ANALYTIC SOLUTION OF THE STEFAN PROBLEM IN FINITE MEDIUMS

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Abstract. The classical Stefan problem is considered in this paper for finite mediums with Dirichlet boundary conditions. Analytic solutions for the temperature distributions and the location of the moving interface are obtained by using Lie group theory and the superposition principle. The existence of analytically exact solutions is established by proving the convergence of the solution.

1. Introduction. The Stefan problem is very interesting because of its wide applications in science and engineering and its nonlinearity, which poses an immense challenge to those who seek analytically exact solutions. In casting industries, the melting and solidification of metals are studied extensively to understand the formation of microstructures. The thawing of soils, the formation of ice, and the cooling of large masses of igneous rock are important in geology. Melting and solidification are also encountered in welding, in crystallization, and in the formation of alloys. Recently, lasers are being increasingly used for cutting, drilling, welding, and developing novel metastable alloys, which involve phase changes and moving boundaries. The application of the Stefan problem to study the formation of metastable alloys during laser-aided materials processing can be found in [1–3]. Tao studied the Stefan problem of a polymorphous material [4] and the formation of binary alloys [5, 6].

Studies on phase change problems were first published by Lamé and Clapeyron in 1831 and then by Stefan in 1889. However, the exact solutions for such phase change problems were discussed by Neumann in his lectures in the 1860s, which were published in 1912. The method of obtaining analytic solutions for phase change problems in semi-infinite mediums can be found in [6, 7], and it can be seen that the solutions are obtained only for a certain class of problems in which the governing equations as well as the boundary conditions are invariant under a similarity variable which transforms the partial differential equations for the temperature distribution into ordinary differential equations. Tao solved the Stefan problem with arbitrary initial and boundary conditions [9] as well as with arbitrary heat flux and initial conditions [6]. In all these studies, the material has been considered semi-infinite, which can be a good approximation for many cases in which the material is so long

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that the boundary at one end remains unaffected for all practical purposes. However, the semi-infinite medium approximation does not hold good when the material is not very long, and in this situation, the Stefan problem has to be solved for the finite medium.

This paper presents a technique for obtaining an analytic solution of the Stefan problem for finite mediums. Although the Stefan problem with Dirichlet boundary conditions is solved in this paper, the proposed technique can be applied to the Stefan problem with arbitrary heat flux and initial conditions by expanding the boundary and initial conditions in appropriate power series as has been done in [9] and [6], respectively.

2. Mathematical analysis. The formulation of the Stefan problem with Dirichlet boundary conditions in semi-infinite mediums can be found in [7] and [8]. For finite mediums, the one-dimensional phase change problem can be written as

$$\alpha_1(\partial^2 T_I / \partial x^2) = \partial T_I / \partial t \quad \text{for } 0 < x < s(t), \quad (1a)$$

$$\alpha_2(\partial^2 T_{II} / \partial x^2) = \partial T_{II} / \partial t \quad \text{for } s(t) < x < w, \quad (1b)$$

$$T_I(0, t) = T_0, \quad T_{II}(w, t) = T_i, \quad T_{II}(x, 0) = T_i, \quad (2)$$

$$T_I(s(t), t) = T_{II}(s(t), t) = T_M, \quad (3a)$$

$$k_1 \left. \frac{\partial T_I}{\partial x} \right|_{s(t)} - k_2 \left. \frac{\partial T_{II}}{\partial x} \right|_{s(t)} = \rho L \frac{ds}{dt}, \quad (3b)$$

$$s(0) = 0, \quad (3c)$$

where x is the location of a point in the Cartesian coordinate system, t is the time variable, α_1 and α_2 are the thermal diffusivities of the solid and liquid phases respectively, T_I and T_{II} are the temperatures of the solid and liquid phases respectively, and k_1 and k_2 are the thermal conductivities of the solid and liquid phases respectively. $s(t)$ is the location of the freezing front, w is the width of the region of interest, ρ is the density of the solid phase, L is the latent heat of freezing, and T_M is the freezing temperature of the liquid phase.

The above governing Eqs. (1a, b), the boundary and the initial conditions (2), and the interface conditions (3a, b, c) refer to the physical situation of the freezing of a liquid which is initially, that is, at $t = 0$, bounded in the region $0 \leq x \leq w$, and at the temperature T_i , which is higher than the freezing temperature, T_M , of the material. Suddenly, the plane $x = 0$ is cooled to a temperature $T_0 < T_M$ so that the solidification takes place.

Group analysis [10, 11] shows that the parabolic Eqs. (1a, b) are invariant under the similarity transformation

$$\eta_i = x / (4\alpha_i t)^{1/2}, \quad i = 1, 2. \quad (4)$$

But the boundary and the initial conditions (2) do not allow such a similarity transform. For this reason, we convert the independent variable x to

$$y = x / (x - w) \quad (5)$$

such that the region $0 \leq x \leq w$ transforms into the half-space $0 \leq y \leq \infty$, which makes the boundary and the initial conditions (2) admit the similarity variable (4). However, under the transformation (5), the governing Eqs. (1a, b) will not admit the similarity variable (4). To obtain a set of similarity variables that will be admitted by the governing Eqs. (1a, b), the boundary and the initial conditions (2), and the interface conditions (3a, b, c), we use the following two group generators from [11],

$$U_1 = T_i \frac{\partial}{\partial T_i} \quad \text{and} \quad U_2 = x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t},$$

where $i = \text{I and II}$, and observe that a linear combination of the Lie groups of a differential equation is also a group for the differential equation. Combining U_1 and U_2 , we obtain a new group generator,

$$U_3 = x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t} - m T_i \frac{\partial}{\partial T_i}, \quad m = 0, 1, 2, \dots, \infty,$$

which yields $Ax/t^{1/2}$ and $B_m T_i t^{m/2}$ as similarity variables, where A and B_m are arbitrary constants, and $i = \text{I and II}$. Using these two similarity variables, we define the following independent and dependent similarity variables in order to obtain nonsimilar solutions for the Stefan problem considered in this study:

$$\eta_1 = yw/(4\alpha_1 t)^{1/2}, \quad (6a)$$

$$\eta_2 = yw/(4\alpha_2 t)^{1/2}, \quad (6b)$$

$$T_I(x, t) = T_1(y(x), t) = \sum_{m=0}^{\infty} t^{m/2} F_m(\eta_1), \quad (7a)$$

$$T_{II}(x, t) = T_2(y(x), t) = \sum_{m=0}^{\infty} t^{m/2} G_m(\eta_2). \quad (7b)$$

Also, we note that as $x \rightarrow s(t)$, $y \rightarrow P(t)$, where $P(t) = s(t)/(s(t) - w)$ because of the transformation (5). Therefore, we can write

$$\dot{s}(t) = -w\dot{P}(t)/(1 - P(t))^2, \quad (8)$$

where $\dot{s}(t) = ds(t)/dt$ and $\dot{P}(t) = dP(t)/dt$. Applying the transformations (5), (6a), (7a), and (5), (6b), (7b), respectively, to the governing Eqs. (1a) and (1b), and equating the coefficients of the same powers of t , we obtain

$$F_m'' + 2\eta_1 F_m' - 2mF_m = \sum_{n=1}^4 u_{n1} \eta_1^{n-1} F_{m-n}' + \sum_{n=1}^4 v_{n1} \eta_1^n F_{m-n}'', \quad (9)$$

and

$$G_m'' + 2\eta_2 G_m' - 2mG_m = \sum_{n=1}^4 u_{n2} \eta_2^{n-1} G_{m-n}' + \sum_{n=1}^4 v_{n2} \eta_2^n G_{m-n}'', \quad (10)$$

for $m = 0, 1, 2, \dots, \infty$, where, for $i = 1$ and 2 , $u_{1i} = 2A_i$, $u_{2i} = -12A_i^2$, $u_{3i} = 24A_i^3$, $u_{4i} = -16A_i^4$, $v_{1i} = 4A_i$, $v_{2i} = -12A_i^2$, $v_{3i} = 16A_i^3$, $v_{4i} = -8A_i^4$,

$u_{ni} = v_{ni} = 0$ for $n > m$, $A_i = \alpha_i^{1/2}/w$, and the single and double primes denote, respectively, the first and second derivatives with respect to η_1 in Eq. (9) and with respect to η_2 in Eq. (10).

Similarly, in terms of the new variables defined by the expressions (5), (6a, b), and (7a, b), the boundary, initial, and interface conditions can be written as follows. Referring to the boundary and initial conditions (2), we find that the boundary condition at $x = 0$ becomes

$$F_m(0) = T_0 \delta_{m0}, \quad m = 0, 1, 2, 3, \dots, \infty, \quad (11a)$$

and the boundary condition at $x = w$ and the initial condition become

$$G_m(\infty) = T_i \delta_{m0}, \quad m = 0, 1, 2, \dots, \infty, \quad (11b)$$

where δ_{m0} is the Kronecker delta. The interface condition (3a) can be written as

$$T_1(P(t), t) = T_2(P(t), t) = T_M. \quad (12a)$$

To simplify (12a), we express $P(t)$ in terms of $t^{1/2}$ as

$$P(t) = 2\alpha_1^{1/2} \sum_{m=0}^{\infty} a_m \tau^{m+1} \quad (12b)$$

where $\tau = t^{1/2}$ and $a_0 \neq 0$. Because of (12b), the interface condition (12a) can be written as $\psi_1(\tau) = \psi_2(\tau) = T_M$, where $\psi_1(\tau) = T_1(P, \tau^2)$ and $\psi_2(\tau) = T_2(P, \tau^2)$. Expanding $\psi_1(\tau)$ and $\psi_2(\tau)$ in Maclaurin series, and equating the coefficients of the same powers of τ , we obtain

$$T_1(P, t)|_{\tau=0} = T_2(P, t)|_{\tau=0} = T_M \quad \text{and} \quad \left. \frac{d^n T_1(P, t)}{d\tau^n} \right|_{\tau=0} = \left. \frac{d^n T_2(P, t)}{d\tau^n} \right|_{\tau=0} = 0$$

for $n = 1, 2, 3, \dots, \infty$, which can be rewritten by using the transformations (7a) and (7b) as

$$F_0(\eta_{1i}) = G_0(\eta_{2i}) = T_M \quad (13a)$$

and

$$\sum_{m=0}^n \frac{d^n}{d\tau^n} \{\tau^m F_m(\eta_1)\} \Big|_{\tau=0, \eta_1=\eta_{1i}} = \sum_{m=0}^n \frac{d^n}{d\tau^n} \{\tau^m G_m(\eta_2)\} \Big|_{\tau=0, \eta_2=\eta_{2i}} = 0, \quad (13b)$$

where $n = 1, 2, 3, \dots, \infty$, and η_{1i} and η_{2i} are, respectively, the values of η_1 and η_2 at the interface, that is, at $y = P(\tau)$. η_{1i} and η_{2i} can be shown to be equal to $w \sum_{m=0}^{\infty} a_m \tau^m$ and $w\alpha^* \sum_{m=0}^{\infty} a_m \tau^m$, respectively, where $\alpha^* = (\alpha_1/\alpha_2)^{1/2}$. The interface condition (3b) can be written as

$$k_1 \tau \left. \frac{\partial T_1}{\partial y} \right|_{y=P(\tau)} - k_2 \tau \left. \frac{\partial T_2}{\partial y} \right|_{y=P(\tau)} = \frac{1}{2} w^2 \rho L \frac{1}{(1-P)^4} \frac{dP}{d\tau}. \quad (14)$$

Expanding $\tau \frac{\partial T_1}{\partial y}$, $\tau \frac{\partial T_2}{\partial y}$, and $\frac{1}{(1-P)^4} \frac{dP}{d\tau}$ in Maclaurin series, using the fact that these three terms are equal to $(w/2\alpha_1^{1/2}) \sum_{m=0}^{\infty} \tau^m F'_m(\eta_1)$, $(w/2\alpha_2^{1/2}) \sum_{m=0}^{\infty} \tau^m G'_m(\eta_2)$, and

$\{2\alpha_1^{1/2}/(1-P)^4\} \sum_{m=0}^{\infty} a_m(m+1)\tau^m$ respectively, and equating the coefficients of the same powers of τ , the interface condition (14) can be written as

$$\begin{aligned} B_1 \frac{d^n}{d\tau^n} \sum_{m=0}^{\infty} \tau^m F'_m(\eta_1) \Big|_{\substack{\tau=0 \\ \eta_1=\eta_{1i}}} - B_2 \frac{d^n}{d\tau^n} \sum_{m=0}^{\infty} \tau^m G'_m(\eta_2) \Big|_{\substack{\tau=0 \\ \eta_2=\eta_{2i}}} \\ = B \frac{d^n}{d\tau^n} \left\{ \frac{1}{(1-P)^4} \sum_{m=0}^{\infty} a_m(m+1)\tau^m \right\}_{\tau=0} \end{aligned} \quad (15)$$

for $n = 0, 1, 2, 3, \dots, \infty$, where $B_1 = wk_1/2\alpha_1^{1/2}$, $B_2 = wk_2/2\alpha_2^{1/2}$, $B = w^2 \rho L \alpha_1^{1/2}$. For $n = 0$, the expression (15) becomes

$$B_1 F'_0(\eta_1) \Big|_{\eta_1=\eta_{1i}} - B_2 G'_0(\eta_2) \Big|_{\eta_2=\eta_{2i}} = Ba_0 \quad (16a)$$

and, for $n = 1, 2, 3, \dots, \infty$, it becomes

$$\begin{aligned} B_1 \sum_{m=0}^n \frac{d^n}{d\tau^n} \{\tau^m F'_m(\eta_1)\} \Big|_{\substack{\tau=0 \\ \eta_1=\eta_{1i}}} - B_2 \sum_{m=0}^n \frac{d^n}{d\tau^n} \{\tau^m G'_m(\eta_2)\} \Big|_{\substack{\tau=0 \\ \eta_2=\eta_{2i}}} \\ = B \frac{d^n}{d\tau^n} \left\{ \frac{1}{(1-P)^4} \sum_{m=0}^{\infty} a_m(m+1)\tau^m \right\}_{\tau=0} \end{aligned} \quad (16b)$$

Also, the condition (3c) becomes $P(0) = 0$, which is already taken into consideration while choosing a polynomial expression for $P(t)$ as given by Eq. (12b). So far, we have expressed the proposed Stefan problem in terms of the similarity variables, and now the solution will be obtained in the next section.

3. Analytic solutions. To solve Eqs. (9) and (10), we note that each of them represents a set of second-order, coupled, linear ordinary differential equations. However, the coupling is such that the inhomogeneous term of the m th differential equation depends on the solutions of the previous differential equations, and therefore, Eqs. (9) and (10) can be solved by successively determining $F_0, F_1, F_2, \dots$ and $G_0, G_1, G_2, \dots$, respectively. Also, the general solution of the m th differential equation can be obtained by taking a sum of the homogeneous and particular solutions.

For the zeroth-order solution, we set $m = 0$ in Eqs. (9) and (10) to obtain $F''_m + 2\eta_1 F'_m = 0$ and $G''_m + 2\eta_2 G'_m = 0$, which can be solved easily [7, 8], and when the boundary conditions (11a, b) and the interface conditions (13a) and (16a) are satisfied, the solutions for F_0 and G_0 become

$$F_0(\eta_1) = T_0 + f_0 \operatorname{erf}(\eta_1),$$

and

$$G_0(\eta_2) = T_i + g_0 \operatorname{erfc}(\eta_2),$$

where $f_0 = (T_m - T_0)/\operatorname{erf}(wa_0)$, $g_0 = (T_m - T_i)/\operatorname{erfc}(w\alpha^* a_0)$, and a_0 is given by the transcendental equation $2B_1 f_0 \exp\{-(wa_0)^2\} + 2B_2 g_0 \exp\{-(w\alpha^* a_0)^2\} = \pi^{1/2} Ba_0$. It should be noted that the zeroth-order solution is identical to the solution of the Stefan problem in a semi-infinite medium with Dirichlet boundary conditions.

For higher-order solutions, we set $m = 1, 2, 3, \dots$ in Eqs. (9) and (10), and the general solutions can be written as

$$F_m(\eta_1) = f_m i^m \operatorname{erfc}(\eta_1) + \bar{f}_m i^m \operatorname{erfc}(-\eta_1) + \phi_{1m}(\eta_1) \quad (17a)$$

where

$$\begin{aligned} \phi_{1m}(\eta_1) = & \sum_{n=0}^{m-1} i^n \operatorname{erfc}(\eta_1) \sum_{p=0}^{\infty} f_{mnp} \eta_1^p + \sum_{n=0}^{m-1} i^n \operatorname{erfc}(-\eta_1) \sum_{p=0}^{\infty} \bar{f}_{mnp} \eta_1^p \\ & + \exp(-\eta_1^2) \sum_{p=0}^{\infty} \bar{f}_{mp} \eta_1^p, \end{aligned} \quad (17b)$$

and

$$G_m(\eta_2) = g_m i^m \operatorname{erfc}(\eta_2) + \bar{g}_m i^m \operatorname{erfc}(-\eta_2) + \phi_{2m}(\eta_2) \quad (17c)$$

where

$$\begin{aligned} \phi_{2m}(\eta_2) = & \sum_{n=0}^{m-1} i^n \operatorname{erfc}(\eta_2) \sum_{p=0}^{\infty} g_{mnp} \eta_2^p + \sum_{n=0}^{m-1} i^n \operatorname{erfc}(-\eta_2) \sum_{p=0}^{\infty} \bar{g}_{mnp} \eta_2^p \\ & + \exp(-\eta_2^2) \sum_{p=0}^{\infty} \bar{g}_{mp} \eta_2^p. \end{aligned} \quad (17d)$$

In each of Eqs. (17a, c), the first two terms on the right-hand side represent the homogeneous solution of the m th equation, which is obtained from [12]. f_m , $\bar{f}_m$, g_m , and $\bar{g}_m$ are constants which have to be determined by satisfying the boundary and interface conditions. $\phi_{1m}(\eta_1)$ and $\phi_{2m}(\eta_2)$ represent the particular solutions of the m th equations, where the constants f_{mnp} , $\bar{f}_{mnp}$, $\bar{f}_{mp}$, g_{mnp} , $\bar{g}_{mnp}$, and $\bar{g}_{mp}$ can be determined by using the method of undetermined coefficients. The number of terms in the particular solutions (17b, d) is purposely taken to be infinite, because it increases as the number m of differential equations increases. Since the number of terms in these series will be finite for a given value of m , the values of f_{mnp} , $\bar{f}_{mnp}$, $\bar{f}_{mp}$, g_{mnp} , $\bar{g}_{mnp}$, and $\bar{g}_{mp}$ will become zero after a certain value of p for each m when the method of undetermined coefficients is applied. For this reason, infinite series are used in the particular solutions (17b, d) to maintain the generality of the solutions.

By applying the boundary condition (11a) to Eq. (17a), we obtain

$$F_m(\eta_1) = f_m \{ i^m \operatorname{erfc}(\eta_1) - i^m \operatorname{erfc}(-\eta_1) \} + \phi_{1m}(\eta_1) - f_m^* i^m \operatorname{erfc}(-\eta_1) \quad (18)$$

where

$$f_m^* = \left\{ \bar{f}_{m0} + \sum_{n=0}^{m-1} f_{mn0} i^n \operatorname{erfc}(0) + \sum_{n=0}^{m-1} \bar{f}_{mn0} i^n \operatorname{erfc}(0) \right\} / i^m \operatorname{erfc}(0).$$

Similarly, applying the boundary condition (11b) to Eq. (17b), and noting that $i^m \operatorname{erfc}(\infty) = 0$ for $m \geq 0$ and $i^m \operatorname{erfc}(-\infty) = \infty$ for $m \geq 1$, $\bar{g}_m$ and $\bar{g}_{mn}$ can be shown to be zero for $n \geq 0$, and we obtain

$$G_m(\eta_2) = g_m i^m \operatorname{erfc}(\eta_2) + \phi'_{2m}(\eta_2) \quad (19)$$

where

$$\phi'_{2m}(\eta_2) = \sum_{n=0}^{m-1} i^n \operatorname{erfc}(\eta_2) \sum_{p=0}^{\infty} g_{mnp} \eta_2^p + \exp(-\eta_2^2) \sum_{p=0}^{\infty} \bar{g}_{mp} \eta_2^p.$$

To determine f_m , g_m , and a_m , we will use the interface conditions (13b) and (16b), which can be simplified as follows. The interface condition (13b) can be rewritten as

$$\sum_{m=0}^n A_{nm} = \sum_{m=0}^n B_{nm} = 0 \quad \text{for } n = 1, 2, \dots, \infty, \quad (20)$$

where

$$\begin{aligned} A_{nm} &= \frac{d^n}{d\tau^n} \{ \tau^m F_m(\eta_1) \} \Big|_{\substack{\tau=0 \\ \eta_1=\eta_{1i}}} \\ &= n! \sum_{r_1=0}^{n-m} \frac{d^{r_1} F_m(\eta_1)}{d\eta_1^{r_1}} \Big|_{\substack{\tau=0 \\ \eta_1=\eta_{1i}}} w^{r_1} \sum_{\gamma_i} \frac{a_1^{\gamma_1} a_2^{\gamma_2} \dots a_{n-m}^{\gamma_{n-m}}}{\gamma_1! \gamma_2! \dots \gamma_{n-m}!}, \end{aligned}$$

$$\gamma_1 + \gamma_2 + \dots + \gamma_{n-m} = r_1,$$

$$\gamma_1 + 2\gamma_2 + \dots + (n-m)\gamma_{n-m} = n-m,$$

and

$$\begin{aligned} B_{nm} &= \frac{d^n}{d\tau^n} \{ \tau^m G_m(\eta_2) \} \Big|_{\substack{\tau=0 \\ \eta_2=\eta_{2i}}} \\ &= n! \sum_{r_2=0}^{n-m} \frac{d^{r_2} G_m(\eta_2)}{d\eta_2^{r_2}} \Big|_{\substack{\tau=0 \\ \eta_2=\eta_{2i}}} (w\alpha^*)^{r_2} \sum_{\beta_i} \frac{a_1^{\beta_1} a_2^{\beta_2} \dots a_{n-m}^{\beta_{n-m}}}{\beta_1! \beta_2! \dots \beta_{n-m}!}, \end{aligned}$$

$$\beta_1 + \beta_2 + \dots + \beta_{n-m} = r_2,$$

and

$$\beta_1 + 2\beta_2 + \dots + (n-m)\beta_{n-m} = n-m.$$

Here we have used Leibnitz's rule for higher derivatives of products and Faá di Bruno's formula [12] for higher derivatives of functions of a function. The higher derivatives of $F_m(\eta_1)$ and $G_m(\eta_2)$ can be determined from Eqs. (18) and (19) respectively.

Similarly, by using Leibnitz's rule and Faá di Bruno's formula, the interface condition (15) can be simplified as

$$B_1 \sum_{m=0}^n C_{nm} - B_2 \sum_{m=0}^n D_{nm} = B \sum_{m=0}^n E_{nm} \quad \text{for } n = 1, 2, \dots, \infty, \quad (21)$$

where

$$\begin{aligned} C_{nm} &= \frac{d^n}{d\tau^n} \{ \tau^m F'_m(\eta_1) \} \Big|_{\substack{\tau=0 \\ \eta_1=\eta_{1i}}} \\ &= n! \sum_{r_3=0}^{n-m} \frac{d^{r_3} F'_m(\eta_1)}{d\eta_1^{r_3}} \Big|_{\substack{\tau=0 \\ \eta_1=\eta_{1i}}} w^{r_3} \sum_{\delta_i} \frac{a_1^{\delta_1} a_2^{\delta_2} \dots a_{n-m}^{\delta_{n-m}}}{\delta_1! \delta_2! \dots \delta_{n-m}!}, \end{aligned}$$

$$\begin{aligned}\delta_1 + \delta_2 + \cdots + \delta_{n-m} &= r_3, \\ \delta_1 + 2\delta_2 + \cdots + (n-m)\delta_{n-m} &= n-m;\end{aligned}$$

$$\begin{aligned}D_{nm} &= \frac{d^n}{d\tau^n} \{ \tau^m G'_m(\eta_2) \} \Big|_{\substack{\tau=0 \\ \eta_2=\eta_{2i}}} \\ &= n! \sum_{r_4=0}^{n-m} \frac{d^{r_4} G'_m(\eta_2)}{d\eta_2^{r_4}} \Big|_{\substack{\tau=0 \\ \eta_2=\eta_{2i}}} (w\alpha^*)^{r_4} \sum_{\mu_i} \frac{a_1^{\mu_1} a_2^{\mu_2} \cdots a_{n-m}^{\mu_{n-m}}}{\mu_1! \mu_2! \cdots \mu_{n-m}!},\end{aligned}$$

$$\begin{aligned}\mu_1 + \mu_2 + \cdots + \mu_{n-m} &= r_4, \\ \mu_1 + 2\mu_2 + \cdots + (n-m)\mu_{n-m} &= n-m;\end{aligned}$$

and

$$\begin{aligned}E_{nm} &= \frac{d^n}{d\tau^n} \left\{ \frac{1}{(1-P)^4} \sum_{m=0}^{\infty} (m+1) a_m \tau^m \right\} \Big|_{\tau=0} \\ &= n!(n-m+1) a_{n-m} \sum_{r_5=0}^m \frac{(3+r_5)!}{3!} (2\alpha_1^{1/2})^{r_5} \sum_{\rho_i} \frac{a_0^{\rho_1} a_1^{\rho_2} \cdots a_{m-1}^{\rho_m}}{\rho_1! \rho_2! \cdots \rho_m!},\end{aligned}$$

$$\begin{aligned}\rho_1 + \rho_2 + \cdots + \rho_m &= r_5, \\ \rho_1 + 2\rho_2 + \cdots + m\rho_m &= m.\end{aligned}$$

The interface conditions (20) and (21) are essentially three conditions which can be used to determine (f_1, g_1, a_1) , (f_2, g_2, a_2) , $\dots$, (f_i, g_i, a_i) , $\dots$ successively. For example, the zeroth-order solutions yield (f_0, g_0, a_0) , which can be used in the interface conditions (20) and (21) to obtain (f_1, g_1, a_1) , and then, using these two sets, the third set (f_2, g_2, a_2) can be determined, and so on. It should be noted that the coefficients, A_{nm} , B_{nm} , C_{nm} , D_{nm} , E_{nm} , are hard to compute because of the nature of the sums involved in the expressions for these coefficients. A technique to evaluate these coefficients will be discussed in the appendix.

4. Existence and convergence. In the last section, we have obtained the solutions (18) and (19) of the proposed Stefan problem, and have given a method for determining the constants f_m , g_m , and a_m . We shall now establish the existence of these solutions by proving [13, 9] convergence of each series that appears in the solutions. Combining (18) with (7a), (19) with (7b), and using the zeroth solutions, we obtain

$$\begin{aligned}T_1 &= F_0(\eta_1) + \sum_{m=1}^{\infty} t^{m/2} f_m \{ i^m \operatorname{erfc}(\eta_1) - i^m \operatorname{erfc}(-\eta_1) \} - \sum_{m=1}^{\infty} t^{m/2} f_m^* i^m \operatorname{erfc}(-\eta_1) \\ &\quad + \sum_{m=1}^{\infty} t^{m/2} \phi_{1m}(\eta_1) \quad \text{for } \eta_1 < \eta_0\end{aligned} \quad (22)$$

and

$$T_{II} = G_0(\eta_2) + \sum_{m=1}^{\infty} i^{m/2} g_m i^m \operatorname{erfc}(\eta_2) + \sum_{m=1}^{\infty} i^{m/2} \phi'_{2m}(\eta_2) \quad \text{for } \eta_2 > \eta_0 \quad (23)$$

where η_0 is the position of the solid-liquid interface at time t_0 .

The existence of the zeroth-order solutions $F_0(\eta_1)$ and $G_0(\eta_2)$ is well established. We note that $i^m \operatorname{erfc}(-\eta)$ and $i^m \operatorname{erfc}(\eta)$ monotonically increase and decrease, respectively, as η increases. This implies that

$$\begin{aligned} |i^m \operatorname{erfc}(\eta_1) - i^m \operatorname{erfc}(-\eta_1)| &\leq |i^m \operatorname{erfc}(\eta_0) - i^m \operatorname{erfc}(-\eta_0)|, \\ i^m \operatorname{erfc}(-\eta_1) &\leq i^m \operatorname{erfc}(-\eta_0) \end{aligned}$$

for $\eta_1 < \eta_0$, and

$$i^m \operatorname{erfc}(\eta_2) \leq i^m \operatorname{erfc}(\eta_0) \quad \text{for } \eta_2 > \eta_0.$$

Now, for $t < t_0$, the convergence of the first and second series in Eq. (22) and the first series in Eq. (23) can be established in the same way as in [9]. The remaining series in Eqs. (22) and (23) involve the particular solutions that represent the effects of the inhomogeneous terms. Since the inhomogeneous terms of the m th differential equations are bounded, and they do not involve any function of the homogeneous solutions of the m th equation, the last series in Eqs. (22) and (23) will converge for $t < t_0$. So T_I and T_{II} are bounded for $t < t_0$. Finally, the existence of the interface position $s(t)$ can be established by following the procedure given in [9].

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Appendix. Let us evaluate the second sum that appears in the expression for A_{nm} because such a sum is hard to compute. So, we are interested in the following sum:

$$S_{nm} = \sum_{\gamma_i} \frac{a_1^{\gamma_1} a_2^{\gamma_2} \cdots a_N^{\gamma_N}}{\gamma_1! \gamma_2! \cdots \gamma_N!} \quad (24)$$

such that

$$\gamma_1 + \gamma_2 + \cdots + \gamma_N = M, \quad (25a)$$

and

$$\gamma_1 + 2\gamma_2 + \cdots + N\gamma_N = N, \quad (25b)$$

where $n - m$ and r_1 of the original expression of A_{nm} have been replaced by N and M , respectively, and $M \leq N$. It should be noted that M , N , and γ_i assume only the integral values $M \geq 0$, $N \geq 0$, and $\gamma_i \geq 0$ for $i = 1, 2, 3, \dots, N$.

The difficulty in computing S_{nm} (see Eq. (24)) arises due to the fact that Eqs. (25a, b) involve more than two unknowns for $N > 2$. For this reason, a trial and error approach has to be taken to determine all possible values of γ_i for $i = 1, 2, \dots, N$ for a given value of N . However, we notice from Eqs. (25a, b) that

$$\operatorname{Max}\{\gamma_i\} \leq \operatorname{Min}\{M, C_i\},$$

where $C_i = \text{Int}\{N/i\}$ for $i = 1, 2, 3, \dots, N$. Here $\text{Int}\{N/i\}$ refers to the integral value of the ratio N/i .

For example, when $N = 2$, the admissible values of M are 0, 1, or 2 because $0 \leq M \leq N$. We will determine all possible values of γ_1 and γ_2 by taking $M = 2$ because this will allow us to select the values of γ_1 and γ_2 for $m = 1$ also. We note that $\text{Max}\{\gamma_1\} \leq 2$ and $\text{Max}\{\gamma_2\} \leq 1$. So the possible values of $\gamma_1 = 0, 1, 2$, and $\gamma_2 = 0, 1$. Now, for a particular value of γ_1 , we use all the values of γ_2 and check which combination satisfies Eq. (25b) and yields an admissible value for M from Eq. (25a). For the present case, we find that $\gamma_1 = 0$ and $\gamma_2 = 1$, and $\gamma_1 = 2$ and $\gamma_2 = 0$ are two possible solutions for $M = 1$ and 2, respectively, which can be verified easily by solving Eqs. (25a, b) for $M = 1$ and 2. This procedure can be extended to any value of N , and can also be applied to determine the coefficients, B_{nm} , C_{nm} , D_{nm} , E_{nm} , because they all involve a sum similar to Eq. (24).

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