HAMEL'S PROBLEM FOR STEADY PLANE ORTHOGONAL MHD FLOWS

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Abstract. Complex variable technique is used to study steady plane orthogonal magnetohydrodynamic flows of a viscous incompressible fluid of finite electrical conductivity. The problem of determining all possible flows such that the streamlines and their orthogonal trajectories form an isometric net is investigated.

1. Introduction. The streamlines and their orthogonal trajectories form an isometric curvilinear coordinate net and are the two families of curves defined by $\text{Re}[f(z)] = \text{const}$ and $\text{Im}[f(z)] = \text{const}$ for any steady plane incompressible irrotational motion when $f(z)$ is some analytic function of z . Hamel [1] determined all possible viscous rotational incompressible plane flows that have the same streamlines as those of an irrotational flow. Jeffrey had suggested this problem in his work [2]. Therefore, we define Hamel's problem for a class of fluid dynamics to be the problem of determining all possible flows such that the streamlines and their orthogonal trajectories form an isometric net. Hamel resolved his problem for viscous incompressible non-MHD flows and determined that the only possible flows are the ones where the streamlines are parallel straight lines, concurrent straight lines, concentric circles, and logarithmic spirals. This result was again confirmed by Martin [3] when he investigated this problem as an illustration of his new approach for the study of steady viscous plane flows.

Prim [4], Garg and Chandna [5], and Chandna and Toews [6] investigated Hamel's problem for rotational gas dynamics, for infinitely conducting orthogonal magnetohydrodynamics, and for infinitely conducting constantly inclined magnetohydrodynamics, respectively.

The present work deals with Hamel's problem of a viscous incompressible electrically conducting fluid when the velocity and magnetic field vectors are mutually perpendicular everywhere in the flow region. The fluid is taken to have finite electrical conductivity, which makes the problem realistic and attractive from both mathematical and physical points of view. Previous to this work, the resolution of a Hamel's problem in any branch of fluid dynamics was achieved by first transforming the flow equations to the natural curvilinear coordinate net. The method used in this paper

Received April 30, 1990.

1991 *Mathematics Subject Classification*. Primary 76.

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does not require such a transformation. This approach involves a change of independent variables from (x, y) to $(z, \bar{z})$ followed by the application of simple results from the theory of complex analysis. Yin [7, 8] successfully used this complex variable technique in the analysis for circulation preserving flows, for viscous non-MHD flows, and for aligned electromagnetofluiddynamic flows in a relatively recent work.

The plan of this paper is as follows: in Sec. 2 we recast the flow equations in terms of the streamfunction. Using the complex variables z and $\bar{z}$ as the new independent variables, we obtain a system of three equations that must be satisfied by the streamfunction and the proportionality function as functions of z and $\bar{z}$. In Sec. 3, we study Hamel's problem and determine the possible flow geometries to be parallel straight lines, concurrent straight lines, and concentric circles.

2. Basic equations. The steady plane flow of a viscous incompressible fluid of finite electrical conductivity is governed by the following system of equations:

$$\operatorname{div} V = 0, \quad (1)$$

$$\rho(V \cdot \nabla)V + \nabla p = \mu \nabla^2 V + \mu^* \operatorname{curl} H \times H, \quad (2)$$

$$\operatorname{curl} \left[V \times H - \frac{1}{\mu^* \sigma} \operatorname{curl} H \right] = 0, \quad (3)$$

$$\operatorname{div} H = 0 \quad (4)$$

where $V = (u, v)$ denotes the velocity vector field, $H = (H_1, H_2)$ the magnetic vector field, p the pressure function, ρ the constant fluid density, μ the constant coefficient of viscosity, μ^* the constant magnetic permeability, and σ the constant electrical conductivity.

In this paper, we consider orthogonal or crossed flows for which the magnetic lines lie in the flow plane and are perpendicular to the streamlines everywhere in the flow region.

From the definition of plane orthogonal flows, we have

$$H = k \times fV \quad (5)$$

where $k = (0, 0, 1)$ and $f(x, y)$ is a scalar function.

We employ this expression for H in Eqs. (2), (3), and (4) and make use of (1) to obtain

$$\operatorname{grad} \left[\frac{u^2 + v^2}{2} + \frac{p}{\rho} \right] - V \times \operatorname{curl} V = \nu \nabla^2 V - \frac{\mu^*}{\rho} (\nabla f \cdot V) fV, \quad (6)$$

$$\frac{\partial}{\partial x} \left[(u^2 + v^2) f - \frac{1}{\mu^* \sigma} (\nabla f \cdot V) \right] = \frac{\partial}{\partial y} \left[(u^2 + v^2) f - \frac{1}{\mu^* \sigma} (\nabla f \cdot V) \right], \quad (7)$$

$$k \cdot \operatorname{curl}(fV) = 0 \quad (8)$$

where ν is the kinematic viscosity and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

Taking the curl of Equation (6), we obtain that the integrability condition for $(p/\rho + (u^2 + v^2)/2)$ is

$$\operatorname{curl}[V \times \omega] + \nu \nabla^2 \omega - \frac{\mu^*}{\rho} \operatorname{curl}[(\nabla f \cdot V) fV] = 0 \quad (9)$$

where

$$\omega = \text{curl } V = \omega k = (v_x - u_y)k. \quad (10)$$

Since $k \neq 0$ and k is parallel to $\text{curl}(fV)$, it follows from Eq. (8) that

$$\text{curl}(fV) = 0. \quad (11)$$

Finally, integrating Eq. (7), we have

$$(u^2 + v^2)f - \frac{1}{\mu^* \sigma}(\nabla f \cdot V) = C \quad (12)$$

where C is an arbitrary constant.

Therefore, the velocity vector field and the scalar function f in steady plane orthogonal flows of a viscous incompressible fluid of finite electrical conductivity must satisfy Eqs. (1), (9), (11), and (12).

Equation (1) implies the existence of a streamfunction $\psi(x, y)$ so that

$$\psi_x = -v, \quad \psi_y = u. \quad (13)$$

From (10) and (13), we obtain

$$\omega = -\nabla^2 \psi k. \quad (14)$$

Using (13) and (14) in (9), (11), and (12), we find that the streamfunction $\psi(x, y)$ and the function $f(x, y)$ must satisfy

$$\begin{aligned} \frac{\partial(\psi, \nabla^2 \psi)}{\partial(x, y)} + \nu \nabla^4 \psi + \frac{1}{2} \frac{\mu^*}{\rho} \left\{ \psi_x \left[\frac{\partial(\psi, f^2)}{\partial(x, y)} \right]_x + \psi_y \left[\frac{\partial(\psi, f^2)}{\partial(x, y)} \right]_y \right. \\ \left. + \frac{\partial(\psi, f^2)}{\partial(x, y)} \nabla^2 \psi \right\} = 0, \end{aligned} \quad (15)$$

$$\psi_x f_x + \psi_y f_y + f \nabla^2 \psi = 0, \quad (16)$$

$$\frac{\partial(\psi, f)}{\partial(x, y)} + \mu^* \sigma (\psi_x^2 + \psi_y^2) f = \mu^* \sigma C. \quad (17)$$

Introducing the variables

$$z = x + iy \quad \text{and} \quad \bar{z} = x - iy$$

where $i = \sqrt{-1}$, we may formally derive the following relations

$$\begin{aligned} m_x &= m_z + m_{\bar{z}}, & m_y &= i(m_z - m_{\bar{z}}), \\ 4m_{z\bar{z}} &= m_{xx} + m_{yy}, & 4m_z m_{\bar{z}} &= m_x^2 + m_y^2, \\ \frac{\partial(m, n)}{\partial(x, y)} &= -4 \text{Im}\{m_z n_{\bar{z}}\}, \\ m_x n_x + m_y n_y &= 4 \text{Re}\{m_z n_{\bar{z}}\} = 4 \text{Re}\{m_{\bar{z}} n_z\} \end{aligned} \quad (18)$$

where m and n are two scalar functions.

If $h(z) = \alpha(x, y) + i\beta(x, y)$ is a complex analytic function, then

$$\begin{aligned} h'(z) &= \alpha_x + i\beta_x = \alpha_x - i\alpha_y, \\ 2\alpha_z &= 2i\beta_z, \\ \overline{h'(z)} &= 2\alpha_{\bar{z}} = -2i\beta_{\bar{z}}. \end{aligned} \quad (19)$$

Applying (18) in (15)–(17), we respectively obtain

$$\psi_{z\bar{z}z\bar{z}} - \frac{1}{\nu} \operatorname{Im}\{\psi_{z\bar{z}z}\psi_z\} - \frac{\mu^*}{\mu} \{\operatorname{Re}\{[\operatorname{Im}\{f f_z \psi_{\bar{z}}\}]_z \psi_{\bar{z}}\} + \psi_{z\bar{z}} \operatorname{Im}\{f f_z \psi_{\bar{z}}\}\} = 0, \quad (20)$$

$$\operatorname{Re}\{f_z \psi_{\bar{z}}\} + f \psi_{z\bar{z}} = 0, \quad (21)$$

$$\operatorname{Im}\{f_z \psi_{\bar{z}}\} - \mu^* \sigma f \psi_z \psi_{\bar{z}} = -\frac{\mu^* \sigma C}{4}. \quad (22)$$

3. Isometric flows. In this section, we prescribe the magnetic lines and the streamlines to coincide with the curves in an isometric net [1]. In other words, we study flows in which the streamlines and magnetic lines are the families of curves defined by the real and imaginary components of an analytic function of z in a manner similar to that of streamlines and equipotential lines for incompressible and irrotational non-MHD flows.

We assume that

$$\psi = \psi(\xi), \quad \psi'(\xi) \neq 0 \quad (23)$$

where the curves $\xi(x, y) = \text{const}$ and their orthogonal trajectories $\eta(x, y) = \text{const}$ generate an isometric net.

Thus,

$$g(z) = \xi(x, y) + i\eta(x, y) \quad (24)$$

is an analytic complex function, and

$$\xi_{xx} + \xi_{yy} = 4\xi_{z\bar{z}} = 0, \quad (25)$$

$$f(x, y) = f(x(\xi, \eta), y(\xi, \eta)) = f^*(\xi, \eta). \quad (26)$$

From (23) and (25), we have

$$\begin{aligned} \psi_{\bar{z}} &= \psi' \xi_{\bar{z}}, & \psi_z &= \psi' \xi_z, & \psi_{z\bar{z}} &= \psi'' \xi_z \xi_{\bar{z}}, \\ \psi_{z\bar{z}z} &= \psi''' \xi_z^2 \xi_{\bar{z}} + \psi'' \xi_{zz} \xi_{\bar{z}}, \\ \psi_{z\bar{z}z\bar{z}} &= \psi^{iv} \xi_z^2 \xi_{\bar{z}}^2 + \psi''' (\xi_z^2 \xi_{\bar{z}z\bar{z}} + \xi_{\bar{z}}^2 \xi_{zzz}) + \psi'' \xi_{zz} \xi_{\bar{z}z\bar{z}} \end{aligned} \quad (27)$$

where prime denotes differentiation with respect to ξ .

From (19) and (26), we have

$$f_z = f_{\xi}^* \xi_z + f_{\eta}^* \eta_z = (f_{\xi}^* - i f_{\eta}^*) \xi_z. \quad (28)$$

Employing (27) and (28) in (20)–(22), we obtain

$$\begin{aligned} &\psi^{iv}(\xi) + 2a\psi'''(\xi) + (a^2 + b^2)\psi''(\xi) - \frac{b}{\nu} \psi'(\xi) \psi''(\xi) \\ &+ \frac{\mu^* \psi'^2(\xi)}{\mu} \left[f_{\xi}^* f_{\eta}^* + f^* f_{\eta\xi}^* + a f^* f_{\eta}^* + 2 \frac{\psi''(\xi)}{\psi'(\xi)} f^* f_{\eta}^* \right] = 0, \end{aligned} \quad (29)$$

$$f_{\xi}^* \psi'(\xi) + f^* \psi''(\xi) = 0, \quad (30)$$

$$[f_{\eta}^* \psi'(\xi) + \mu^* \sigma f^* \psi'^2(\xi)] \xi_z \xi_{\bar{z}} = \frac{\mu^* \sigma C}{4} \quad (31)$$

where

$$a = \operatorname{Re} \left\{ \frac{\xi_{zz}}{\xi_z^2} \right\} = \frac{1}{2} \left[\frac{\xi_{zz}}{\xi_z^2} + \frac{\xi_{\bar{z}\bar{z}}}{\xi_{\bar{z}}^2} \right], \quad (32)$$

$$b = \operatorname{Im} \left\{ \frac{\xi_{zz}}{\xi_z^2} \right\} = \frac{1}{2i} \left[\frac{\xi_{zz}}{\xi_z^2} - \frac{\xi_{\bar{z}\bar{z}}}{\xi_{\bar{z}}^2} \right], \quad (33)$$

$$a^2 + b^2 = \frac{\xi_{zz} \xi_{\bar{z}\bar{z}}}{\xi_z^2 \xi_{\bar{z}}^2}. \quad (34)$$

Applying (19) in (24) and making use of (32) and (33), we get

$$2 \frac{g''(z)}{g'^2(z)} = \frac{\xi_{zz}}{\xi_z^2} = a + ib. \quad (35)$$

Since $g(z)$ is an analytic function of $z = x + iy$, it follows from (35) that $a + ib$ is an analytic function of z , and since z is an analytic function of $\xi + i\eta$, it follows that $a + ib$ is also an analytic function of $\xi + i\eta$. Therefore, we have the relations

$$a_{\xi} = b_{\eta}, \quad (36)$$

$$a_{\eta} = -b_{\xi}. \quad (37)$$

Conversely, $g(z) = \xi + i\eta$ is an analytic function of $a + ib$.

Equation (30) yields

$$f^*(\xi, \eta) = \frac{A(\eta)}{\psi'(\xi)} \quad (38)$$

where $A(\eta)$ is an arbitrary function of η .

Using (38) in (29) and (31), we have

$$\psi^{\text{iv}}(\xi) + 2a\psi'''(\xi) + (a^2 + b^2)\psi''(\xi) - \frac{b}{\nu}\psi'(\xi)\psi''(\xi) + \frac{\mu^*}{\mu}aA(\eta)A'(\eta) = 0, \quad (39a)$$

$$[A'(\eta) + \mu^* \sigma \psi'(\xi)A(\eta)] \xi_z \xi_{\bar{z}} = \frac{\mu^* \sigma C}{4}. \quad (39b)$$

We let

$$B(\eta) = \frac{A(\eta)}{\mu^* \sigma C}, \quad \varphi(\xi) = \mu^* \sigma \psi(\xi), \quad (40)$$

and write equation (39b) as

$$4\xi_z \xi_{\bar{z}} = e^{-w} \quad (41)$$

where

$$w(\xi, \eta) = B'(\eta) + B(\eta)\varphi'(\xi). \quad (42)$$

Differentiating (41) with respect to z , we get

$$4\xi_{zz} \xi_{\bar{z}} = -w_z e^{-w} = -e^{-w}(w_{\xi} - iw_{\eta})\xi_z = -4\xi_z^2 \xi_{\bar{z}}(w_{\xi} - iw_{\eta}).$$

Hence,

$$2 \frac{g''(z)}{g'^2(z)} = \frac{\xi_{zz}}{\xi_z^2} = -w_\xi + iw_\eta. \quad (43)$$

Equation (43) shows that $-w_\xi$ and w_η are the real and imaginary parts of the complex analytic function $g''(z)/g'^2(z)$. Therefore, $w(\xi, \eta)$ is a plane harmonic function of its arguments. Furthermore, from Eqs. (35) and (43), we have

$$a = -w_\xi, \quad b = w_\eta. \quad (44)$$

We now proceed to analyse two different flows: (i) irrotational flow and (ii) rotational flow.

(i) $\varphi''(\xi) = 0$ (irrotational flow). In this flow, we have

$$\varphi'(\xi) = k \quad (45)$$

where k is an arbitrary real constant. Hence, we conclude from Eq. (42) that w is a function of η only; therefore, from the identities (44) and the Cauchy-Riemann equations for a and b , we get

$$a = 0, \quad b = w'(\eta) = \text{const.} \quad (46)$$

The second equation of (46) yields

$$w(\eta) = b\eta + B_1 \quad (47)$$

where B_1 is an arbitrary real constant.

Equation (43) reduces to

$$2 \frac{g''(z)}{g'^2(z)} = \frac{\xi_{zz}}{\xi_z^2} = ib, \quad (48)$$

which shows that the flow may have two possible configurations: (1) straight parallel lines ($b = 0$) and (2) concurrent straight lines ($b \neq 0$).

Subcase 1. $b = 0$. Equations (47) and (48) reduce to

$$w(\eta) = B_1, \quad (49)$$

$$2 \frac{g''(z)}{g'^2(z)} = \frac{\xi_{zz}}{\xi_z^2} = 0. \quad (50)$$

Integrating Eq. (50) twice with respect to z , we get

$$g(z) = \xi + i\eta = \frac{\overline{F}}{F\overline{F}}z + F_0$$

where $F = F_1 + iF_2$ and $F_0 = F_3 + iF_4$ are arbitrary complex constants.

Hence, we obtain

$$\xi(x, y) = \frac{F_1}{F_1^2 + F_2^2}x + \frac{F_2}{F_1^2 + F_2^2}y + F_3, \quad (51)$$

$$\eta(x, y) = \frac{F_1}{F_1^2 + F_2^2}y - \frac{F_2}{F_1^2 + F_2^2}x + F_4, \quad (52)$$

$$4\xi_z\xi_{\overline{z}} = \xi_x^2 + \xi_y^2 = \frac{1}{F\overline{F}}. \quad (53)$$

Using Eqs. (45) and (49) in (42) and integrating the resulting equation with respect to η , we have

$$B(\eta) = B_3 e^{-k\eta} + B_2 \quad (54)$$

where $B_2 = e^{B_1}/k$ and B_3 is an arbitrary real constant.

Using Eqs. (49) and (53) in (41), we find that $B_2 = F\bar{F}/4k$. Hence, from Eqs. (40) and (38), we obtain

$$A(\eta) = B_4 e^{-k\eta} + B_5, \quad (55)$$

$$\psi(\xi) = k_1 \xi + k_2,$$

$$f^*(\eta) = B_6 e^{-k\eta} + B_7 \quad (56)$$

where $B_4 = \mu^* \sigma C B_3$, $B_5 = \mu^* \sigma C F\bar{F}/4k$, $B_6 = B_4/k_1$, $B_7 = B_5/k_1$, $k_1 = k/\mu^* \sigma$, and k_2 is an arbitrary real constant.

Using $\psi(\xi)$, $f(\eta)$, a , and b given above in (39a), we find that Eq. (39a) is identically satisfied.

The solution in the physical plane is

$$\psi(x, y) = F_5 x + F_6 y + F_7, \quad (57)$$

$$f(x, y) = B_6 \exp[F_8 y + F_9 x + F_{10}] + B_7, \quad (58)$$

$$\mathbf{V} = (\psi_y - \psi_x) = (F_6, -F_5),$$

$$\mathbf{H} = B_6 [\exp(F_8 y + F_9 x + F_{10}) + B_7] (F_5, F_6),$$

$$p(x, y) = F_{11} [B_6 \exp(F_8 y + F_9 x + F_{10}) + 2B_7] \\ \times \exp(F_8 y + F_9 x + F_{10}) - \frac{\rho}{2} (F_5^2 + F_6^2) + F_{12} \quad (59)$$

where $F_5 = k_1 F_1 / (F_1^2 + F_2^2)$, $F_6 = k_1 F_2 / (F_1^2 + F_2^2)$, $F_7 = k F_3 + k_3$, $B_6 = B_4/k_1$, $B_7 = B_5/k_1$, $F_8 = -k F_1 / (F_1^2 + F_2^2)$, $F_9 = k F_2 / (F_1^2 + F_2^2)$, $F_{10} = -k F_4$, $F_{11} = \mu^* B_6 F_5 (F_6 F_9 - F_5 F_8) / 2 F_8$, and F_{12} is an arbitrary constant.

Subcase 2. $b \neq 0$. Proceeding as in the preceding subcase, we obtain

$$\xi(x, y) = \frac{2}{b} \tan^{-1} \left(\frac{x + G_4}{y + G_3} \right) + G_5,$$

$$\eta(x, y) = \frac{1}{b} \ln[(x + G_4)^2 + (y + G_3)^2] + G_6,$$

$$\psi(\xi) = k_1 \xi + k_2,$$

$$f^*(\eta) = \begin{cases} (B_{11} e^{k\eta} + B_{12} e^{-k\eta})/k_1, & b + k \neq 0, \\ (B_{13} \eta + B_{14}) e^{-k\eta}, & b + k = 0. \end{cases}$$

The solution in the physical plane is

$$\begin{aligned}
 \psi(x, y) &= k_3 \tan^{-1} \left(\frac{x + G_4}{y + G_3} \right) + k_4, \\
 f(x, y) &= \begin{cases} B_{15}[(x + G_4)^2 + (y + G_3)^2] + B_{16}[(x + G_4)^2 + (y + G_3)^2]^{-k/b}, & b + k \neq 0, \\ \{B_{17} \ln[(y + G_3)^2 + (x + G_4)^2] + B_{18}\} \\ \quad \times [(x + G_4)^2 + (y + G_3)^2]^{-k/b}, & b + k = 0, \end{cases} \\
 \mathbf{V} = (\psi_y, -\psi_x) &= -k_3 \left(\frac{x + G_4}{(x + G_4)^2 + (y + G_3)^2}, \frac{y + G_3}{(x + G_4)^2 + (y + G_3)^2} \right), \\
 \mathbf{H} &= \begin{cases} k_3 \{B_{16}[(x + G_4)^2 + (y + G_3)^2]^{-k/b-1} + B_{15}\} \\ \quad \times ((y + G_3), -(x + G_4)), & b + k \neq 0, \\ k_3 \{B_{17} \ln[(x + G_4)^2 + (y + G_3)^2] + B_{18}\} \\ \quad \times ((y + G_3), -(x + G_4)), & b + k = 0, \end{cases} \\
 p(x, y) &= \mu^* \int j H_2 dx + j H_1 dy - \frac{\rho k_3^2}{2} [(x + G_4)^2 + (y + G_3)^2]^{-1} + B_{19} \quad (60)
 \end{aligned}$$

where

$$j = \begin{cases} 2k_3 \{ \frac{k}{b} B_{16} [(x + G_4)^2 + (y + G_3)^2]^{-k/b-1} - B_{15} \}, & b + k \neq 0, \\ -2k_3 \{ B_{17} \ln[(x + G_4)^2 + (y + G_3)^2] + (B_{17} + B_{18}) \}, & b + k = 0, \end{cases}$$

and $G_3, G_4, G_5, G_6, k_1, k_2, k_3, k_4, B_{11}, B_{12}, B_{13}, B_{14}, B_{15}, B_{16}, B_{17}, B_{18}$, and B_{19} are arbitrary real constants.

(ii) $\varphi''(\xi) \neq 0$ (rotational flow). Equation (42) can be written as

$$w(\xi, \eta) = \ln B(\eta) + \ln \left[\frac{B'(\eta)}{B(\eta)} + \varphi'(\xi) \right], \quad (61)$$

which yields

$$w_\xi = \frac{\varphi''(\xi)}{\frac{B'(\eta)}{B(\eta)} + \varphi'(\xi)}, \quad w_\eta = \frac{B'(\eta)}{B(\eta)} + \frac{\left(\frac{B'(\eta)}{B(\eta)} \right)'}{\frac{B'(\eta)}{B(\eta)} + \varphi'(\xi)}, \quad (62)$$

$$w_{\xi\xi} = \frac{\varphi'''(\xi)}{\frac{B'(\eta)}{B(\eta)} + \varphi'(\xi)} - \frac{\varphi''(\xi)^2}{\left[\frac{B'(\eta)}{B(\eta)} + \varphi'(\xi) \right]^2}, \quad (63)$$

$$w_{\eta\eta} = \left(\frac{B'(\eta)}{B(\eta)} \right)' + \frac{\left(\frac{B'(\eta)}{B(\eta)} \right)''}{\frac{B'(\eta)}{B(\eta)} + \varphi'(\xi)} - \frac{\left(\frac{B'(\eta)}{B(\eta)} \right)' ^2}{\left[\frac{B'(\eta)}{B(\eta)} + \varphi'(\xi) \right]^2}. \quad (64)$$

Since

$$w_{\xi\xi} + w_{\eta\eta} = 0,$$

it follows from (63) and (64) that

$$\begin{aligned} \varphi'''(\xi) \left[\frac{B'(\eta)}{B(\eta)} + \varphi'(\xi) \right] - \varphi''^2(\xi) + \left(\frac{B'(\eta)}{B(\eta)} \right)' \left[\frac{B'(\eta)}{B(\eta)} + \varphi'(\xi) \right]^2 \\ + \left(\frac{B'(\eta)}{B(\eta)} \right)'' \left[\frac{B'(\eta)}{B(\eta)} + \varphi'(\xi) \right] - \left(\frac{B'(\eta)}{B(\eta)} \right)'^2 = 0. \end{aligned} \quad (65)$$

Differentiating Eq. (65) with respect to ξ and then dividing by $\varphi''(\xi)$, we obtain

$$\frac{\varphi^{iv}(\xi)}{\varphi''(\xi)} \left[\frac{B'(\eta)}{B(\eta)} + \varphi'(\xi) \right] - \varphi'''(\xi) + 2 \left(\frac{B'(\eta)}{B(\eta)} \right)' \left[\frac{B'(\eta)}{B(\eta)} + \varphi'(\xi) \right] + \left(\frac{B'(\eta)}{B(\eta)} \right)'' = 0. \quad (66)$$

Repeating the above step, we have

$$\frac{1}{\varphi''(\xi)} \left(\frac{\varphi^{iv}(\xi)}{\varphi''(\xi)} \right)' \left[\frac{B'(\eta)}{B(\eta)} + \varphi'(\xi) \right] + 2 \left(\frac{B'(\eta)}{B(\eta)} \right)' = 0. \quad (67)$$

Differentiating (67) with respect to ξ , we get

$$\left[\frac{1}{\varphi''(\xi)} \left(\frac{\varphi^{iv}(\xi)}{\varphi''(\xi)} \right)' \right]' \left(\frac{B'(\eta)}{B(\eta)} + \varphi'(\xi) \right) + \left(\frac{\varphi^{iv}(\xi)}{\varphi''(\xi)} \right)' = 0. \quad (68)$$

If $[(1/\varphi''(\xi))(\varphi^{iv}(\xi)/\varphi''(\xi))]' \neq 0$, then it follows from Eq. (68) that $(\ln B(\eta))' = B'(\eta)/B(\eta)$ is a function of ξ only. But since $(\ln B(\eta))'$ is a function of η , $(\ln B_\eta)'$ must be a real constant q_1 . Hence we have

$$B(\eta) = q_1 e^{q_1 \eta} \quad (69)$$

where q_1 is a real constant.

Using (69) in Eq. (65), we find

$$\varphi'''(\xi)(q_1 + \varphi'(\xi)) - \varphi''^2(\xi) = 0 \quad (70)$$

or

$$\left[\ln \left(\frac{\varphi''(\xi)}{q_1 + \varphi'(\xi)} \right) \right]' = 0. \quad (71)$$

Equation (71) gives

$$\frac{\varphi''(\xi)}{q_1 + \varphi'(\xi)} = p_1 \quad (72)$$

where p_1 is an arbitrary real constant. But by the assumption $\varphi''(\xi) \neq 0$, p_1 is restricted to be a nonzero constant. Integrating Eq. (72) with respect to ξ , we obtain

$$\varphi'(\xi) = p_2 e^{p_1 \xi} - q_1 \quad (73)$$

where p_2 is an arbitrary real constant.

From (44), (62), (69), and (73), we obtain

$$-a = w_\xi = p_1, \quad b = w_\eta = q_1. \quad (74)$$

Using (74) in (43) and integrating with respect to z , we have

$$\xi_z = \frac{1}{2}M \exp \left[-\frac{1}{2}(p_1\xi + q_1\eta) + \frac{i}{2}(q_1\xi - p_1\eta) \right] \quad (75)$$

where $M = M_1 + iM_2$ is an arbitrary complex constant.

Hence, using (75) in Eq. (41) and making use of (42), (69), and (73), we find that $p_2 = 1/M\overline{M}q_2$. From (40), (69), and (73), we obtain

$$\psi'(\xi) = p_3 e^{p_1\xi} - \frac{q_1}{\mu^* \sigma}, \quad A(\eta) = \mu^* \sigma C q_2 e^{q_1\eta} \quad (76)$$

where $p_3 = 1/M\overline{M}\mu^* \sigma q_2$.

Using (74) and (76) in Eq. (39a), we find that

$$q_1 \psi''(\xi) \left[q_1 - \frac{1}{\nu} \psi'(\xi) \right] = \frac{\mu^* p_1 q_1}{\mu} A^2(\eta). \quad (77)$$

Equation (77) holds only if the left and right sides reduce to a constant. However, since $A(\eta) = \mu^* \sigma C q_2 \exp(q_1\eta)$, it follows that the above equation is only possible for $q_1 = 0$. Therefore, Eq. (75) reduces to

$$g'(z) = 2\xi_z = M \exp \left[-\frac{1}{2}p_1(\xi + i\eta) \right]. \quad (78)$$

Integrating (78) with respect to z , we get

$$g(z) = \xi + i\eta = \frac{2}{p_1} \ln(z + z_0) + \frac{2}{p_1} \ln \left(\frac{p_1}{2} M \right) \quad (79)$$

where $z_0 = x_0 + iy_0$ is an arbitrary complex constant.

From (79), we conclude that the only possible streamlines are concentric circles and the solution for the flow is

$$\psi(\xi) = p_4 e^{p_1\xi} + p_5, \quad (80)$$

$$f^*(\xi) = \frac{\mu^* \sigma C q_2}{p_3} e^{-p_1\xi} \quad (81)$$

where $p_4 = p_3/p_1$ and p_5 is an arbitrary real constant.

The preceding proof is based on the assumption that $[(1/\varphi''(\xi))(\varphi^{iv}(\xi)/\varphi''(\xi))']' \neq 0$. Now if

$$\left[\frac{1}{\varphi''(\xi)} \left(\frac{\varphi^{iv}(\xi)}{\varphi''(\xi)} \right)' \right]' = 0$$

then we have

$$\frac{1}{\varphi''(\xi)} \left(\frac{\varphi^{iv}(\xi)}{\varphi''(\xi)} \right)' = N$$

where N is an arbitrary real constant. Equation (67) reduces to

$$N \left[\frac{B'(\eta)}{B(\eta)} + \varphi'(\xi) \right] + 2 \frac{B'(\eta)}{B(\eta)} = 0. \quad (82)$$

Equation (82) implies that either $N = 0$, in which case $B'(\eta)/B(\eta) = q_1$ follows as before, or $N \neq 0$ and $\phi'(\xi)$ is constant, which violates the assumption $\phi''(\xi) \neq 0$. Therefore, the result obtained is also valid when $[(1/\phi''(\xi))(\phi^{iv}(\xi)/\phi''(\xi))]' = 0$.

Proceeding as before, we obtain the solution in the physical plane from (79), (80), and (81):

$$\begin{aligned}\psi(x, y) &= p_6[(x + x_0)^2 + (y + y_0)^2] + p_5, \\ f(x, y) &= p_7[(x + x_0)^2 + (y + y_0)^2]^{-1}, \\ \mathbf{V} &= (\psi_y, -\psi_x) = 2p_6(y + y_0, -(x + x_0)), \\ \mathbf{H} &= \frac{p_7}{(x + x_0)^2 + (y + y_0)^2}(x + x_0, y + y_0), \\ p(x, y) &= 2\rho p_6^2[(x + x_0)^2 + (y + y_0)^2] + p_8\end{aligned}\tag{83}$$

where p_6 , p_7 , and p_8 are arbitrary real constants.

In summary, we have the following result.

THEOREM. If the family of streamlines of a steady plane orthogonal MFD flow of a viscous incompressible fluid of finite electrical conductivity coincides with one of the two families of curves of an orthogonal isometric net, then the streamlines are restricted to straight parallel lines, concurrent straight lines, and concentric circles, whose solutions are given by (59), (60), and (83), respectively.

Acknowledgments. The authors are grateful to the referees for their useful comments and suggestions. The work reported in this paper has been supported by Grant No. A7161 of N.S.E.R.C. of Canada.

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