

$$(\Delta_p + k^2)U_p = (\Delta_s + k^2 + \varepsilon^2 L_\varepsilon)U_p, \quad (1.01)$$

where L_ε is an operator, ε is a parameter, U_p is the scattering potential for the prolate spheroid, and Δ_p and Δ_s are the Laplacian operators for the spheroid and sphere, respectively.

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We look for a solution of the Helmholtz equation

$$(\Delta_p + k^2)U_p = 0 \quad (1.02)$$

as

$$U_p = \sum_{q=0}^{\infty} \varepsilon^{2q} Q_{2q}(r, \theta, \psi), \quad (1.03)$$

where Q_{2q} is to be computed.

The motivation for (1.03) stems from the asymptotic expansion for the solution U of $(\Delta + k^2)U = 0$. Sommerfeld and Rung [16] proposed the function $e^{ik\tau}\Phi$ as an asymptotic expression for U . Later Birkhoff in 1933 [5] generalized this procedure by proposing U to be of the form

$$e^{ik\tau} \sum_{\nu=0}^{\infty} \Phi_{\nu}(ik)^{-\nu}. \quad (1.03)^+$$

These asymptotic expansions, popularly known as the ray method with respect to geometric considerations, and further generalizations have been used quite often in recent years to obtain formal solutions to many problems including scattering (cf. Keller et al. [10]). Some asymptotic solutions of this form are presented in papers by Anyanwu and Keller [2, 4]; Anyanwu [3]; and Lynn and Keller [11].

In the process of developing the theory we look at the separation of variable technique of the reduced wave equation $\nabla^2 U_p + k^2 U_p = 0$ as studied by Senior and Uslenghi in [6]. However, we consider a parameter expansion to improve their approximation (see the Appendix). Later, however, we employ the method of regular perturbation so that certain a priori conditions are avoided. The natural choice of ε for our expansion is $\varepsilon = l/a$, which gives rise to (1.03). Here a is the radius of the sphere to which the spheroid approximates and l is one half the interfocal distance.

In Sec. 3, an error estimate of the asymptotic approximation is presented.

An application of this method [17] shows how the radiation from a spherical source near a scattering sphere could be retrieved from a situation where the spherical source is near a scattering prolate spheroid.

1. Transformation from spheroidal to spherical polar coordinates in the limit and derivation of the wave equation for the prolate spheroidal obstacle. The prolate spheroidal coordinates can be related to the Cartesian coordinates thus:

$$\begin{aligned} x &= l \sinh \mu \sin \nu \cos \phi, \\ y &= l \sinh \mu \sin \nu \sin \phi, \\ z &= l \cosh \mu \cos \nu, \end{aligned} \quad (1.1)$$

where $0 \leq \mu < \infty$, $0 \leq \nu < \pi$, and $0 \leq \phi < 2\pi$.

We obtain a prolate spheroid by rotating an ellipse about the major axis (Fig. 1.1, about the z -axis); and an oblate spheroid by rotating about the minor axis (Fig. 1.1, about the y -axis).

For small values of μ prolate spheroids are rod-shaped and so can be regarded as an approximation to a wire antenna. In the case of oblate spheroids the surface $\mu = 0$ is a disc $y = 0$, $x^2 + z^2 = l^2$.

FIG. 1.1. The geometric representation of the prolate spheroid.

From the coordinate system (1.1), we obtain the Laplacian as

$$\nabla_p^2 U_p = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial \mu} \left(\frac{h_1 h_2}{h_3} \frac{\partial U_p}{\partial \mu} \right) + \frac{\partial}{\partial \nu} \left(\frac{h_3 h_1}{h_2} \frac{\partial U_p}{\partial \nu} \right) + \frac{\partial}{\partial \phi} \left(\frac{h_2 h_3}{h_1} \frac{\partial U_p}{\partial \phi} \right) \right\}, \quad (1.2)$$

where h_1 , h_2 , h_3 are metric coefficients given by

$$h_i^2 = \left(\frac{\partial x}{\partial u_i} \right)^2 + \left(\frac{\partial y}{\partial u_i} \right)^2 + \left(\frac{\partial z}{\partial u_i} \right)^2, \quad i = 1, 2, 3,$$

so that

$$\begin{aligned} \nabla_p^2 U_p &= \frac{1}{l^2 \{ (\cosh \mu)^2 - (\cos \nu)^2 \}} \\ &\cdot \left\{ \frac{\partial^2 U_p}{\partial u^p} + \frac{\partial^2 U_p}{\partial \nu^2} + \coth \mu \frac{\partial U_p}{\partial \mu} + \cos \nu \frac{\partial U_p}{\partial \nu} \right\} \\ &+ \frac{1}{l^2 (\sinh \mu \sin \nu)^2} \frac{\partial^2 U_p}{\partial \phi^2}. \end{aligned} \quad (1.3)$$

FIG. 1.2. Confocal ellipse.

Let us now look at the spherical polar coordinates given by

$$\begin{aligned} x &= r \sin \theta \cos \psi, \\ y &= r \sin \theta \sin \psi, \\ z &= r \cos \theta, \end{aligned} \quad (1.4)$$

where $0 \leq r < \infty$, $0 \leq \theta \leq \pi$, and $0 \leq \psi \leq 2\pi$.

Our immediate task would be to obtain the Laplacian of the prolate spheroid in spherical coordinates. Our expectation is that such a Laplacian would be the sum of that of a sphere and a small term. To do this, we relate the spheroidal coordinates to the polar spherical coordinates.

First, we want to find a suitable parameter that would characterize our small term. We recall that the prolate spheroid is generated by a rotation of an ellipse about its major axis. The equation of an ellipse with centre at the origin will then be

$$\sqrt{(x-l)^2 + y^2} + \sqrt{(x+l)^2 + y^2} = 2a \quad (1.5)$$

($l = l(a, b)$, a constant), that is,

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-l^2/a^2)} = 1. \quad (1.6)$$

The ellipse collapses into a circle in the limit as $l/a \rightarrow 0$. We may then conclude that the prolate spheroid collapses into a sphere in the same limit $l/a \rightarrow 0$, since a sphere is generated by a circle. Thus the parameter suitable for our expansion will be l/a , where l is one half the interfocal distance and a (which is finite) is the radius of the approximating sphere.

FIG. 1.3. An elliptic cross section in the Cartesian plane.

Comparing Eqs. (1.1) and (1.4), we find

$$\frac{y}{x} = \tan \phi \quad \text{and} \quad \frac{y}{x} = \tan \psi$$

so that

$$\psi \rightarrow \phi. \quad (1.7)$$

We wish to relate (μ, ν, ϕ) to $(r, \theta, \psi; \varepsilon)$ for l/a small such that

$$\begin{aligned} \mu &= \mu(r, \theta, \psi; \varepsilon), \\ \nu &= \nu(r, \theta, \psi; \varepsilon), \\ \phi &= \phi(r, \theta, \psi; \varepsilon). \end{aligned} \quad (1.8)$$

From (1.1) and (1.4) we have already obtained $\phi = \psi$. Also,

$$l \sinh \mu \sin \nu = r \sin \theta, \quad l \cosh \mu \cos \nu = r \cos \theta, \quad (1.9)$$

and for $\mu \gg 1$, looking at the dominant terms we let

$$\begin{aligned} e^\mu &= \frac{1}{\varepsilon} f_{-1}(r, \theta) + f_0(r, \theta) + \varepsilon^2 f_1(r, \theta) + \varepsilon^2 f_2(r, \theta) + \dots, \\ \nu &= g_0(r, \theta) + \varepsilon g_1(r, \theta) + \varepsilon^2 g_2(r, \theta) + \varepsilon^3 g_3(r, \theta) + \dots. \end{aligned} \quad (1.10)$$

That is,

$$e^{-\mu} = \frac{\varepsilon}{f_{-1}} - \frac{\varepsilon^2 f_0}{f_{-1}^2} - \frac{\varepsilon^3 f_1}{f_{-1}^2} - \dots, \quad (1.11)$$

so that

$$\begin{aligned} \cosh \mu = & \frac{1}{2\varepsilon} f_{-1} + \frac{1}{2} f_0 + \frac{1}{2} \left(f_1 + \frac{1}{f_{-1}} \right) \varepsilon \\ & + \frac{1}{2} \left(f_2 - \frac{f_0}{f_{-1}^2} \right) \varepsilon^2 + \frac{1}{2} \left(f_3 - \frac{f_1}{f_{-1}^2} \right) \varepsilon^3 + \dots, \end{aligned} \quad (1.12a)$$

$$\begin{aligned} \sinh \mu = & \frac{1}{2\varepsilon} f_{-1} + \frac{1}{2} f_0 + \frac{1}{2} \left(f_1 - \frac{1}{f_{-1}} \right) \varepsilon + \frac{1}{2} \left(f_2 + \frac{f_0}{f_{-1}^2} \right) \varepsilon^2 \\ & + \frac{1}{2} \left(f_3 + \frac{f_1}{f_{-1}^2} \right) \varepsilon^3 + \dots. \end{aligned} \quad (1.12b)$$

Also,

$$\begin{aligned} \cos \nu = & \cos g_0 - g_1 \sin g_0 \cdot \varepsilon + \left[-\frac{g_1}{2} \cos g_0 - g_2 \sin g_0 \right] \varepsilon^2 \\ & + \left[-g_1 g_2 \cos g_0 + \sin g_0 \left(\frac{g_1^3}{6} - g_3 \right) \right] \varepsilon^3 + \dots, \end{aligned} \quad (1.13a)$$

$$\begin{aligned} \sin \nu = & \sin g_0 + g_1 \cos g_0 \cdot \varepsilon + \left(-\frac{g_1^2}{2} \sin g_0 + g_2 \cos g_0 \right) \varepsilon^2 \\ & + \left(-g_1 g_2 \sin g_0 + \left(g_3 - \frac{g_1^3}{6} \right) \cos g_0 \right) \varepsilon^3 + \dots. \end{aligned} \quad (1.13b)$$

From (1.9) we have

$$\begin{aligned} a\varepsilon \left\{ \frac{1}{2\varepsilon} f_{-1} + \frac{1}{2} f_0 + \frac{1}{2} \left(f_1 - \frac{1}{f_{-1}} \right) \varepsilon \right. \\ \left. + \frac{1}{2} \left(f_2 - \frac{f_0}{f_{-1}^2} \right) \varepsilon^2 + \frac{1}{2} \left(f_3 + \frac{f_1}{f_{-1}^2} \right) \varepsilon^3 + \dots \right\} \\ \cdot \left\{ \sin g_0 + g_1 \cos g_0 \cdot \varepsilon + \left(-\frac{g_1^2}{2} \sin g_0 + g_2 \cos g_0 \right) \cdot \varepsilon^2 \right. \\ \left. + \left(-g_1 g_2 \sin g_0 + \left(g_3 - \frac{g_1^3}{6} \right) \cos g_0 \right) \varepsilon^3 + \dots \right\} \\ = r \sin \theta \end{aligned} \quad (1.14)$$

and

$$\begin{aligned}
 a\varepsilon & \left\{ \frac{1}{2\varepsilon}f_{-1} + \frac{1}{2}f_0 + \frac{1}{2}\left(f_1 + \frac{1}{f_{-1}}\right)\varepsilon \right. \\
 & \quad \left. + \frac{1}{2}\left(f_2 - \frac{f_0}{f_{-1}^2}\right)\varepsilon^2 + \frac{1}{2}\left(f_3 - \frac{f_1}{f_{-1}^2}\right)\varepsilon^3 + \dots \right\} \\
 & \quad \cdot \left\{ \cos g_0 - g_1 \sin g_0 \cdot \varepsilon + \left[-\frac{g_1}{2}\cos g_0 - g_2 \sin g_0\right]\varepsilon^2 \right. \\
 & \quad \left. + \left[-g_1g_2\cos g_0 + \sin g_0\left(\frac{g_1^3}{6} - g_3\right)\right]\varepsilon^3 + \dots \right\} \\
 & = r \cos \theta.
 \end{aligned} \tag{1.15}$$

From (1.14) and (1.15) we then obtain for,

$$\begin{aligned}
 O(1): \quad & \frac{a}{2}f_{-1}\cos g_0 = r \cos \theta, \\
 & \frac{a}{2}f_{-1}\sin g_0 = r \sin \theta, \\
 & g_0 = \theta, \\
 & f_{-1} = \frac{2r}{a},
 \end{aligned} \tag{1.16}$$

$$\begin{aligned}
 O(\varepsilon): \quad & \frac{a}{2}f_0\cos \theta - rg_1\sin \theta = 0, \\
 & \frac{a}{2}f_0\sin \theta + rg_1\cos \theta = 0, \\
 \Rightarrow & f_0 = 0, \\
 & g_1 = 0,
 \end{aligned} \tag{1.17}$$

$$\begin{aligned}
 O(\varepsilon^2): \quad & \frac{a}{2}f_1\cos \theta - rg_2\sin \theta = -\frac{a^2}{4r}\cos \theta, \\
 & \frac{a}{2}f_1\sin \theta + rg_2\cos \theta = \frac{a^2}{4r}\sin \theta, \\
 \Rightarrow & f_1 = -\frac{a}{2r}\cos 2\theta, \\
 & g_2 = \left(\frac{a}{2r}\right)^2\sin 2\theta,
 \end{aligned} \tag{1.18}$$

$$\begin{aligned}
 O(\varepsilon^3): \quad & \frac{a}{2}f_2\cos \theta - rg_3\sin \theta = 0, \\
 & \frac{a}{2}f_2\sin \theta + rg_3\cos \theta = 0, \\
 \Rightarrow & f_2 = 0, \\
 & g_3 = 0,
 \end{aligned} \tag{1.19}$$

etc.

On substitution of (1.16)–(1.19) into (1.10), we get

$$\begin{aligned} e^\mu &= \frac{2r}{a} \cdot \frac{1}{\varepsilon} - \frac{a}{2r} \cos 2\theta \cdot \varepsilon + O(\varepsilon^3), \\ \nu &= \theta + \left(\frac{a}{2r}\right)^2 \sin 2\theta \cdot \varepsilon^2 + O(\varepsilon^4), \end{aligned} \quad (1.20)$$

and also $\phi = \psi$.

We shall now transform the reduced wave equation (1.3) using (1.20). To do this, we need the following relations which we may obtain from (1.20):

$$\frac{\partial r}{\partial \mu} = r \left\{ 1 - 2 \left(\frac{a}{2r}\right)^2 \cos 2\theta \cdot \varepsilon^2 + O(\varepsilon^4) \right\}, \quad (1.21a)$$

$$\frac{\partial r}{\partial \nu} = -\frac{a^2}{2r} \sin 2\theta \cdot \varepsilon^2 + O(\varepsilon^4), \quad (1.21b)$$

$$\frac{\partial \theta}{\partial \mu} = \frac{1}{2} \left(\frac{a}{r}\right)^2 \sin 2\theta \cdot \varepsilon^2 + O(\varepsilon^4), \quad (1.21c)$$

$$\frac{\partial \theta}{\partial \nu} = 1 - 2 \left(\frac{a}{2r}\right)^2 \cos 2\theta \cdot \varepsilon^2 + O(\varepsilon^4), \quad (1.21d)$$

Also,

$$\frac{\partial^2 r}{\partial \mu^2} = r \left\{ 1 - \left(\frac{a}{r}\right)^2 \cos 2\theta \cdot \varepsilon^2 + O(\varepsilon^4) \right\}, \quad (1.22a)$$

$$\frac{\partial^2 r}{\partial \nu^2} = -\left(\frac{a}{r}\right)^2 \cos 2\theta \cdot \varepsilon^2 + O(\varepsilon^4), \quad (1.22b)$$

$$\frac{\partial^2 \theta}{\partial \mu^2} = -\left(\frac{a}{r}\right)^2 \sin 2\theta \cdot \varepsilon^2 + O(\varepsilon^4), \quad (1.22c)$$

$$\frac{\partial^2 \theta}{\partial \nu^2} = \left(\frac{a}{r}\right)^2 \sin 2\theta \cdot \varepsilon^2 + O(\varepsilon^4). \quad (1.22d)$$

We also require the relation

$$\begin{aligned} \frac{\partial^2 U_p}{\partial u_i^2} &= \frac{\partial^2 U_p}{\partial r^2} \left(\frac{\partial r}{\partial u_i}\right)^2 + 2 \frac{\partial^2 U_p}{\partial r \partial \theta} \cdot \frac{\partial r}{\partial u_i} \frac{\partial \theta}{\partial u_i} \\ &\quad + \frac{\partial^2 U_p}{\partial \theta^2} \left(\frac{\partial \theta}{\partial u_i}\right)^2 + \frac{\partial U_p}{\partial r} \frac{\partial^2 r}{\partial u_i^2} + \frac{\partial U_p}{\partial \theta} \cdot \frac{\partial^2 \theta}{\partial u_i^2} \end{aligned} \quad (1.23)$$

for $i = 1, 2$ and $u_1 = \mu$, $u_2 = \nu$. Also, using (1.12) and (1.13) in conjunction with (1.16)–(1.19), we have

$$l^2 \{ \cosh^2 \mu - \cos^2 \nu \} = r^2 \left\{ 1 - \left(\frac{a}{r}\right)^2 \cos 2\theta \cdot \varepsilon^2 + O(\varepsilon^4) \right\}, \quad (1.24a)$$

$$\coth \mu = \frac{\cosh \mu}{\sinh \mu} = 1 + \frac{1}{2} \left(\frac{a}{r}\right)^2 \cdot \varepsilon^2 + O(\varepsilon^4), \quad (1.24b)$$

$$l^2 (\sinh^2 \mu \sin^2 \nu) = r^2 \sin^2 \theta \{ 1 + O(\varepsilon^4) \}, \quad (1.24c)$$

$$\cot \nu = \frac{\cos \nu}{\sin \nu} = \cot \theta \left\{ 1 - 2 \left(\frac{a}{2r}\right)^2 \cdot \varepsilon^2 + O(\varepsilon^4) \right\}, \quad (1.24d)$$

and, employing (1.21) and (1.22), we obtain the following:

$$\begin{aligned} \frac{\partial U_p}{\partial \mu} &= \frac{\partial U_p}{\partial \theta} \left\{ \frac{1}{2} \left(\frac{a}{r} \right)^2 \sin 2\theta \cdot \varepsilon^2 + O(\varepsilon^4) \right\} \\ &\quad + \frac{\partial U_p}{\partial r} \left\{ r \left(1 - 2 \left(\frac{a}{r} \right)^2 \cos 2\theta \cdot \varepsilon^2 + O(\varepsilon^4) \right) \right\}, \end{aligned} \quad (1.25a)$$

$$\begin{aligned} \frac{\partial U_p}{\partial \nu} &= \frac{\partial U_p}{\partial \theta} \left\{ 1 - 2 \left(\frac{a}{2r} \right)^2 \cos 2\theta \cdot \varepsilon^2 + O(\varepsilon^4) \right\} \\ &\quad + \frac{\partial U_p}{\partial r} \left\{ -\frac{a^2}{2r} \sin 2\theta \cdot \varepsilon^2 + O(\varepsilon^4) \right\}. \end{aligned} \quad (1.25b)$$

With (1.23), we get

$$\begin{aligned} \frac{\partial^2 U_p}{\partial \mu^2} &= \frac{\partial^2 U_p}{\partial r^2} \left\{ r^2 \left(1 - \left(\frac{a}{r} \right)^2 \cos 2\theta \cdot \varepsilon^2 + O(\varepsilon^4) \right) \right\} \\ &\quad + \frac{\partial^2 U_p}{\partial r \partial \theta} \left\{ r \left(\frac{a}{r} \right)^2 \sin 2\theta \cdot \varepsilon^2 + O(\varepsilon^4) \right\} + \frac{\partial^2 U_p}{\partial \theta^2} \{ O(\varepsilon^4) \} \\ &\quad + \frac{\partial U_p}{\partial r} \left\{ r \left(1 - \left(\frac{a}{r} \right)^2 \cos 2\theta \cdot \varepsilon^2 + O(\varepsilon^4) \right) \right\} \\ &\quad + \frac{\partial U_p}{\partial \theta} \left\{ -\left(\frac{a}{r} \right)^2 \sin 2\theta \cdot \varepsilon^2 + O(\varepsilon^4) \right\} \end{aligned} \quad (1.26a)$$

and

$$\begin{aligned} \frac{\partial^2 U_p}{\partial \nu^2} &= \frac{\partial^2 U_p}{\partial r^2} \{ O(\varepsilon^4) \} + \frac{\partial^2 U_p}{\partial r \partial \theta} \{ O(\varepsilon^4) \} \\ &\quad + \frac{\partial^2 U_p}{\partial \theta^2} \left\{ 1 - \left(\frac{a}{r} \right)^2 \cos 2\theta \cdot \varepsilon^2 + O(\varepsilon^4) \right\} \\ &\quad + \frac{\partial u_p}{\partial \theta} \left\{ \left(\frac{a}{r} \right)^2 \sin 2\theta \cdot \varepsilon^2 + O(\varepsilon^4) \right\} \\ &\quad + \frac{\partial U_p}{\partial r} \left\{ -\left(\frac{a}{r} \right)^2 \cos 2\theta \cdot \varepsilon^2 + O(\varepsilon^4) \right\}. \end{aligned} \quad (1.26b)$$

Now imploring (1.24)–(1.26) we then obtain the Laplacian for the prolate spheroidal

scatterer (1.3) in the form

$$\begin{aligned} \nabla_p^2 U_p = & \left\{ 1 - \left(\frac{a}{r} \right)^2 \cos 2\theta \cdot \varepsilon^2 + O(\varepsilon^4) \right\}^{-1} \\ & \cdot \left\{ \frac{\partial^2 U_p}{\partial r^2} + \frac{2}{r} \frac{\partial U_p}{\partial r} + \frac{1}{r^2} \frac{\partial^2 U_p}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial U_p}{\partial \theta} \right. \\ & + \left(\frac{a}{r} \right)^2 \left(-\cos 2\theta \frac{\partial^2 U_p}{\partial r^2} + \frac{\sin 2\theta}{r} \frac{\partial^2 U_p}{\partial r \partial \theta} - \frac{\cos 2\theta}{r^2} \frac{\partial^2 U_p}{\partial \theta^2} \right. \\ & \left. \left. - \frac{3(r+2)}{2r^2} \cos 2\theta \frac{\partial U_p}{\partial r} + \frac{1}{2r^2} (1 + 2 \sin 2\theta - 2 \cot \theta) \frac{\partial U_p}{\partial \theta} \right) \varepsilon^2 + O(\varepsilon^4) \right\} \\ & + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 U_p}{\partial \psi^2} + O(\varepsilon^4). \end{aligned} \quad (1.27)$$

Thus the reduced wave equation for the prolate spheroidal scatterer, namely,

$$(\nabla_p^2 + k^2) U_p = 0, \quad (1.28)$$

becomes

$$(\nabla_s^2 + k^2 + \varepsilon^2 L_\varepsilon) U_p = 0, \quad (1.29)$$

where

$$\nabla_s^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \psi^2} \quad (1.30a)$$

and

$$\begin{aligned} L_\varepsilon = \left(\frac{a}{r} \right)^2 & \left\{ -\cos 2\theta \frac{\partial^2}{\partial r^2} + \frac{1}{r} \sin 2\theta \frac{\partial^2}{\partial r \partial \theta} - \frac{\cos 2\theta}{r^2} \frac{\partial^2}{\partial \theta^2} \right. \\ & - \frac{3(r+2)}{2r^2} \cos 2\theta \frac{\partial}{\partial r} + \frac{1}{2r^2} (1 + 2 \sin 2\theta - 2 \cot \theta) \frac{\partial}{\partial \theta} \\ & \left. + \frac{1}{r^2} (\csc^2 \theta - 2) \frac{\partial^2}{\partial \psi^2} + k^2 \cos 2\theta \right\} + O(\varepsilon^2). \end{aligned} \quad (1.30b)$$

It is obvious from (1.29) that we retrieve the reduced wave equation for the spherical scatterer, as expected, when we set $\varepsilon = 0$.

It is worth mentioning that the independent variable U_p under the transformation (1.20) is invariant, consistent with the geometrical theory of wave propagation. This can easily be verified by substituting $U_p = J^\alpha U_p^*$ into (1.28), where J , the Jacobian of the transformation, is given by

$$J = \frac{1}{\rho} \left\{ 1 + 2 \left(\frac{a}{2\rho} \right)^2 \cos 2\theta \cdot \varepsilon^2 + O(\varepsilon^4) \right\}, \quad (1.31)$$

and U_p^* is the transformed independent variable. As $\varepsilon \rightarrow 0$, we should obtain the Helmholtz equation for the spherical scatterer and such is the case when $\alpha \equiv 0$.

2. The asymptotic expansion of the scattered field for the prolate spheroidal scatterer. The scattered field U_p of the prolate spheroidal scatterer may be expressed as

$$U_p \sim \sum_{q=0}^{\infty} \varepsilon^{2q} Q_{2q}(r, \theta, \psi). \quad (2.1)$$

The motivation for this choice of U_p was stated earlier in the introduction. The scattered field U_s by the sphere for an incident plane wave in the direction θ_0 and ψ_0 has the exact value given by [6, p. 358]

$$U_s = - \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} i^n \varepsilon_m \frac{(n-m)!}{(n+m)!} (2n+1) \frac{j_n(ka)}{h_n^{(1)}(ka)} h_n^{(1)}(kr) \cdot P_n^m(\cos \theta) P_n^m(\cos \theta_0) \cos m(\psi - \psi_0), \quad (2.2)$$

and the expansion for the incident wave is

$$U_i = - \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} i^n \varepsilon_m \frac{(n-m)!}{(n+m)!} j_n(kr) P_n^m(\cos \theta) \cdot P_n^m(\cos \theta_0) \cos m(\psi - \psi_0), \quad (2.3)$$

where ε_m = Neumann symbol ($\varepsilon_0 = 1$, $\varepsilon_m = 2$ for $m = 1, 2, \dots$) and the other symbols have their usual meanings. Equation (1.29) may be written as

$$(\Delta_s + k^2 + \varepsilon^2 L_\varepsilon)[Q_0 + \varepsilon^2 Q_2 + \varepsilon^4 Q_4 + \dots] = 0, \quad (2.4)$$

that is,

$$(\Delta + k^2 + \varepsilon^2 L_\varepsilon)Q_0 + \varepsilon^2(\Delta + k^2 + \varepsilon^2 L_\varepsilon)Q_2 + \varepsilon^4(\Delta + k^2 + \varepsilon^2 L_\varepsilon)Q_4 + \dots = 0, \quad (2.5)$$

where the subscript s on the Δ (that is, ∇^2) operator has been dropped for the sake of brevity. We shall now employ the method of regular perturbation on Eq. (2.5). The constant terms (i.e., terms of order unity) are

$$(\Delta_s + k^2)Q_0 = 0. \quad (2.6)$$

The above Eq. (2.6) is the Helmholtz equation for the spherical scatterer. Considering the fact that $U_p \rightarrow U_s$ when $\varepsilon \rightarrow 0$ in Eq. (2.1), it becomes obvious that the most suitable choice for Q_0 is

$$Q_0 = U_s. \quad (2.7)$$

We now consider terms of order ε^2 in (2.5):

$$(\Delta + k^2)Q_2 = -L_{\varepsilon^0}Q_0, \quad (2.8)$$

where $L_{\varepsilon^{2n}}$ ($n = 0, 1, 2, \dots$) represent terms in the operator (1.30b) of up to order $2n$ (that is, terms of $O(\varepsilon^{2n})$), so that generally we obtain Q_{2n} from

$$(\Delta + k^2)Q_{2n} = - \sum_{i=0}^{n-1} L_{\varepsilon^{2i}} Q_{2(n-1-i)}. \quad (2.9)$$

Let us apply the boundary condition on the surface, namely

$$U_i + U_p = 0, \quad (2.10)$$

that is,

$$U_i = \sum_{q=0}^{\infty} \varepsilon^{2q} Q_{2q}(r, \theta, \psi) = 0. \quad (2.11)$$

Now, since on the surface $r = a$ and $Q_0 = U_s$ (where U_i and U_s are defined in (2.2) and (2.3)), it follows that

$$[Q_{2n}]_{\text{surface}} = 0. \quad (2.12)$$

Also U_p satisfies the radiation condition, namely

$$U_p \sim \frac{c}{r} \quad \text{as } r \rightarrow \infty, \quad (2.13)$$

where c is a constant. We shall now construct a suitable Green's function for our problem, taking into consideration the boundary condition at the surface and the radiation condition. The Green's function would satisfy the homogeneous equation

$$\Delta G + k^2 G = 0 \quad \text{except at } (r_0, \theta_0, \psi_0), \quad (2.14)$$

and for $R = |r - r_0|$

$$G \sim \frac{1}{4\pi R} \quad \text{as } R \rightarrow 0 \text{ (i.e., } r \rightarrow r_0) \quad (2.15)$$

and

$$G \sim \frac{c}{r} \quad \text{as } r \rightarrow \infty \text{ (} c = \text{constant)}. \quad (2.16)$$

With a Green's function determined satisfying the above conditions (2.14)–(2.16), we may then write the solution of (2.9) as

$$Q_{2q}(r_0, \theta_0, \psi_0) = \int_0^{2\pi} \int_0^\pi \int_{r_0}^\infty G(r, \theta, \psi; r_0, \theta_0, \psi_0) \cdot \left(\sum_{i=0}^{q-1} L_{\varepsilon^{2i}} Q_{2(q-1-i)} \right) r^2 \sin \theta \, dr \, d\theta \, d\psi. \quad (2.17)$$

The Green's function in spherical coordinates for the reduced wave equation (Morse and Feshbach [12, p. 887]) is

$$G = \frac{e^{ikR}}{4\pi R} = \frac{ik}{4\pi} \sum_{\substack{n=0 \\ m=0}}^{\infty} \varepsilon_m (2n+1) \frac{(n-m)!}{(n+m)!} \cos m(\psi - \psi_0) \cdot P_n^m(\cos \theta) P_n^m(\cos \theta) j_n(kr_0) h_n(kr), \quad r \geq r_0, \quad (2.18)$$

where $G = G(r, \theta, \psi; r_0, \theta_0, \psi_0)$.

Let us now obtain Q_2 . To do this we shall first of all compute $L_{\epsilon^0}Q_0$:

$$\begin{aligned}
 L_{\epsilon^0}Q_0 = \left(\frac{a}{r}\right)^2 \left\{ k^2 \cos 2\theta - \frac{m^2}{r^2} (\csc^2 \theta - 2) \right. \\
 - \cos 2\theta (h_n^{(1)}(kr))^{-1} \frac{d^2}{dr^2} [h_n^{(1)}(kr)] \\
 + \frac{\sin 2\theta}{r} (P_n^m(\cos \theta) h_n^{(1)}(kr))^{-1} \frac{d}{dr} [h_n^{(1)}(kr)] \frac{d}{d\theta} [P_n^m(\cos \theta)] \\
 - \frac{\cos 2\theta}{r^2} (P_n^m(\cos \theta))^{-1} \frac{d^2}{d\theta^2} [P_n^m(\cos \theta)] \\
 - \frac{3(r+2)}{2r^2} \cos 2\theta (h_n^{(1)}(kr))^{-1} \frac{d}{dr} [h_n^{(1)}(kr)] \\
 + \frac{1}{2r^2} (1 + 2 \sin 2\theta - 2 \cot \theta) (P_n^m(\cos \theta))^{-1} \\
 \left. \cdot \frac{d}{d\theta} [P_n^m(\cos \theta)] \right\} U_s, \quad (2.19a)
 \end{aligned}$$

where

$$\frac{d}{d\theta} [P_n^m(\cos \theta)] = n \cot \theta P_n^m(\cos \theta) - \frac{(n+m)}{\sin \theta} P_{n-1}^m(\cos \theta), \quad (2.19b)$$

$$\begin{aligned}
 \frac{d^2}{d\theta^2} [P_n^m(\cos \theta)] = (n \cot^2 \theta + n + n^2 \cot^2 \theta \sin^2 \theta) P_n^m(\cos \theta) \\
 + ((n-1) \sin^2 \theta \cot \theta - (n+m) \cos \theta \\
 - (n+m) \cot \theta \csc \theta) P_{n-1}^m(\cos \theta) \\
 - (n+m-1) \cos \theta P_{n-2}^m(\cos \theta), \quad (2.19c)
 \end{aligned}$$

$$\frac{d}{dr} [h_n^{(1)}(kr)] = k h_{n-1}^{(1)}(kr) - \frac{(n+1)}{r} h_n^{(1)}(kr), \quad (2.19d)$$

and

$$\begin{aligned}
 \frac{d^2}{dr^2} [h_n^{(1)}(kr)] = k^2 h_{n-2}^{(1)}(kr) - \frac{k}{2} (n + (n+1)k) h_{n-1}^{(1)}(kr) \\
 + \frac{k(n+1)(n+2)}{r^2} h_n^{(1)}(kr). \quad (2.19e)
 \end{aligned}$$

Using (2.18) and (2.19) we may now evaluate (2.17) for Q_2 . We shall employ the trigonometric expansions for the associated Legendre polynomials which we may write as (see p. 335 of [1])

$$P_n^m(\cos \theta) = S_k(m, n) (\sin \theta)^m \sin[(m+n+2k+1)\theta], \quad k = 0, 1, \dots, \quad (2.20a)$$

where

$$S_k(m, n) = \pi^{-1/2} 2^{m+1} \frac{\Gamma(m+n+1)}{\Gamma(n+3/2)} \sum_{k=0}^{\infty} \frac{(m+1/2)_k (m+n+1)_k}{k! (n+3/2)_k}. \quad (2.20b)$$

Now, imploring [1] and [7], we may then integrate to obtain

$$Q_2 = \frac{k\pi}{4a} \sum_{\substack{n=0 \\ m=0}}^{\infty} \frac{i^{n+1}}{2^{2m}} \varepsilon_m \left\{ (2n+1) \frac{(n-m)!}{(n+m)!} P_n^m(\cos \theta_0) \cdot S_p(m, n) j_n(ka) \right\}^2 h_n^{(1)}(ka) \\ \cdot \left\{ -16 \left(\frac{m^2}{(2n+3)} + \frac{n}{2n+3} \right) \binom{2(m-1)}{m-1} \right. \\ + \left(\frac{15(ka)^2}{2(2n+1)} - \frac{m^2}{2(2n+3)} + \frac{15k(n+1)(n+2)}{2(2n+3)} + \frac{8n(n+1)}{(2n+3)} \right. \\ \left. \left. - \frac{15n}{(2n+3)} - \frac{4n^2}{(2n+3)} - \frac{15(n+1)}{2} \left[\frac{3}{2(2n+3)} - \frac{1}{2n+4} \right] + \frac{8n}{2n+3} \right) \right. \\ \left. \cdot \binom{2m}{m} - \frac{2n^2}{(2n+3)} \binom{2(m+1)}{m+1} \right\}, \quad (2.21)$$

where

$$\binom{2m}{m} = \frac{2m(2m-1)(2m-2) \cdots (2m-m+1)}{1 \cdot 2 \cdot 3 \cdots m},$$

$S_p(m, n)$ is as given in (2.20b), and all the other symbols have their usual meanings.

With Q_2 determined the scattered field U_p of the prolate spheroidal scatterer may then be given as in (2.1).

3. Error estimate in the asymptotic approximation. We shall now estimate the error in our asymptotic approximation. For convenience we shall simply write U_p as U , so that

$$U = \sum_{q=0}^{\infty} \varepsilon^{2q} Q_{2q}. \quad (3.1)$$

Now let

$$U_N = \sum_{q=0}^N \varepsilon^{2q} Q_{2q}, \quad (3.2)$$

and let the error be

$$R_N = U_N - U. \quad (3.3)$$

We invoke Eq. (1.29), which is

$$(\Delta + k^2 + \varepsilon^2 L_\varepsilon)U = 0,$$

so that

$$(\Delta + k^2 + \varepsilon^2 L_\varepsilon)(U_N - R_N) = 0 \quad (3.4)$$

or

$$(\Delta + k^2 + \varepsilon^2 L_\varepsilon)U_N = (\Delta + k^2 + \varepsilon^2 L_\varepsilon)R_N. \quad (3.5)$$

The left-hand side of this equation yields

$$(\Delta + k^2 + \varepsilon^2 L_\varepsilon)U_N = (\Delta + k^2 + \varepsilon^2 L_\varepsilon) \left\{ \sum_{q=0}^N \varepsilon^{2q} Q_{2q} \right\}. \quad (3.6)$$

Employing (3.1) we write

$$(\Delta + k^2 + \varepsilon^2 L_\varepsilon)U_N = -(\Delta + k^2) \left\{ \sum_{q=N+1}^{\infty} \varepsilon^{2q} Q_{2q} \right\} - \varepsilon^2 L_\varepsilon \left\{ \sum_{q=N+1}^{\infty} \varepsilon^{2q} Q_{2q} \right\}$$

so that, using (2.9), we get

$$(\Delta + k^2 + \varepsilon^2 L_\varepsilon)U_N = \sum_{q=N+1}^{\infty} \varepsilon^{2q} L_{\varepsilon^{2(q-1)}} Q_{2(q-1)} - \sum_{q=N+1}^{\infty} \varepsilon^{2(q+1)} L_\varepsilon Q_{2q},$$

that is,

$$(\Delta + k^2 + \varepsilon^2 L)U_N = \varepsilon^{2(N+1)} L_{\varepsilon^{2N}} Q_{2N} + \sum_{q=N+1}^{\infty} \varepsilon^{2(q+1)} (L_{\varepsilon^{2q}} - L_\varepsilon) Q_{2q} \quad (3.7)$$

or

$$(\Delta + k^2 + \varepsilon^2 L_\varepsilon)U_N = (L_{\varepsilon^{2N}} Q_{2N}) \cdot \varepsilon^{2(N+1)} + O(\varepsilon^{2(N+2)}). \quad (3.8)$$

On substitution into (3.5) we obtain

$$(\Delta + k^2)R_N = -\varepsilon^2 L_\varepsilon R_N + (L_{\varepsilon^{2N}} Q_{2N}) \cdot \varepsilon^{2(N+1)} + O(\varepsilon^{2(N+2)}), \quad (3.9)$$

but

$$L_\varepsilon R_N = L_\varepsilon \left\{ \sum_{q=N+1}^{\infty} \varepsilon^{2q} Q_{2q} \right\} = \sum_{q=N+1}^{\infty} \varepsilon^{2q} L_\varepsilon Q_{2q}$$

or

$$L_\varepsilon R_N = \varepsilon^{2(N+1)} L_{\varepsilon^0} Q_{2N} + O(\varepsilon^{2(N+1)}) \quad (3.10)$$

so that from (3.9) we then get

$$(\Delta + k^2)R_N = (L_{\varepsilon^{2N}} Q_{2N}) \cdot \varepsilon^{2(N+1)} + O(\varepsilon^{2(N+2)}). \quad (3.11)$$

This is a reduced wave equation in R_N for which Δ is the spherical operator. We may then obtain R_N as

$$R_N = \varepsilon^{2(N+1)} \int_{\Omega} G(r, \theta, \psi; r', \theta', \psi') [L_{\varepsilon^{2N}} Q_{2N}] d\Omega + O(\varepsilon^{2(N+2)}) \quad (3.12)$$

so that

$$|R_N| \leq \varepsilon^{2(N+1)} \int_{\Omega} |G(r, \theta, \psi; r', \theta', \psi')| |L_{\varepsilon^{2N}} Q_{2N}| d\Omega + |O(\varepsilon^{2(N+2)})|. \quad (3.13)$$

But

$$|S_N| = |L_{\varepsilon^{2N}} Q_{2N}| = O(1) \quad (3.14a)$$

and from (2.18)

$$|G(r, \theta, \psi; r', \theta', \psi')| = O(1). \quad (3.14b)$$

Thus from (3.13) we may conclude that

$$R_N = O|\varepsilon^{2(N+1)}|. \quad (3.15)$$

This error function suggests that the error would be quite small for $\varepsilon \rightarrow 0$, i.e., $O(\varepsilon^4)$ if the first two terms of the expansion in (2.1) are taken as an approximation.

4. Appendix. The scattering potential for the prolate spheroid that departs only infinitesimally from the sphere as a perturbation of the solution for a sphere was given in [6]. Separation of variables of (1.28) for

$$\frac{l^2 \cos^2 \theta}{r^2} = \frac{l^2}{a^2} \quad (= \varepsilon^2), \quad (4.1)$$

where ε is a small parameter, yields a result that is close to the result in [6, p. 437]. The *raison d'être* for setting $l^2 \cos^2 \theta / r^2 = \varepsilon^2$ is that $|\cos^2 \theta| \leq 1$ and on the spherical surface ($r = a$) and outside the body (sphere)

$$\frac{l^2 \cos^2 \theta}{r^2} < \frac{l^2}{a^2} = \varepsilon^2. \quad (4.2)$$

With this approximation, we assume that the solution of (1.28) is of the form

$$U_p = R(r)T(\psi)H(\theta)$$

so that on separating (1.28) we get

$$R'' + \frac{2}{r}R' + \left[k^2(1 - \varepsilon^2) - \frac{n(n+1)}{r^2} \right] R = 0, \quad (4.3a)$$

$$H'' + \cot \theta H' + \left[n(n+1) - \frac{m^2}{\sin^2 \theta} \right] H = 0, \quad (4.3b)$$

$$T'' + \frac{m^2}{(1 + \varepsilon)} T = 0. \quad (4.3c)$$

Solutions to these equations take the form

$$H = P_n^m(\cos \theta), \quad (4.4a)$$

$$T = a_m \cos m\rho\psi + b_m \sin m\rho\psi, \quad (4.4b)$$

$$R = c_n j_n(k_p r) + d_n h_n^{(1)}(k_p r), \quad (4.4c)$$

respectively, where the symbols have their usual meaning and

$$\rho = (1 + \varepsilon)^{-1/2}, \quad k_p = k(1 - \varepsilon^2)^{1/2}. \quad (4.5)$$

Thus the scattered potential U_p becomes

$$U_p = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} a_{nm} h_n(k_p r) P_n^m(\cos \theta_0) P_n^m(\cos \theta) \cos m\rho(\psi - \psi_0). \quad (4.6)$$

But the expansion of the incident wave for the spheroid is given by

$$U^i = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} i^n \varepsilon_m (2n+1) \frac{(n-m)!}{(n+m)!} j_n(k_p r) P_n^m(\cos \theta_0) \cdot P_n^m(\cos \theta) \cos m\rho(\psi - \psi_0). \quad (4.7)$$

On application of the Dirichlet boundary condition $U^i + U_p = 0$ on the surface, we obtain with the use of the orthogonality condition for trigonometric and Legendre functions that

$$a_{mn} = \varepsilon_m i^n (2n+1) \frac{(n-m)!}{(n+m)!} \frac{j_n(k_p a)}{h_n(k_p a)}. \quad (4.8)$$

Using (4.5) and the expansion for the spherical functions $j_n(k_p a)$ and $h_n(k_p a)$ we obtain

$$\begin{aligned}
 a_{mn} h_n^{(1)}(k_p r) = & \varepsilon_m i^n \frac{(n-m)!}{(n+m)!} \frac{j_n(ka)}{h_n^{(1)}(ka)} \\
 & \cdot \left\{ 1 + \frac{\varepsilon}{2h_n^{(1)}(ka)} \left[\frac{(ka)j_{n+1}(ka)h_n^{(1)}(ka)}{j_n(ka)} - (ka)j_{n+1}(ka) \right. \right. \\
 & \quad \left. \left. + i(2y_{n-1}(ka) - y_n(ka)) + nh_n^{(1)}(ka) \right. \right. \\
 & \quad \left. \left. \frac{(kr)j_n(kr)h_n^{(1)}(ka)}{h_n^{(1)}(kr)} + i \frac{y_n(kr)h_n^{(1)}(ka)}{h_n^{(1)}(kr)} \right. \right. \\
 & \quad \left. \left. - i \frac{y_{n-1}(kr) \cdot h_n^{(1)}(ka)}{h_n^{(1)}(kr)} \right] \right\} + O(\varepsilon^2),
 \end{aligned} \tag{4.9}$$

which on substitution into (4.6) gives U_p .

Conclusion. The method developed in this work has been applied to other work by the author [17]. Apart from the usual powerful advantage of a parameter expansion, this method has a special advantage over the traditional methods in that the scattering potentials for the spheroid may be obtained in terms of simpler functions (spherical functions) instead of the more complicated spheroidal functions. Furthermore, scattering and diffraction problems whose solutions are known for the sphere can very easily be transcribed to the spheroidal shapes, and vice versa through a careful limiting process. The scattered potential U_p for the prolate spheroid could stand for any other wave function provided linearity in the initial transformation is maintained.

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