THE SINGULARITY OF THE STRESS FIELD OF TWO NEARBY HOLES IN A PLANAR ELASTIC MEDIUM

By

CONSTANTINE J. CALLIAS (*University of Crete*)

AND

XANTHIPPI MARKENSCOFF (*University of California at San Diego, La Jolla, CA*)

The problem we address in this note has received considerable attention for some time [L, DKH, Z]. Recently Zimmerman [Z] argued that the “hoop” stress at the minimum distance points on the hole boundaries and in the direction of the loading (uniaxial tension T_∞), is asymptotically of the form $C/\sqrt{\varepsilon}$, in the limit as the distance ε of the two holes goes to zero. He finds that $C \approx 1.94$, in slight disagreement with the estimate $C \approx 2.13$ of [DKH]. Zimmerman’s calculation is based on an ad hoc summation of a divergent series

$$\sum_{n=1}^{\infty} (-1)^n = -\frac{1}{2}.$$

We will show how his calculation can be justified rigorously, based on an approach to singular asymptotic problems that has been used effectively in other problems of elasticity [CM, CM1]. Because our method is much more refined, we obtain a full asymptotic expansion of the hoop stress to all orders in ε . We also obtain an asymptotic expansion of the hoop stress at points other than the points of minimum distance [see Eq. (0.1) below].

The formula for the stress field was given by Ling in [L]: In terms of a suitable curvilinear coordinate β , the field $T_{\beta\beta}$ on the hole boundary is given by

$$\frac{T_{\beta\beta}(\alpha, \beta)}{T_\infty} = 2(\cosh \alpha - \cos \beta) \cdot K \cdot \sinh \alpha \left(1 + 4L - 2 \frac{\sinh \alpha}{\sinh 2\alpha} \right),$$

where

$$K = K(\alpha) = \left(\frac{1}{2} + \tanh \alpha \sinh^2 \alpha - 4M(\alpha) \right)^{-1},$$

$$M(\alpha) = \sum_{n=2}^{\infty} \frac{e^{-n\alpha} \sinh n\alpha + n^2 \sinh^2 \alpha + n \sinh \alpha \cosh \alpha}{n(n^2 - 1)(\sinh 2n\alpha + n \sinh 2\alpha)},$$

Received October 16, 1991.

This research was supported by the National Science Foundation grant #MSS-9002804.

©1993 Brown University

and

$$L = L(\alpha, \beta) = \sum_{n=2}^{\infty} \frac{\sinh n\alpha}{\sinh 2n\alpha + n \sinh 2\alpha} \cos n\beta.$$

The $\beta = \text{constant}$ curves are normal to the boundary of the hole. The parameter α is related to the interhole distance. The two holes tend to touch each other in the limit $\alpha \rightarrow 0$.

The arguments of [Z] lead to the result

$$T_{\beta\beta}(\alpha, \pi) = T_{\infty} \cdot I \cdot \alpha^{-1} + o(\alpha^{-1}),$$

where

$$I^{-1} = 2 \int_0^{\infty} \frac{\sinh^2 w - w^2}{w^3 (\sinh 2w + 2w)} dw,$$

based partially on an estimate of $M(\alpha)$ given in [L]. We will rederive the results in [Z] and in [L] and obtain the stronger result

$$T_{\beta\beta}(\alpha, \beta) \sim T_{\infty} \cdot I \cdot (1 - \cos \beta) \left(-\frac{1}{2} - \cos \beta \right) \alpha^{-1} + \sum_{k=0}^{\infty} (T_k + S_k \ln \alpha) \alpha^k, \quad (0.1)$$

for $\cos \beta \neq 1$, in the sense of asymptotic series.

To compute the asymptotic behavior of $T_{\beta\beta}$ we need the asymptotics of $M(\alpha)$ and $L(\alpha, \beta)$. We show how this is done to all orders in α and explicitly compute the coefficients needed for the leading terms. The result (0.1) is a direct corollary of (1.3) and (1.2) for L and (2.4), (2.5) for M in the following sections.

Section 1 settles the questionable part of the calculation in [Z] and is based on the tools of [CM1]. To tackle the problem of the asymptotic behavior of $M(\alpha)$, we had to develop the theory sketched in [CM1] a little further. This is done in Sec. 2.1. We present a reparametrization invariant approach to singular asymptotics of contour integrals, which we believe will be of value in other asymptotic problems.

1. The oscillatory series. The truly problematic part in Zimmerman's calculation is the asymptotics of $L(\alpha, \beta)$. Following the general procedure of Sec. 1(C) of [CM1], we define

$$H(y, x) = \frac{\sinh y}{\sin 2y + (1/x) \sin 2y},$$

$$\varphi(z) = \cos \beta z \cot \pi z,$$

in terms of which we can write

$$L(\alpha, \beta) = \frac{1}{2i} \int_C H\left(\alpha z, \frac{1}{z}\right) \varphi(z) dz.$$

We break up the contour C into three pieces, as in [CM1]: $C = C_0 \cup C_+ \cup C_-$. We let $\chi \in C_0^\infty[0, \infty)$, $\chi \equiv 1$ on $[0, 2]$, and $\chi_1 = 1 - \chi$. For $\text{Re } z \geq 2$, $C_\pm$ is parametrized by

$$(0, 1] \ni x \mapsto \pm i\varepsilon + 1 + 1/x \equiv z_\pm(x).$$

We then write

$$L(\alpha, \beta) = \frac{1}{2i} \int_C H(\alpha z, 1/z) \varphi(z) \chi(\operatorname{Re} z) dz \\ - \sum_{\pm} (\pm) \frac{1}{2i} \int_0^{\infty} \alpha^{-2} h_{\pm} \left(\frac{\alpha}{x}, x \right) \varphi_{\pm}(x) \chi_1 \left(\frac{1}{x} \right) dx,$$

where

$$h_{\pm}(y, x) = y^2 H(xy \cdot z_{\pm}(x), 1/z_{\pm}(x)), \\ \varphi_{\pm}(x) = \varphi(z_{\pm}(x)).$$

It is almost obvious that $h_{\pm}(y, x)$ is of class A. Because $\varphi_{+}(x)$ is not regular at $x = 0$, we regularize. The fourth-order regularization used in Sec. 1(C) of [CM1] suffices. Noting that $H(0, 1/z) = \frac{1}{4}$, we obtain

$$L(\alpha, \beta) = \frac{1}{2i} \left\{ \int_C \frac{1}{4} \cdot \varphi(x) \chi(\operatorname{Re} z) dz + O(\alpha) \right. \\ \left. + \alpha^{-2} \sum_{\pm} \left[(\pm) \alpha^2 \int_y \frac{1}{2} (x \partial_x - y \partial_y)^4 h_{\pm}(y, x) |_{y=0} \right. \right. \\ \left. \left. \cdot \varphi_{\pm(4)}(x) \chi_1(1/x) dx + O(\alpha^3) \right] \right\}.$$

Now $(y \partial_y)^j h_{\pm}(y, x) = 2^j \cdot \frac{1}{4} y^2 + O(y^3)$, and this relation can be differentiated with respect to x any number of times. This gives, for $\cos \beta \neq 1$,

$$L(\alpha, \beta) = \frac{1}{4} \cdot \frac{1}{2i} \left\{ \int_C \varphi(z) \chi(\operatorname{Re} z) dz \right. \\ \left. + \int 2^4 \chi_1(1/x) (\varphi_{-(4)}(x) - \varphi_{+(4)}(x)) dx \right\} + O(\alpha) \\ = -\frac{1}{4} \left(\cos \beta + \frac{1}{2} \right) + O(\alpha) \quad (1.2)$$

where we used (1.12) of [CM1]. Coupled with the results of [Z] and [L] for $M(\alpha)$, we obtain the leading term in (0.1). The same answer is obtained if we use the asymptotic analysis of the next section. The singular asymptotics lemma, given in [CM] for renormalized integrals, also yields an asymptotic series

$$L(\alpha, \beta) \sim \sum_{j=0}^{\infty} L_j(\beta) \alpha^j \quad (1.3)$$

for $\cos \beta \neq 1$.

2. The β -independent series. In this section we will prove the existence of an asymptotic expansion

$$M(\alpha) \sim \sum_{k=0}^{\infty} M_k \alpha^k + \sum_{k=2}^{\infty} N_k \alpha^k \ln \alpha \quad (2.4)$$

and compute a few of the leading coefficients that are relevant to our estimate of the behavior of the hoop stress:

$$\begin{aligned} M_0 &= \sum_{n=2}^{\infty} \frac{1}{2n(n^2-1)} = \frac{1}{8}, \\ M_1 &= 0, \\ M_2 &= - \int_0^{\infty} \frac{\sinh^2 w - w^2}{w^3(\sinh 2w + 2w)} dw, \\ N_2 &= 0. \end{aligned}$$

These values confirm the results of [L] and [Z].

2.1. Singular asymptotics, revisited. To compute the first few asymptotic coefficients in the expansion of $M(\alpha)$, we will need some ideas on asymptotic expansions of certain types of contour integrals.

Let Γ be a semi-infinite curve in the complex plane. Specifically, suppose that Γ is parametrized by

$$[0, \infty) \ni t \mapsto z(t) \in \mathbb{C},$$

where $z(t)$ is a smooth complex-valued function of t ; $z(0) = 0$; $z'(t) \neq 0$ for all $t \in [0, \infty)$, $z'(t)$ is bounded; $\lim_{t \rightarrow \infty} z(t) = \infty$.

For a function $f: \Gamma \rightarrow \mathbb{C}$ we can define an “intrinsic” complex derivative at $z_0 = z(t_0)$ by

$$(\partial_z f)(z_0) = \frac{1}{z'(t_0)} \lim_{t \rightarrow t_0} \frac{1}{t - t_0} (f(z(t)) - f(z(t_0))).$$

Since $z'(t) \neq 0$ for all t , f is z -differentiable if and only if $f(z(t))$ is differentiable with respect to t .

Now suppose that Γ is contained in the interior of a closed cone

$$K = \{z \in \mathbb{C} \mid \theta_1 \leq \arg z \leq \theta_2\},$$

where $\theta_2 - \theta_1 < 2\pi$, so that $K \neq \mathbb{C}$. Let $\bar{K}$ be the complex conjugate cone

$$\bar{K} = \{z \in \mathbb{C} \mid -\theta_2 \leq \arg z \leq -\theta_1\}.$$

We will state a result on singular asymptotics of integrals along Γ . The expansion of [CM] will correspond to the special case $\Gamma = \bar{R}_+$. The proof, however, can be given along the lines of the integration by parts argument of [CM], so we will not give the details.

For the statement of the theorem and the calculations that follow the following definitions will be helpful. We are dealing with functions $f(\xi)$ that have asymptotic expansions as $\xi \rightarrow 0$ with respect to the functions $\xi^k (\ln \xi)^j$, for $k \in \mathbb{C}$ and $j \in \mathbb{Z}_+$. For a function of this kind we let $\mathcal{D}_{k,j} f(0)$ denote the coefficient of $\xi^k (\ln \xi)^j$ in the asymptotic expansion of f . Further, if $f(z)$ is in $C^\infty(\Gamma)$, we let

$$r_m^f(z) = r_m^z f(z) = f(z) - \sum_{j=0}^{m-1} \frac{1}{j!} f^{(j)}(0) z^j,$$

the remainder in the Taylor series for $f(z)$ about $z = 0$ after m terms.

THEOREM 2.1. Let $h(z, w) \in C^\infty(\Gamma \times \overline{K})$, and suppose that $h(z, w)$ is holomorphic in w , for w in the interior of $\overline{K}$. Suppose $h(z, w)$ has compact z -support and

$$|\partial_z^k h(x, w)| \leq |w|^k \cdot h_k(|w|),$$

for all $z \in \Gamma$, $w \in \overline{K}$, $k = 0, 1, 2, \dots$, where the h_k satisfy $\int_0^1 h_k(1/r) dr < \infty$. For $s > 0$ define

$$F(s) = \int_{\Gamma} h\left(\frac{s}{z}, z\right) dz.$$

Then

$$F(s) \sim \sum_{k=0}^{\infty} F_k s^k + \sum_{k=1}^{\infty} G_k s^k \ln s,$$

where

$$F_0 = \int_{\Gamma} h(0, z) dz,$$

$$F_m = u_m[h] + W_m[h] + \frac{1}{m!(m-1)!} \partial_z^{m-1} \partial_z^m h(0, 0) \sum_{j=1}^{m-1} \frac{1}{j},$$

$$G_m = -\frac{1}{m!(m-1)!} \partial_z^{m-1} \partial_w^m h(0, 0),$$

where

$$u_m[h] = -\frac{1}{m!(m-1)!} \int_{\Gamma} \log_0 z \partial_z^m \partial_w^m h(z, 0) dz,$$

$$W_m[h] = -\frac{1}{(m-1)!} \lim_{\varepsilon \rightarrow 0+} \left\{ - \int_{\substack{\varepsilon e^{i\theta} \\ \arg \omega = \theta}}^{\infty} \omega^{m-1} \partial_z^{m-1} (r_{m+1}^w h) \left(0, \frac{1}{\omega}\right) d\omega \right. \\ \left. + \log_0(\varepsilon e^{i\theta}) \frac{1}{m!} \partial_z^{m-1} \partial_w^m h(0, 0) \right\}$$

with $\theta_1 \leq \theta \leq \theta_2$. In the formulas for both u_m and W_m , $\log_0 z$ represents a branch of $\log z$ that is holomorphic for $z \in \overline{K} \setminus \{0\}$. (Note that $u_m + W_m$ is independent of the choice of branch with this property.)

An alternative formula for F_m is

$$F_m = Z_m[h] + W_m[h],$$

where

$$Z_m[h] = -\frac{1}{m!} \int_{\Gamma} \log_0 z \partial_z (z^{-m+1} \partial_w^m (r_m^z h)(z, 0)) dz.$$

This is proved by a simple integration by parts; the integrated terms cancel out the sum in the third term in the formula for F_m in the statement of the theorem.

If $\partial_\omega \partial_z^{m-1} h(0, 1/\omega)$ is sufficiently well behaved for ω near 0, we also have

$$W_m[h] = -\frac{1}{(m-1)!} \int_{\arg \omega = \theta}^{\infty} \log_0 \omega \partial_\omega (\omega^m \partial_z^{m-1} (r_{m+1}^w h)(0, 1/\omega)) d\omega.$$

In [CM1] we showed how to apply the method of singular asymptotics to integrals of the form

$$\Phi(s) = \int_{\Gamma} h(z, s/z) \varphi(z) dz,$$

when $\Gamma = \mathbf{R}_+$ and $\varphi(z)$ is not necessarily smooth up to $z = 0$, while h satisfies the hypotheses of Theorem 2.1. The idea can be easily extended to the present setting. We have [CM1]

$$\Phi(s) = \int_{\Gamma_{\pm}} (D_m^w - D_m^z)^k h_j(z, w)|_{w=s/z} \varphi_{(k)}(z) dz, \quad (2.6)$$

where $D_m^z = \xi \partial_\xi - m$, for a constant $m \in \mathbf{C}$, and $\varphi_{(k)}$ is the k th-order regularization of φ along Γ :

$$\begin{aligned} \varphi_{(0)} &= \varphi \\ \varphi_{(k)}(z) &= \frac{1}{z} \int_{\Gamma}^z \varphi_{(k-1)}(z') dz'. \end{aligned}$$

We will see momentarily how the redundancy of the parameter m is exploited. We have the following adaptation of Theorem 2.1 to renormalized integrals.

THEOREM 2.2. Let Γ , K , h , $\log_0$, and θ be as in Theorem 2.1. Let $\varphi \in C^\infty(\Gamma \setminus \{0\})$ be bounded, and suppose $\varphi_{(k)} \in C^{k-1}(\Gamma)$ for each $k > 0$. Let $\tilde{h}(z, w) = h(z, w)\varphi(z)$ and

$$\Phi(s) = \int_{\Gamma} \tilde{h}(z, s/z) dz.$$

Then

(a)

$$\Phi(s) \sim \sum_{k=0}^{\infty} \Phi_k s^k + \sum_{k=1}^{\infty} \Psi_k s^k \ln s.$$

The asymptotic coefficients Φ_k and Ψ_k are given by

$$\Phi_0 = \int_{\Gamma} h(z, 0) \varphi(z) dz,$$

$$\Phi_m = Z_m[h, \varphi; k] + W_m[h, \varphi; k],$$

$$\Psi_m = -\mathcal{D}_{m-1,0}^z \mathcal{D}_{m,0}^w ((w \partial_w - z \partial_z)^k h(z, w) \varphi_{(k)}(z))$$

for each k sufficiently large, where

$$Z_m[h, \varphi; k] = - \int_{\Gamma} \log_0 z \partial_z (z^{-m} r_m^z (D_m^z)^k \mathcal{D}_m^w h(z, 0) \varphi_{(k)}(z)) dz,$$

$$W_m[h, \varphi; k] = W_m[(w \partial_w - z \partial_z)^k h(z, w) \varphi_{(k)}(z)].$$

(b) If in addition $\varphi_{(k)}^{(j)}(0) = 0$ for $1 \leq j < k$, we have

$$W_m[h, \varphi; k] = W_m[h(z, w)] \cdot \varphi_{(k)}(0).$$

(c) If $\mathcal{D}_{j,0}^z h(0, w) = 0$ for $j < m$, then

$$Z_m[h, \varphi; k] = - \int_{\Gamma} \log_0 z \partial_z (z^{-m+1} \mathcal{D}_m^w h(z, 0) \varphi(z)) dz.$$

Proof. This is an immediate corollary of Theorem 2.1 if we note that

$$\mathcal{D}_{m,0}(D_m^z)^k f(z) = 0,$$

if f is C^∞ in a neighborhood of 0 and $k > 0$. Parts (b) and (c) follow by repeated integrations by parts. $\square$

2.2. Application to the asymptotics of $M(\alpha)$. We will compute the asymptotics of $M(\alpha)$ by using the representation

$$M(\alpha) = \sum_{n=2}^{\infty} H\left(\frac{1}{n}, \alpha n, \alpha\right),$$

where

$$H(z, w, \alpha) = H_1(z, w) + \frac{\sinh^2 \alpha}{\alpha^2} H_2(z, w) + \cosh \alpha \frac{\sinh \alpha}{\alpha} H_3(z, w) \quad (2.7)$$

and

$$\begin{aligned} H_1(z, w) &= \frac{z^3}{1 - z^2} \cdot \frac{e^{-w} \sinh w}{\sinh 2w + z^{-1} \sinh 2zw}, \\ H_2(z, w) &= \frac{z^3}{1 - z^2} \cdot \frac{w^2}{\sinh 2w + z^{-1} \sinh 2zw}, \\ H_3(z, w) &= \frac{z^3}{1 - z^2} \cdot \frac{w}{\sinh 2w + z^{-1} \sinh 2zw}. \end{aligned}$$

The asymptotic coefficients are clearly combinations of the coefficients of

$$M_j(\alpha) = \sum_{n=2}^{\infty} H_j\left(\alpha n, \frac{1}{n}\right) = \frac{1}{2i} \left\{ \int_C H_j\left(\frac{1}{\zeta}, \alpha \zeta\right) \varphi\left(\frac{1}{\zeta}\right) \chi(\operatorname{Re} \zeta) d\zeta + \sum_{\pm} (\mp) I_{j\pm}(\alpha) \right\} \quad (2.8)$$

for $j = 1, 2, 3$, where

$$I_{j\pm}(\alpha) = \mp \int_{C_{\pm}} H_j\left(\alpha \zeta, \frac{1}{\zeta}\right) \chi_1(\operatorname{Re} \zeta) \varphi\left(\frac{1}{\zeta}\right) d\zeta.$$

Here $\varphi(z) = \cot \pi/z$ and χ, χ_1 are as in the last section. We let $\Gamma_{\pm}$ be a curve, of the type described in Sec. 2.1, that is, a semi-infinite extension of the image of $C_{\pm}$ under $\zeta \rightarrow z = 1/\zeta$, oriented with the origin as the initial point. The nontrivial part of our calculation is the asymptotics of each of the integrals

$$I_{j\pm}(\alpha) = \int_{\Gamma_{\pm}} h_j\left(z, \frac{\alpha}{z}\right) \cdot \varphi(z) dz,$$

in the second term of (2.8), where

$$h_j(z, w) = \frac{1}{z^2} H_j(z, w) \cdot \chi_1 \left(\operatorname{Re} \frac{1}{z} \right).$$

The asymptotics of the first term on the right in (2.8) is trivial. It suffices to do a Taylor expansion of the integrand with respect to α :

$$\mathcal{D}_{k,0} \int_C H_j \left(\frac{1}{\zeta}, \alpha \zeta \right) \varphi \left(\frac{1}{\zeta} \right) \chi(\operatorname{Re} \zeta) d\zeta = \int_C \mathcal{D}_{k,0}^w H_j(z, 0) \cdot \chi(\operatorname{Re} \zeta) \cot \pi \zeta d\zeta. \quad (2.9)$$

Theorem 2.2 cannot be applied directly to $I_{j\pm}$; $\varphi(z)$ is rapidly oscillatory near $z = 0$, along $\Gamma_{\pm}$, and needs to be regularized. We let $\varphi_{\pm(k)}$ be the k th-order regularization of φ along $\Gamma_{\pm}$:

$$\begin{aligned} \varphi_{\pm(0)} &= \varphi|_{\Gamma_{\pm}}, \\ \varphi_{\pm(k)}(z) &= \frac{1}{z} \int_{\Gamma_{\pm}}^z \varphi_{\pm(k-1)}(z') dz'. \end{aligned}$$

For the calculations we need

LEMMA 2.1. For $k \geq 1$, $\varphi_{\pm(k)}(z)$ is C^{k-1} for $z \in \Gamma_{\pm}$ and

$$\begin{aligned} \varphi_{\pm(k)}(0) &= \mp i \\ \varphi_{\pm(k)}^{(j)}(0) &= 0, \quad j = 1, 2, \dots, k-1. \end{aligned}$$

Proof. This is essentially done in [CM1], except that we regularized with respect to the parameter of the curve $\Gamma_{\pm}$ there. $\square$

This lemma allows us to use Theorem 2.2. Since the H_j are holomorphic for z and w in a cone around the interval $[2, \infty)$ on the real axis, it is not hard to see that the hypotheses of the theorem are satisfied. We proceed with the calculations.

Observe that $\partial_z^k h_j(z, w)|_{z=0} = 0$ for k odd. We then obtain the asymptotic expansion (2.4) from Theorem 2.2. We will compute M_k , for $k \leq 2$, and N_2 , to prove (2.5). From (2.7) we deduce

$$\mathcal{D}_{0,0} M(0) = \sum_{j=1}^3 \mathcal{D}_{0,0} M_j(0), \quad k = 0, 1, \quad (2.10)$$

$$\mathcal{D}_{2,0} M(0) = \sum_{j=1}^3 \mathcal{D}_{2,0} M_j(0) + \frac{1}{3} \mathcal{D}_{0,0} M_2(0) + \frac{2}{3} \mathcal{D}_{0,0} M_3(0), \quad (2.11)$$

where we used

$$\begin{aligned} \sinh^2 \alpha &= \alpha^2 + \frac{1}{3} \alpha^4 + O(\alpha^6), \\ \cosh \alpha \cdot \sinh \alpha &= \alpha + \frac{2}{3} \alpha^3 + O(\alpha^5). \end{aligned}$$

Further,

$$\mathcal{D}_{2,1}M(0) = \sum_{j=1}^3 \mathcal{D}_{2,1}M_j(0).$$

We start with

$$\mathcal{D}_{k,0}M_j(0) = \int_C \chi(\operatorname{Re} z) \mathcal{D}_{k,0}^w H_j \left(\frac{1}{z}, 0 \right) \cot \pi z \, dz + \sum_{\pm} \mathcal{D}_{k,0} I_{j\pm}(0)$$

for $k = 0, 1$. We need

$$\begin{aligned} \mathcal{D}_{0,0}^w H_1(z, 0) &= \mathcal{D}_{0,0}^w H_3(z, 0) = \frac{1}{4} \frac{z}{1-z^2}, \\ \mathcal{D}_{0,0}^w \frac{1}{z^2} H_2(z, 0) &= 0, \\ \mathcal{D}_{1,0}^w \frac{1}{z^2} H_1(z, 0) &= -\frac{1}{4} \frac{z}{1-z^2}, \\ \mathcal{D}_{1,0}^w \frac{1}{z^2} H_2(z, 0) &= \frac{1}{4} \frac{z}{1-z^2}, \\ \mathcal{D}_{1,0}^w \frac{1}{z^2} H_3(z, 0) &= 0, \\ \mathcal{D}_{1,0}^z \frac{1}{z^2} H_1(z, w) \Big|_{z=0} &= \frac{e^{-w} \sinh w}{\sinh 2w + 2w}, \\ \mathcal{D}_{1,0}^z \frac{1}{z^2} H_2(z, w) \Big|_{z=0} &= \frac{w^2}{\sinh 2w + 2w}, \\ \mathcal{D}_{1,0}^z \frac{1}{z^2} H_3(z, w) \Big|_{z=0} &= \frac{w}{\sinh 2w + 2w}. \end{aligned}$$

Using Theorem 2.2 we then find

$$\begin{aligned} \mathcal{D}_{0,0} I_{2\pm}(0) &= 0 = \mathcal{D}_{0,0} M_2(0), \\ \mathcal{D}_{0,0} I_{3\pm}(0) &= \int_{\Gamma_{\pm}} \frac{1}{4} \frac{z}{1-z^2} \cot \frac{\pi}{z} \chi_1 \left(\operatorname{Re} \frac{1}{z} \right) dz, \end{aligned} \quad (2.12)$$

$$\mathcal{D}_{0,0} \sum_{j=1}^3 I_{j\pm}(0) = \int_{\Gamma_{\pm}} \frac{1}{2} \frac{z}{1-z^2} \cot \frac{\pi}{z} \chi_1 \left(\operatorname{Re} \frac{1}{z} \right) dz, \quad (2.13)$$

$$\mathcal{D}_{2,0} \sum_{j=1}^3 I_{j\pm}(0) = 0. \quad (2.14)$$

From (2.12) and (2.9) we obtain

$$\mathcal{D}_{0,0} M_3(0) = \int_C \frac{1}{4} \frac{1}{\zeta(\zeta^2 - 1)} \cot \pi \zeta \, d\zeta, \quad (2.15)$$

which we need in (2.11). Similarly from (2.13) we find

$$\mathcal{D}_{0,0} M(0) = \frac{1}{2} \int \frac{1}{\zeta(\zeta^2 - 1)} \cot \pi \zeta \, d\zeta = \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n(n^2 - 1)},$$

which proves the first of the formulas (2.5). For the $\mathcal{D}_{2,j}$ -coefficients we need

$$\begin{aligned}\mathcal{D}_{2,0}^w \frac{1}{z^2} H_1(z, 0) &= \frac{1}{12} z, \\ \mathcal{D}_{2,0}^w \frac{1}{z^2} H_2(z, 0) &= 0,\end{aligned}\tag{2.16}$$

$$\mathcal{D}_{2,0}^w \frac{1}{z^2} H_3(z, 0) = -\frac{1}{12} \frac{z(1+z^2)}{1-z^2},\tag{2.17}$$

and

$$\begin{aligned}r_2^w \mathcal{D}_{1,0}^z \frac{1}{z^2} (H_1 + H_2 + H_3)(z, w) \Big|_{z=0} &= \frac{e^{-w} \sinh w + w + w^2}{\sinh 2w + 2w} - \frac{1}{2} \\ &= \frac{1}{2} \frac{1 + 2w^2 - \cosh 2w}{\sinh 2w + 2w} \\ &= \frac{w^2 - \sinh^2 w}{\sinh 2w + 2w}.\end{aligned}\tag{2.18}$$

Referring to Theorem 2.2(b), we have

$$\begin{aligned}\mathcal{D}_{2,0} \sum_{j=1}^3 I_{j\pm}(0) &= \varphi_{(k)}(0) \cdot W_2 \left[\frac{1}{z^2} (H_1 + H_2 + H_3)(z, w) \right] \\ &\quad + Z_2[h_1 + h_2 + h_3, \varphi|_{\Gamma_{\pm}}; k].\end{aligned}\tag{2.19}$$

From (2.16) we have

$$\mathcal{D}_{2,0}^w (h_1 + h_2 + h_3)(z, 0) = -\frac{1}{6} \frac{z^3}{1-z^2} \chi_1 \left(\operatorname{Re} \frac{1}{z} \right)$$

so that, by Theorem 2.2(c),

$$\begin{aligned}z_2[h_1 + h_2 + h_3, \varphi|_{\Gamma_{\pm}}; k] &= - \int_{\Gamma_{\pm}} \log_0 z \partial_z \left(z^{-1} \chi_1 \left(\operatorname{Re} \frac{1}{z} \right) \cdot \varphi(z) \cdot \left(-\frac{1}{6} \frac{z^3}{1-z^2} \right) \right) dz \\ &= \pm \int_{C_{\pm}} \frac{1}{\zeta(\zeta^2 - 1)} \cdot \chi_1(\operatorname{Re} \zeta) \cot \pi \zeta d\zeta.\end{aligned}\tag{2.20}$$

Further, from (2.18) we obtain

$$\begin{aligned}W_2 \left[\frac{1}{z^2} (H_1 + H_2 + H_3)(z, w) \right] &= - \int_0^{\infty} \log_0 \omega \partial_{\omega} \cdot \left(\omega^{+2} \cdot \frac{(1/\omega)^2 - \sinh^2(1/\omega)}{\sinh(2/\omega) + (2/\omega)} \right) d\omega \\ &= - \int_0^{\infty} w^{-3} \frac{\sinh^2 w - w^2}{\sinh 2w + 2w} dw.\end{aligned}\tag{2.21}$$

Substituting (2.20) and (2.21) back into (2.19) and then using this result, Lemma 2.1, and (2.9) in (2.8), we find

$$\mathcal{D}_{2,0} \sum_{j=1}^3 M_j(0) = -\frac{1}{6} \cdot \int_C \frac{1}{\zeta(\zeta^2 - 1)} \cot \pi \zeta \frac{d\zeta}{2i} - \int_0^{\infty} w^{-3} \frac{\sinh^2 w - w^2}{\sinh 2w + 2w} dw.$$

Finally, substituting this formula and (2.15) into (2.11), we obtain the formula for M_2 in (2.5).

The coefficient N_2 in (2.4) is 0 by Theorem 2.2, because

$$\mathcal{D}_{1,0}^z \mathcal{D}_{2,0}^w (h_1 + h_2 + h_3)(0, 0) = 0$$

by (2.16).

This completes the proof of (2.5).

REFERENCES

- [CM] C. Callias and X. Markenscoff, *Singular asymptotics of integrals and the near-field radiated from nonuniformly moving dislocations*, Arch. Rational Mech. Anal. **102**, 273–285 (1988)
- [CM1] C. Callias and X. Markenscoff, *Singular asymptotics analysis for the singularity at a hole near a boundary*, Quart. Appl. Math. **47**, 233–245 (1989)
- [DKH] Z. P. Duan, R. Kienzler, and G. Herrmann, *An integral equation method and its application to defect mechanics*, J. Mech. Phys. Solids. **34**, 539–561 (1986)
- [L] C. B. Ling, *On the stresses in a plate containing two circular holes*, J. Appl. Phys. **19**, 77 (1948)
- [Z] R. W. Zimmerman, *Stress singularity around two nearby holes*, Mech. Res. Commun. **55**, 87–90 (1988)