AN APPROXIMATION FRAMEWORK FOR EQUATIONS IN LINEAR VISCOELASTICITY WITH STRONGLY SINGULAR KERNELS

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Abstract. In this paper we consider equations that arise in linear viscoelastic models. Within the context of linear semigroup theory we present an approximation framework for these equations. A relevant convergence result is proved using the Trotter-Kato theorem. This work extends previous results that did not apply to equations with strongly singular kernels.

1. Introduction. We consider the following integro-differential equation in the Hilbert space H :

$$\ddot{u}(t) + A \left[Eu(t) + \int_{-r}^0 a(\theta) \dot{u}(t + \theta) d\theta \right] = f(t). \quad (1.1)$$

Here A is a positive definite, selfadjoint unbounded operator on H , and $f(t)$ is a locally integrable H -valued function. Also, E and r are positive constants. We assume that the function $a(\theta)$ satisfies $a(\theta) = \int_{-r}^0 g(\xi) d\xi$, and we refer to the function $g(\theta)$ as the history kernel. Equations of this type arise in the modeling of viscoelastic beams and plates and have been the focus of much interest (see [2, 4, 7, 10, 13, 14]). In Eq. (1.1) we may distinguish between weakly and strongly singular kernels g , in which the difference has to do with the nature of the singularity of $g(\theta)$ at $\theta = 0$. More precisely, weakly singular kernels satisfy:

(WS.a) $g(\theta) > 0$ for $\theta \in (-r, 0)$;

(WS.b) $g(\theta) \in L^1(-r, 0)$;

(WS.c) $g \in H^1(-r, -\alpha)$ for all $\alpha > 0$, and $g'(\theta) \geq 0$ for $\theta \in (-r, 0)$.

This class includes the function $g(\theta) = e^{\gamma\theta}/|\theta|^p$ for $0 \leq p < 1$ and $\gamma > 0$. Strongly

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singular kernels satisfy:

- (SS.a) $a(-r) = 0$, and $a(\theta) \in L^1(-r, 0) \cap H^2(-r, -\alpha)$ for all $\alpha > 0$;
- (SS.b) $g(\theta) = a'(\theta) > 0$ for $\theta \in (-r, 0)$;
- (SS.c) $g'(\theta) \geq 0$ on $(-r, 0)$.

In [11] we discussed well-posedness and convergence analysis for an approximation scheme for (1.1) with a weakly singular kernel. These results have proven useful for applications to parameter estimation and optimal control for viscoelastic structures (see [2, 4, 6]). In [12] we discussed more regularity results for Eq. (1.1) with a weakly singular kernel, and we considered a well-posed state space formulation for Eq. (1.1) with a strongly singular kernel. In this paper we present an approximation framework for Eq. (1.1) with a strongly singular kernel. We make the following assumption on $g(\theta)$:

$$g(\theta) = \bar{g}(\theta)/|\theta|^p, \quad 1 \leq p < 2, \quad \text{where } \bar{g}(\theta) \text{ is nonnegative, continuous, and increasing on } [-r, 0]. \quad (\text{A1})$$

In Sec. 2 we describe this approximation framework, and in Sec. 3 we prove the relevant convergence results within the context of the Trotter-Kato Theorem of linear semigroup theory. In Sec. 4 we will discuss a few numerical examples.

Next we will introduce some notation and reformulate Eq. (1.1) as an abstract Cauchy problem. Assume that V and H are Hilbert spaces and $V \subset H$ with continuous dense injection. Let V^* denote the strong dual space of V . We identify H with its dual, so that $V \subset H = H^* \subset V^*$. The dual product $\langle \phi, \psi \rangle_{V^*, V}$ on $V \times V^*$ is the unique extension by continuity of the scalar product $\langle \phi, \psi \rangle_H$ of H restricted to $V \times H$. Consider a symmetric sesquilinear form σ on V such that

$$\begin{aligned} |\sigma(u, v)| &\leq C|u|_V|v|_V \quad \text{for } u, v \in V, \\ \operatorname{Re} \sigma(u, u) &\geq \omega|u|_V^2 \quad \text{for } u \in V, \end{aligned} \quad (1.2)$$

where $C, \omega > 0$. Let $A \in L(V, V^*)$ be defined by

$$\sigma(u, v) = \langle Au, v \rangle_{V^*, V} \quad \text{for all } u, v \in V. \quad (1.3)$$

It follows from [17, Theorem 2.2.3] that the restriction of A on H with

$$\operatorname{dom} A = \{u \in V : Au \in H\},$$

where we will use the same symbol A for such a restriction, defines a positive definite and selfadjoint operator on H , $\operatorname{dom} A^{1/2} = V$, and

$$\sigma(u, v) = \langle A^{1/2}u, A^{1/2}v \rangle_H \quad \text{for all } u, v \in V.$$

Thus, V can be equipped with the scalar product $\langle u, v \rangle_V = \sigma(u, v)$. Next, following the ideas found in [18] and [9], let $W = L_g^2(-r, 0; V)$ be the Hilbert space of all V -valued square integrable functions on the measure space $([-r, 0], \text{Borel sets}, g d\theta)$ equipped with norm

$$|w|_W^2 = \int_{-r}^0 g(\theta) |w(\theta)|_V^2 d\theta.$$

Let Z denote the Hilbert space $V \times H \times W$ equipped with norm

$$|(u, v, w)|_Z^2 = E|u|_V^2 + |v|_H^2 + \int_{-r}^0 g(\theta)|w(\theta)|_V^2 d\theta.$$

We define the operator $D: \text{dom } D \subset W \rightarrow W$ on the domain

$$\text{dom } D = \left\{ w \in W: \begin{array}{l} w \text{ is locally absolutely continuous} \\ w(0) = 0, \text{ and } \frac{dw}{d\theta} \in W \end{array} \right\}$$

by $Dw = dw/d\theta$. Again following [9] and [18], we introduce the state functions

$$\begin{aligned} v(t) &= \dot{u}(t), \\ w(t, \theta) &= u(t) - u(t + \theta). \end{aligned}$$

Observe that Eq. (1.1) can now be written as

$$\ddot{u}(t) + A \left[Eu(t) - \int_{-r}^0 a(\theta) Dw(t) d\theta \right] = f(t),$$

where $\dot{u}(t + \theta) = du(t + \theta)/d\theta = -Dw(t)$. Following an integration by parts, this may be rewritten as

$$\ddot{u}(t) + A \left[Eu(t) + \int_{-r}^0 g(\theta)g(\theta)w(t) d\theta \right] = f(t).$$

Observing this and setting $z(t) = (u(t), v(t), w(t))$, one can formulate Eq. (1.1) as the following abstract Cauchy problem on Z :

$$\dot{z}(t) = \mathcal{A} z(t) + (0, f(t), 0). \quad (1.4)$$

Here $\mathcal{A}$ is defined on the domain

$$\text{dom } \mathcal{A} = \left\{ (u, v, w) \in Z: \begin{array}{l} w \text{ is locally a.c.,} \\ Dw + v \in W, \quad w(0) = 0, \\ Eu + \int_{-r}^0 g(\theta)w(\theta) d\theta \in \text{dom } A \end{array} \right\}$$

by

$$\mathcal{A}(u, v, w) = \left(v, -A \left(Eu + \int_{-r}^0 g(\theta)w(\theta) d\theta \right), Dw + v \right).$$

In [12] we showed the well-posedness of the Cauchy problem (1.4) by showing that the operator $\mathcal{A}$ is the infinitesimal generator of a contraction semigroup $T(t)$ on Z . In the next section we will discuss construction of an approximation scheme for Eq. (1.4).

2. Approximation framework. The construction of an approximation scheme for Eq. (1.4) involves choosing finite-dimensional subspaces of the state space Z and an approximation of the state operator $\mathcal{A}$. This involves an approximation in the spatial variable and an approximation in the delay variable.

For the spatial variable let V^N be any sequence of finite-dimensional subspaces of V satisfying the approximation condition:

$$\text{for any } \varphi \in V, \text{ there exists a sequence } \varphi^N \in V^N \text{ such that} \quad (C1) \\ |\varphi^N - \varphi|_V \rightarrow 0 \text{ as } N \rightarrow \infty.$$

Let P_H^N and P_V^N denote orthogonal projections of H and V onto V^N , respectively. Then condition (C1) guarantees strong convergence to the identity of these projections. In addition, define continuous linear operators $A^N: V^N \rightarrow V^N$ by

$$\langle A^N x, y \rangle_H = \sigma(x, y) \quad \text{for } x, y \in V^N. \quad (2.1)$$

This completes the approximation in the spatial variable. (This part of the approximation framework is straightforward; for example, standard finite element methods may be used.)

Next we consider approximation in the delay variable. This involves constructing finite-dimensional subspaces $W^{N,M}$ of W , as well as an operator $D^{N,M}$ which approximates the operator D . In [11] we showed that the averaging scheme developed in [1] could be used successfully for discretizing the delay variable for equations with weakly singular kernels. However, for strongly singular kernels the averaging scheme is insufficient, as will be explained. We consider a modification of the averaging scheme in which a linear basis function replaces the characteristic basis function on the "first subinterval" of $[-r, 0]$. More specifically, define a partition of $[-r, 0]$ by $\theta_j^M = -jr/M$, $j = 0, 1, \dots, M$. Let $E_i^M(\theta) = \chi[\theta_i^M, \theta_{i-1}^M)$, $i = 2, \dots, M$, where χ is the usual characteristic function. The difficulty with the averaging scheme is that, since $g \notin L^1(-r, 0)$, the usual characteristic basis function $\chi[\theta_1^M, 0]$ is not an element of W . Thus, the following basis function is used:

$$E_1^M(\theta) = \begin{cases} -M\theta/r, & \theta_1^M \leq \theta \leq 0, \\ 0 & \text{elsewhere.} \end{cases}$$

Let B_i^M , $i = 2, \dots, M$, be the usual first-order spline elements defined by

$$B_i^M(\theta) = \begin{cases} M(\theta - \theta_{i+1}^M)/r, & \theta_{i+1}^M \leq \theta \leq \theta_i^M, \\ M(\theta_{i-1}^M - \theta)/r, & \theta_i^M \leq \theta \leq \theta_{i-1}^M, \\ 0 & \text{elsewhere,} \end{cases}$$

and set

$$B_1^M(\theta) = \begin{cases} M(\theta - \theta_2^M)/r, & \theta_2^M \leq \theta \leq \theta_1^M, \\ (M\theta/r)^2, & \theta_1^M \leq \theta \leq 0, \\ 0 & \text{elsewhere.} \end{cases}$$

For integers N, M , define the following subspaces of W :

$$\begin{aligned} \widetilde{W}^{N,M} &= \left\{ w \in W : w = \sum_{i=1}^M b_i^M B_i^M, \quad b_i^M \in V^N \right\}, \\ W^{N,M} &= \left\{ w \in W : w = \sum_{i=1}^M a_i^M E_i^M, \quad a_i^M \in V^N \right\}. \end{aligned}$$

Let $P_g^{N,M}$ be the orthogonal projection of W onto $W^{N,M}$.

In order to define an approximation of the operator D , consider the operator $\tilde{D}^{N,M}: \tilde{W}^{N,M} \rightarrow W^{N,M}$ defined by $\tilde{D}\tilde{w}^{N,M} = d\tilde{w}^{N,M}/d\theta$. Also, define an isomorphism $i^{N,M}: \tilde{W}^{N,M} \rightarrow W^{N,M}$ by $i^{N,M}(\sum_{i=1}^M b_i^M B_i^M) = \sum_{i=1}^M b_i^M E_i^M$. Define $D^{N,M}: W^{N,M} \rightarrow W^{N,M}$ by $D^{N,M} = \tilde{D}^{N,M}(i^{N,M})^{-1}$. Since $\tilde{D}^{N,M}(\sum_{i=1}^M a_i^M B_i^M) = (M/r) \sum_{i=2}^M (a_{i-1}^M - a_i^M) E_i^M - 2Ma_1^M E_1^M/r$, we have that

$$D^{N,M} \left(\sum_{i=1}^M a_i^M E_i^M \right) = \frac{M}{r} \sum_{i=2}^M (a_{i-1}^M - a_i^M) E_i^M - 2\frac{M}{r} a_1^M E_1^M. \quad (2.2)$$

Note that this is exactly like the averaging approximation of the operator D in [1] except for the term resulting from the new basis element E_1^M . Next define the finite-dimensional space $Z^{N,M} = V^N \times V^N \times W^{N,M}$, and let $P_Z^{N,M}: Z \rightarrow Z^{N,M}$ be the orthogonal projection. Finally, define the operator $\mathcal{A}^{N,M}: Z^{N,M} \rightarrow Z^{N,M}$ by

$$\mathcal{A}^{N,M}(u, v, w) = \left(v, -A^N \left(Eu + \int_{-r}^0 g(\theta) w(\theta) d\theta \right), D^{N,M} w + f^M(\theta) v \right).$$

Here, $f^M(\theta)$ is given by

$$f^M(\theta) = \sum_{i=2}^M E_i^M + \alpha^M E_1^M, \quad \alpha^M = \frac{-\int_{-r/M}^0 \theta g(\theta) d\theta}{(M/r) \int_{-r/M}^0 \theta^2 g(\theta) d\theta}.$$

This is derived by “formally” projecting the constant function 1 (“formally” because the constant function 1 is not in W) onto $W^{N,M}$ according to the projection equation

$$\int_{-r}^0 g(\theta) \cdot 1 \cdot w^{N,M} d\theta = \int_{-r}^0 g(\theta) \cdot f^M(\theta) \cdot w^{N,M} d\theta \quad \forall w^{N,M} \in W. \quad (2.3)$$

Note that the integrals used to define α^M are finite. To see this for the numerator of α^M , observe that $\int_{-r}^0 -\theta g(\theta) d\theta = \int_{-r}^0 \int_{-r}^{\theta} g(\varepsilon) d\varepsilon d\theta = \int_{-r}^0 a(\theta) d\theta < \infty$. The argument for the denominator is similar. Thus, we have defined a finite-dimensional state space $Z^{N,M}$ and an approximate state operator $\mathcal{A}^{N,M}: Z^{N,M} \rightarrow Z^{N,M}$. Letting $z^{N,M}(t) = (u^N(t), v^N(t), w^{N,M}(t))$, we approximate Eq. (1.4) by the following differential equation on $Z^{N,M}$:

$$\frac{d}{dt} z^{N,M}(t) = \mathcal{A}^{N,M} z^{N,M}(t) + \text{Col}(0, P_H^N f(t), 0). \quad (2.4)$$

In the next section we will prove convergence results for this approximation framework.

3. Convergence results. Our objective is to use the Trotter-Kato Theorem (see [16]) in order to show that $S^{N,M}(t)P_Z^{N,M}z \rightarrow S(t)z \quad \forall z \in Z$, where $S^{N,M}(t)$ and $S(t)$ are the semigroups generated by $\mathcal{A}^{N,M}$ and $\mathcal{A}$, respectively. We will follow many of the ideas found in [11], making appropriate modifications of some estimates to account for the fact that the strongly singular kernel $g(\theta)$ under consideration is not in $L^1(-r, 0)$.

LEMMA 3.1. The operator D is the infinitesimal generator of a C_0 -semigroup of contractions.

Proof. It follows from the arguments used in [12] to show that $\mathcal{A}$ is the infinitesimal generator of a C_0 -semigroup that D is dissipative on W . Further, for any $w \in W$, $(\lambda I - D) \int_{\theta}^0 e^{\lambda(\theta-\xi)} w(\xi) d\xi = w$ so that $\text{range}(I - D) = W$. Hence, it follows from [16, Theorem 1.4.6] that $\text{dom}(D)$ is dense in W , and the lemma is proved. $\square$

LEMMA 3.2. If condition (A1) holds, then $P_g^{N,M} h \rightarrow h$ for all $h \in W$.

Proof. Following [11] Lemma 3.1, for $w \in \text{dom } D^2$, let

$$w^{N,M}(\theta) = \sum_{i=1}^M w^N(\theta_{i-1}^M) E_i^M(\theta),$$

where $w^N = P_V^N w$. Observing that $w^N(0) = 0$, it follows that

$$|w^N - w^{N,M}|_W^2 \leq \frac{M}{3} r |D^2 w^N|_W^2 \left(\frac{r}{M} \right)^3.$$

Hence, for $w \in \text{dom}(D^2)$,

$$|w - P_g^{N,M} w|_W \leq |w - w^N|_W + K |D^2 w|_W^2 \cdot \frac{1}{M}, \quad (3.1)$$

which converges to zero as $N, M \rightarrow \infty$ since the first term converges to zero by the dominated convergence theorem. Since $\text{dom}(D^2)$ is dense in W and $|P_g^{N,M}| \leq 1$, the result follows (K is independent of N, M). $\square$

LEMMA 3.3. The operator $\mathcal{A}^{N,M}$ is dissipative in $Z^{N,M}$.

Proof. For $z^{N,M} = (u, v, w^{N,M}) \in Z^{N,M}$, with $w^{N,M} = \sum_{i=1}^M a_i^M E_i^M$, we have

$$\begin{aligned} \langle \mathcal{A}^{N,M} z^{N,M}, z^{N,M} \rangle_Z &= E\sigma(u, v) - E\sigma(v, u) - \int_{-r}^0 g(\theta) \sigma(v, w^{N,M}) d\theta \\ &\quad + \int_{-r}^0 g(\theta) \sigma(w^{N,M}, D^{N,M} w^{N,M} + f^M(\theta) v) d\theta \\ &= \int_{-r}^0 g(\theta) \sigma(w^{N,M}, D^{N,M} w^{N,M}) d\theta \\ &= \sum_{i=2}^M \frac{M}{r} \langle a_{i-1}^M - a_i^M, a_i^M \rangle_V g_i^M \\ &\quad - 2 \left(\frac{M}{r} \right)^3 |a_1^M|_V^2 \int_{-r/M}^0 g(\theta) \cdot \theta^2 d\theta \end{aligned}$$

where $g_i^M = \int_{\theta_i^M}^{\theta_{i-1}^M} g(\theta) d\theta$, $i = 2, \dots, M$. Continuing, we have

$$\begin{aligned} \langle \mathcal{A}^{N,M} z^{N,M}, z^{N,M} \rangle_Z &= \frac{1}{2} \sum_{i=2}^M \frac{M}{r} g_i^M (|a_{i-1}^M|_V^2 - |a_i^M|_V^2 - |a_i^M - a_{i-1}^M|_V^2) \\ &\quad - 2 \left(\frac{M}{r} \right)^3 \int_{-r/M}^0 \theta^2 g(\theta) d\theta |a_1^M|_V^2 \\ &= \frac{1}{2} \frac{M}{r} \sum_{i=2}^{M-1} |a_i^M|_V^2 (g_{i+1}^M - g_i^M) - \frac{1}{2} \frac{M}{r} g_M^M |a_M^M|_V^2 \\ &\quad - \frac{1}{2} \frac{M}{r} \sum_{i=2}^M g_i^M |a_{i-1}^M - a_i^M|_V^2 \\ &\quad + \left[\frac{1}{2} \frac{M}{r} g_2^M - 2 \left(\frac{M}{r} \right)^3 \int_{-r/M}^0 \theta^2 g(\theta) d\theta \right] |a_1^M|_V^2. \end{aligned}$$

The first three terms are nonpositive (note that $g_i^M \geq g_{i+1}^M$ since g is nondecreasing). Also, using assumption (A1), we have (for the case $p \neq 1$)

$$g_2^M \leq \frac{\bar{g}(-M/r)}{p-1} \left(\frac{M}{r} \right)^{p-1} \left[1 - \left(\frac{1}{2} \right)^{p-1} \right]$$

and

$$\int_{-r/M}^0 \theta^2 g(\theta) d\theta \geq \left(\frac{r}{M} \right)^{3-p} \left(\frac{1}{3-p} \right) \bar{g} \left(-\frac{M}{r} \right). \quad (3.2)$$

Hence, for $p \neq 1$ the above inequalities yield

$$\begin{aligned} \langle \mathcal{A}^{N,M} z^{N,M}, z^{N,M} \rangle_Z &\leq \left[\left(\frac{1}{2(p-1)} \left(1 - \left(\frac{1}{2} \right)^{p-1} \right) - \frac{2}{3-p} \right) \bar{g} \left(-\frac{M}{r} \right) \right] \left(\frac{M}{r} \right)^p |a_1^M|_V^2 \leq 0. \end{aligned}$$

In the case where $p = 1$ it follows that $g_2^M \leq (\log 2) \bar{g}(-M/r)$, and we get

$$\langle \mathcal{A}^{N,M} z^{N,M}, z^{N,M} \rangle_Z \leq (\log 2)/2 - 1 < 0. \quad \square$$

LEMMA 3.4. For $\lambda > 0$, $(\lambda I - D^{N,M})^{-1} P_g^{N,M} h \rightarrow (\lambda I - D)^{-1} h$ for all $h \in W$.

Proof. Observe that $\langle D^{N,M} w^{N,M}, w^{N,M} \rangle_W = \int_{-r}^0 g(\theta) \sigma(D^{N,M} w^{N,M}, w^{N,M}) d\theta \leq 0$ (the last inequality follows from the arguments in Lemma 3.3). Hence, $D^{N,M}$ is dissipative, so that $|(\lambda I - D^{N,M})^{-1}|_W \leq 1/\lambda$. Let $w^{N,M} = (\lambda I - D^{N,M})^{-1} P_g^{N,M} h$

and $w = (\lambda I - D)^{-1}h$, so $w \in \text{dom}(D)$. We must show that $|w - w^{N,M}|_W \rightarrow 0$ as $N, M \rightarrow \infty$. Following Lemma 3.3 in [11], let $\hat{w}^{N,M} = \int_0^\theta P_g^{N,M} Dw d\xi \in \widetilde{W}^{N,M}$ and $\tilde{w}^{N,M} = (i^{N,M})^{-1}w^{N,M} \in \widetilde{W}^{N,M}$. Then

$$|w - w^{N,M}|_W \leq |w - i^{N,M}\hat{w}^{N,M}|_W + |i^{N,M}\hat{w}^{N,M} - w^{N,M}|_W.$$

But, as in [11] it follows that

$$\begin{aligned} |i^{N,M}\hat{w}^{N,M} - w^{N,M}|_W &= |i^{N,M}(\hat{w}^{N,M} - \tilde{w}^{N,M})| \\ &\leq |P_g^{N,M}(w - i^{N,M}\hat{w}^{N,M})|_W; \end{aligned} \quad (3.3)$$

hence,

$$|w - w^{N,M}|_W \leq 2|w - i^{N,M}\hat{w}^{N,M}|_W. \quad (3.4)$$

Next, observe that

$$|w - i^{N,M}\hat{w}^{N,M}|_W \leq \left| \int_0^\theta (Dw - P_g^{N,M}(Dw)) d\xi \right|_W + |i^{N,M}\hat{w}^{N,M} - \hat{w}^{N,M}|_W.$$

Considering these last two terms, we have

$$|i^{N,M}\hat{w}^{N,M} - \hat{w}^{N,M}|_W \leq \left(\frac{r}{M} \right) |Dw|_W \rightarrow 0 \quad \text{as } M \rightarrow \infty, \quad (3.5)$$

and

$$\begin{aligned} \left| \int_0^\theta [Dw - P_g^{N,M}(Dw)] d\xi \right|_W &= \int_{-r}^0 \left| \int_\theta^0 \sqrt{\frac{g(\theta)}{g(\xi)}} \sqrt{g(\xi)} [Dw - P_g^{N,M}(Dw)] d\xi \right|^2 d\theta \\ &\leq \frac{r^2}{2} |Dw - P_g^{N,M}Dw|_W^2 \rightarrow 0 \quad [\text{by Lemma 3.2}]. \end{aligned} \quad (3.6)$$

The result follows from Eqs. (3.4)–(3.6). $\square$

Before giving the desired semigroup convergence result, we discuss some preliminary resolvent calculations. Let $z = (\varphi, \psi, h) \in Z$, and let $P_Z^{N,M}z = (\varphi^N, \psi^N, h^{N,M}) \in Z^{N,M}$. Also, let $(u, v, w) = (\lambda I - \mathcal{A})^{-1}z$ and $(u^N, v^N, w^{N,M}) = (\lambda I - \mathcal{A}^{N,M})^{-1}P_Z^{N,M}z$. If $(\lambda I - \mathcal{A})(u, v, w) = z$, then

$$\lambda u - v = \varphi, \quad (3.7a)$$

$$\lambda v + A \left(Eu + \int_{-r}^0 g(\theta)w(\theta) d\theta \right) = \psi, \quad (3.7b)$$

$$\lambda w - (Dw + v) = h. \quad (3.7c)$$

From Eq. (3.7c), $\lambda(w + \theta v) - D(w + \theta v) = h + \lambda\theta v$, or, equivalently,

$$\begin{aligned} w &= -\theta v + (\lambda I - D)^{-1}(h + \lambda\theta v) \\ &= (\lambda I - D)^{-1}h + \frac{1}{\lambda}(e^{\lambda\theta} - 1)v. \end{aligned} \quad (3.8)$$

Solving Eq. (3.7a) for v and substituting in Eq. (3.7b) yields

$$\begin{aligned} \lambda^2 u + A \left[E - \int_{-r}^0 g(\theta)(e^{\lambda\theta} - 1) d\theta \right] u \\ = \psi + \lambda\phi - A \int_{-r}^0 g(\theta) \left[(\lambda I - D)^{-1}h + \frac{1}{\lambda}(e^{\lambda\theta} - 1)\phi \right] d\theta. \end{aligned} \quad (3.9)$$

Hence,

$$\Delta(\lambda)u = \psi + \lambda\phi - A \int_{-r}^0 g(\theta) \left[(\lambda I - D)^{-1}h + \frac{1}{\lambda}(e^{\lambda\theta} - 1)\phi \right] d\theta \quad (3.10)$$

where $\Delta(\lambda) = \lambda^2 I + A(E - \int_{-r}^0 g(\theta)(e^{\lambda\theta} - 1) d\theta) \in L(V, V^*)$. We define the sesquilinear form μ on V by

$$\begin{aligned} \mu(u, v) &= \langle \Delta(\lambda)u, v \rangle_{V^*, V}, \quad \text{for } u, v \in V \\ &= \lambda^2 \langle u, v \rangle_H + \left(E - \int_{-r}^0 g(\theta)(e^{\lambda\theta} - 1) d\theta \right) \sigma(u, v). \end{aligned}$$

In [12] it is shown that μ is V -bounded and coercive. Proceeding in a similar manner for $(\lambda I - \mathcal{A}^{N,M})(u^N, v^N, w^{N,M}) = P_Z^{N,M} z$, we get

$$\lambda u^N - v^N = \phi^N, \quad (3.11a)$$

$$\lambda v^N + A^N \left[E^N + \int_{-r}^0 g(\theta)w^{N,M}(\theta) d\theta \right] = \psi^N, \quad (3.11b)$$

$$\lambda w^{N,M} - (D^{N,M}w^{N,M} + f^M(\theta)v) = h^{N,M}. \quad (3.11c)$$

From Eq. (3.11c),

$$w^{N,M} = (\lambda I - D^{N,M})^{-1}(h^{N,M} + f^M(\theta)v). \quad (3.12)$$

Substituting Eqs. (3.11a) and (3.12) into Eq. (3.11b) yields

$$\begin{aligned} \lambda^2 u^N + A^N \left[E u^N + \int_{-r}^0 g(\theta)(\lambda I - D^{N,M})^{-1} f^M(\theta) \lambda u d\theta \right] \\ = -A^N \int_{-r}^0 g(\theta) \left[(\lambda I - D^{N,M})^{-1} h^{N,M} - (\lambda I - D^{N,M})^{-1} f^M(\theta) \phi^N \right] d\theta \\ + \psi^M + \lambda \phi^N. \end{aligned} \quad (3.13)$$

Hence,

$$\Delta^M(\lambda)u^N = -A^N \int_{-r}^0 g(\theta) \left[(\lambda I - D^{N,M})^{-1} h^{N,M} + \frac{1}{\lambda} e^M(\lambda\theta) \varphi^N \right] d\theta + \psi^N + \lambda \varphi^N \quad (3.14)$$

where

$$\Delta^M(\lambda) = \lambda^2 I + A^N \left[E - \int_{-r}^0 g(\theta) e^M(\lambda\theta) d\theta \right]$$

and

$$e^M(\lambda\theta) = -(\lambda I - D^{N,M})^{-1} \lambda f^M(\theta). \quad (3.15)$$

We now define the sesquilinear form μ^M on V by

$$\mu^M(u, v) = \lambda^2 \langle u, v \rangle_H + \left(E - \int_{-r}^0 g(\theta) e^M(\lambda\theta) d\theta \right) \sigma(u, v) \quad \text{for } u, v \in V$$

and note that μ^M is coercive. Observe that u satisfies

$$\mu(u, \chi) = -\sigma \left(\int_{-r}^0 g(\theta) \left[(\lambda I - D)^{-1} h + \frac{1}{\lambda} (e^{\lambda\theta} - 1) \varphi \right] d\theta, \chi \right) + \langle \psi + \lambda \varphi, \chi \rangle_H \quad (3.16)$$

for all $\chi \in V$, and u^N satisfies

$$\begin{aligned} \mu^M(u^N, \chi^N) &= -\sigma \left(\int_{-r}^0 g(\theta) \left[(\lambda I - D^{N,M})^{-1} h^{N,M} + \frac{1}{\lambda} e^M(\lambda\theta) \varphi^N \right] d\theta, \chi^N \right) \\ &\quad + \langle \psi^N + \lambda \varphi^N, \chi^N \rangle_H \end{aligned} \quad (3.17)$$

for all $\chi^N \in V$.

LEMMA 3.5. If assumption (A1) holds, then $\int_{-r}^0 g(\theta) |(e^{\lambda\theta} - 1) - e^M(\lambda\theta)| d\theta \rightarrow 0$ as $M \rightarrow \infty$.

Proof. By definition $e^M(\lambda\theta) = -\lambda(\lambda I - D^{N,M})^{-1} f^M(\theta)$, where $f^M(\theta) = \sum_{j=2}^M E_j^M + \alpha^M E_1$. In general, we have the following formula for $(\lambda I - D^{N,M})^{-1}$:

$$(\lambda I - D^{N,M})^{-1} \left(\sum_{i=1}^M h_i E_i \right) = \sum_{i=1}^M a_i E_i,$$

where

$$a_i = \sum_{j=2}^i \frac{r}{M} \left(1 + \frac{\lambda r}{M} \right)^{j-i-1} h_j + \frac{r}{M} \left(1 + \frac{\lambda r}{M} \right)^{-(i-1)} \left(2 + \frac{\lambda r}{M} \right)^{-1} h_1. \quad (3.18)$$

Hence, direct calculation yields the equalities

$$\begin{aligned}
 e^M(\lambda\theta) &= -\lambda \sum_{i=1}^M \left[\sum_{j=2}^i \frac{r}{M} \left(1 + \frac{\lambda r}{M}\right)^{j-i-1} + \frac{r}{M} \left(1 + \frac{\lambda r}{M}\right)^{-(i-1)} \left(2 + \frac{\lambda r}{M}\right)^{-1} \alpha^M \right] E_i \\
 &= -\lambda \sum_{i=1}^M \left[\sum_{j=1}^i \frac{r}{M} \left(1 + \frac{\lambda r}{M}\right)^{j-i-1} \right. \\
 &\quad \left. + \frac{r}{M} \left(1 + \frac{\lambda r}{M}\right)^{-(i-1)} \left(2 + \frac{\lambda r}{M}\right)^{-1} \alpha^M - \frac{r}{M} \left(1 + \frac{\lambda r}{M}\right)^{-i} \right] E_i \\
 &= -\lambda \sum_{i=1}^M \left[\frac{1}{\lambda} \left(1 - \left[1 + \frac{\lambda r}{M}\right]^{-i}\right) \right. \\
 &\quad \left. + \frac{r}{M} \left(1 + \frac{\lambda r}{M}\right)^{-i} \left[\left(1 + \frac{\lambda r}{M}\right) \left(2 + \frac{\lambda r}{M}\right)^{-1} \alpha^M - 1 \right] \right] E_i \\
 &= \sum_{i=2}^M \left\{ \left[\left(1 + \frac{\lambda r}{M}\right)^{-i} - 1 \right] \right. \\
 &\quad \left. - \frac{\lambda r}{M} \left(1 + \frac{\lambda r}{M}\right)^{-i} \left[\left(1 + \frac{\lambda r}{M}\right) \left(2 + \frac{\lambda r}{M}\right)^{-1} \alpha^M - 1 \right] \right\} E_i \\
 &\quad - \frac{\lambda r}{M} \left(2 + \frac{\lambda r}{M}\right)^{-1} \alpha^M E_1.
 \end{aligned}$$

To proceed, we consider the integral under consideration on two subintervals as follows:

$$\begin{aligned}
 \int_{-r}^0 g(\theta) |(e^{\lambda\theta} - 1) - e^M(\lambda\theta)| d\theta &= \int_{-r}^{-r/M} g(\theta) |(e^{\lambda\theta} - 1) - e^M(\lambda\theta)| d\theta \\
 &\quad + \int_{-r/M}^0 g(\theta) |(e^{\lambda\theta} - 1) - e^M(\lambda\theta)| d\theta.
 \end{aligned} \tag{3.19}$$

Considering the first integral in Eq. (3.19), we have that

$$\begin{aligned}
 &\int_{-r}^{-r/M} g(\theta) |(e^{\lambda\theta} - 1) - e^M(\lambda\theta)| d\theta \\
 &\leq \sum_{i=2}^M \int_{\theta_i^M}^{\theta_{i-1}^M} g(\theta) \left| e^{\lambda\theta} - \left(1 + \frac{\lambda r}{M}\right)^{-i} \right| d\theta \\
 &\quad + \sum_{i=2}^M \int_{\theta_i^M}^{\theta_{i-1}^M} g(\theta) \left| \frac{\lambda r}{M} \left(1 + \frac{\lambda r}{M}\right)^{-i} \left[\left(1 + \frac{\lambda r}{M}\right) \left(2 + \frac{\lambda r}{M}\right)^{-1} \alpha^M - 1 \right] \right| d\theta \\
 &= S_1 + S_2.
 \end{aligned}$$

It is not difficult to show that α^M is bounded for sufficiently large M ; hence, we can argue that

$$S_2 \leq \frac{K}{M} \int_{-r}^{-r/M} g(\theta) d\theta \quad \text{for all sufficiently large } M,$$

where K is independent of M . Furthermore, by using (A1), we have

$$S_2 \leq \frac{K}{M} \left(\frac{\bar{g}(0)}{p-1} \right) \left(\left(\frac{r}{M} \right)^{1-p} - r^{1-p} \right) \rightarrow 0 \quad \text{as } M \rightarrow \infty.$$

Also,

$$\begin{aligned} S_1 &\leq \sum_{i=2}^M \int_{\theta_i^M}^{\theta_{i-1}^M} g(\theta) \left\{ \left| e^{\lambda\theta} - e^{-\lambda r i/M} \right| + \left| e^{-\lambda r i/M} - \left(1 + \frac{\lambda r}{M} \right)^{-i} \right| \right\} d\theta \\ &\leq \sum_{i=2}^M \int_{\theta_i^M}^{\theta_{i-1}^M} g(\theta) \left\{ \left| e^{-\lambda r(i-1)/M} - e^{-\lambda r i/M} \right| + \frac{(\lambda r i/M)^2}{2i} \right\} d\theta \end{aligned}$$

where we used the inequality $|e^{-x} - (1 + x/N)^{-N}| \leq x^2/(2N)$ (see [15, p. 482]). Continuing, we have that

$$\begin{aligned} S_1 &\leq \sum_{i=2}^M \int_{\theta_i^M}^{\theta_{i-1}^M} g(\theta) \left\{ \frac{r}{M} + \frac{\lambda^2 r^2 i}{2M^2} \right\} d\theta \\ &\leq \left(\frac{\lambda^2 r^2}{2} + r \right) \frac{1}{M} \int_{-r}^{-r/M} g(\theta) d\theta \rightarrow 0 \quad \text{as } M \rightarrow \infty. \end{aligned}$$

Next, considering the second integral in Eq. (3.19), we have

$$\begin{aligned} &\int_{-r/M}^0 g(\theta) \left| (e^{\lambda\theta} - 1) - e^M(\lambda\theta) \right| d\theta \\ &= \int_{-r/M}^0 g(\theta) \left| (e^{\lambda\theta} - 1) - \lambda \left(2 + \frac{\lambda r}{M} \right)^{-1} \alpha^M \theta \right| d\theta \\ &\leq \int_{-r/M}^0 g(\theta) (1 - e^{\lambda\theta}) d\theta + K \int_{-r/M}^0 g(\theta) (-\theta) d\theta \end{aligned} \quad (3.20)$$

where $|\lambda(2 + \lambda r/M)^{-1} \alpha^M| \leq K$, and K is independent of M . Finally, observe that integrating by parts gives

$$\begin{aligned} \int_{-r}^0 g(\theta) (1 - e^{\lambda\theta}) d\theta &= \lim_{\varepsilon \rightarrow 0^-} (1 - e^{\lambda\varepsilon}) a(\varepsilon) + \int_{-r}^0 \lambda e^{\lambda\theta} a(\theta) d\theta \\ &= \int_{-r}^0 \lambda e^{\lambda\theta} a(\theta) d\theta < \infty. \end{aligned}$$

Thus $g(\theta)(1 - e^{\lambda\theta}) \in L^1(-r, 0)$. Since it is also true that $g(\theta)(-\theta) \in L^1(-r, 0)$, both integrals in Eq. (3.20) converge to zero as $M \rightarrow \infty$, and the result follows. $\square$

We now prove the main convergence result.

THEOREM 3.1. For all $z \in Z$, $S^{N,M}(t)P_Z^{N,M}z \rightarrow S(t)z$ as $N, M \rightarrow \infty$, uniformly on bounded t -intervals.

Proof. In Lemma 3.3 we established that the stability hypothesis of the Trotter-Kato Theorem is satisfied. It remains for us to show that $(\lambda I - \mathcal{A}^{N,M})^{-1}P_Z^{N,M}z \rightarrow (\lambda I - A)^{-1}z$. Assume that $z = (\varphi, \psi, h) \in Z$. Following the notation used above (Eqs. (3.7)–(3.17)), it is sufficient to show that $u^N \rightarrow u$. If we set $\hat{u}^N = P_V^N u$, it is

sufficient to argue that $|\hat{u}^N - u^N|_V \rightarrow 0$. Letting $\chi^N = \hat{u}^N - u^N$, we have (using the coerciveness of μ^M)

$$\begin{aligned}
 |\hat{u}^N - u^N|_V^2 &\leq \mu^M(\hat{u}^N - u^N, \hat{u}^N - u^N) \\
 &\leq |\mu^M(\hat{u}^N - u, \chi^N)| + |\mu^M(u, \chi^N) - \mu(u, \chi^N)| \\
 &\quad + |\mu(u, \chi^N) - \mu^M(u^N, \chi^N)| \\
 &\leq K_1 |\hat{u}^N - u|_V |\chi^N|_V \\
 &\quad + K_2 \int_{-r}^0 g(\theta) |(e^{\lambda\theta} - 1) - e^M(\lambda\theta)| d\theta \cdot |u|_V \cdot |\chi^N|_V \\
 &\quad + |(\psi + \lambda\phi) - (\psi^N + \lambda\phi^N)|_H |\chi^N|_V \\
 &\quad + K_3 \int_{-r}^0 g(\theta) |(\lambda I - D^{N,M})^{-1} P_g^{N,M} h - (\lambda I - D)^{-1} h|_V d\theta |\chi^N|_V \\
 &\quad + K_4 \cdot \frac{1}{\lambda} \int_{-r}^0 g(\theta) |(e^{\lambda\theta} - 1)\phi - e^M(\lambda\theta)\phi^N| d\theta |\chi^N|_V.
 \end{aligned} \tag{3.21}$$

After cancelling $|\chi^N|_V$ from each side of the inequality, every term on the right-hand side $\rightarrow 0$ as $N, M \rightarrow \infty$. This is clear to see for every term except the middle integral in Eq. (3.21). Thus, to finish the proof we must show that

$$\int_{-r}^0 g(\theta) |(\lambda I - D^{N,M})^{-1} P_g^{N,M} h - (\lambda I - D)^{-1} h|_V d\theta \rightarrow 0 \quad \text{as } N, M \rightarrow \infty. \tag{3.22}$$

In [11] the analogous step was carried out by using the Cauchy-Schwarz inequality and the fact that $|(\lambda I - D^{N,M})^{-1} P_g^{N,M} h - (\lambda I - D)^{-1} h|_W \rightarrow 0$. However, that argument requires $g \in L^1(-r, 0)$, which is not the case here. Proceeding differently, assume that h has support in $[-r, -2\tau]$ for $\tau > 0$. Then for M sufficiently large we have $\gamma(\theta) = 0$ and $\gamma^{N,M}(\theta) = 0$ for $\theta \in [-\tau, 0)$, where $\gamma = (\lambda I - D)^{-1} h$ and $\gamma^{N,M} = (\lambda I - D^{N,M})^{-1} P_g^{N,M} h$. Thus,

$$\begin{aligned}
 \int_{-r}^0 g(\theta) |\gamma^{N,M}(\theta) - \gamma(\theta)|_V d\theta &\leq \int_{-r}^{-\tau} g(\theta) |\gamma^{N,M}(\theta) - \gamma(\theta)|_V d\theta \\
 &\leq \int_{-r}^{-\tau} g(\theta) d\theta \cdot |\gamma^{N,M} - \gamma|_W,
 \end{aligned}$$

which goes to 0 as $N, M \rightarrow \infty$. Next observe that, given $h \in W$, $|h_\tau - h|_W \rightarrow 0$ as $\tau \rightarrow 0$, where $h_\tau(\theta) = h(\theta) \cdot \chi_\tau(\theta)$ and $\chi_\tau(\theta)$ is the characteristic function on the interval $[-r, -2\tau]$. Hence, Eq. (3.22) follows since $(\lambda I - D)^{-1}$ and $(\lambda I - D^{N,M})^{-1} P_g^{N,M}$ are bounded uniformly in N and M . $\square$

4. Examples. In this section we discuss two numerical examples illustrating the convergence behavior of this approximation scheme.

EXAMPLE 4.1. We consider the following model of a cantilevered viscoelastic beam:

$$\frac{\partial^2 u}{\partial t^2}(t, x) + \frac{\partial^2}{\partial x^2} \left[Eu_{xx}(t, x) + \int_{-r}^0 a(\theta) u_{xxt}(t + \theta, x) d\theta \right] = f(t, x) \quad (4.1)$$

where $0 < x < 1$, $t \geq 0$ and

$$\begin{aligned} u(t, 0) &= 0, & \left[Eu_{xx}(t, x) + \int_{-r}^0 a(\theta) u_{xxt}(t + \theta, x) d\theta \right] \Big|_{x=1} &= 0, \\ u_x(t, 0) &= 0, & \frac{\partial}{\partial x} \left[Eu_{xx}(t, x) + \int_{-r}^0 a(\theta) u_{xxt}(t + \theta, x) d\theta \right] \Big|_{x=1} &= 0. \end{aligned}$$

In this equation $u(t, x)$ represents transverse displacement at time t and position x along the beam of length 1. Also, $a(\theta) = \int_{-r}^0 g(\xi) d\xi$, where $g(\xi) = \alpha e^{\beta\xi}/(-\xi)^{3/2}$. We refer the reader to [11] for a discussion of a similar example for a weakly singular kernel. The system (4.1) can be reformulated as an abstract Cauchy problem (1.1) and approximated by a finite-dimensional system (2.4). In this example we compare the eigenvalues of Eq. (1.1) (i.e., the eigenvalues of $\mathcal{A}$) with eigenvalues of Eq. (2.4) (i.e., the eigenvalues of $\mathcal{A}^{N,M}$). Note that for the case $a(\theta) \equiv 0$, Eq. (4.1) is an elastic Euler-Bernoulli beam model and the eigenvalues of $\mathcal{A}$ lie on the imaginary axis. The introduction of damping causes these eigenvalues to “bend back” off the imaginary axis and into the left half-plane. However, the complete distribution of the eigenvalues is not entirely clear. For weakly singular kernels it is known that the eigenvalues lie in a logarithmic sector [12], and we expect this behavior for strongly singular kernels also. In any case, one way to observe the convergence behavior of the approximation scheme is to first compute several eigenvalues of $\mathcal{A}$ by numerically finding roots of the characteristic equation $\Delta(\lambda) = 0$ (recall Eq. (3.10)). Then compute the eigenvalues of $\mathcal{A}^{N,M}$ and observe the behavior as N and M increase.

In this example we set $E = 40$, $r = 1$, $\alpha = 0.1$, and $\beta = 5$ and compute the following eigenvalues of $\mathcal{A}$: $\lambda_1 = -0.29504 + 22.3942i$, $\lambda_2 = -5.0854 + 143.241i$. Next we compute the eigenvalues of $\mathcal{A}^{N,M}$ for $N = 6$ and several values of M . In Fig. 4.1, we show the distribution of eigenvalues in the left half-plane for the first two modes for $M = 64$. Observe that there is one eigenvalue of $\mathcal{A}^{N,M}$ near each of the “true” eigenvalues of $\mathcal{A}$. For large values of M it becomes computationally impractical to compute all of the eigenvalues of $\mathcal{A}^{N,M}$. Instead we track only the eigenvalues $\lambda_1^{N,M}$ and $\lambda_2^{N,M}$ of $\mathcal{A}^{N,M}$ that are near the “true” eigenvalues of $\mathcal{A}$. This is done by computing roots of the characteristic equation $\Delta^M(\lambda) = 0$ (recall (3.14)–(3.15)). In Table 1 we list these eigenvalues for various values of M . We remark that, while it is clear that convergence is attained, large values of M are required. This may be an unreasonable computational burden in some applications (i.e., Potter’s method in optimal control theory). For this reason we are currently investigating the possibility of a nonuniform mesh for strongly singular kernels analogous to that used in [6] and [11] for weakly singular kernels.

FIG. 4.1. Eigenvalues for $N = 5$, $M = 64$.

TABLE 1

M	$\lambda_1^{N,M}$	$\lambda_2^{N,M}$
50	$-0.394 + 22.479i$	$-3.171 + 145.93i$
200	$-0.384 + 22.417i$	$-6.256 + 145.31i$
800	$-0.351 + 22.393i$	$-6.645 + 144.05i$
4000	$-0.322 + 22.386i$	$-6.064 + 143.42i$
16000	$-0.309 + 22.384i$	$-5.626 + 143.28i$

EXAMPLE 4.2. In this example, we consider the following scalar delay equation:

$$\begin{aligned}
 \ddot{u}(t) + Eu(t) - \int_{-1}^0 \frac{\alpha e^{\beta s}}{(-s)^{3/2}} u(t+s) ds &= f(t); \\
 u(0) &= u_0, \\
 \dot{u}(0) &= u_1, \\
 u(s) &= h(s) \quad \text{for } -r \leq s \leq 0.
 \end{aligned} \tag{4.2}$$

Here E is a positive constant and $u(t) \in \mathbb{R}$. All of the above theory applies to Eq. (4.2) except that the Hilbert spaces H and V are each replaced by the Hilbert space $\mathbb{R}$ (i.e., there is no spatial variable). We observe that if $\lambda = a + bi$ is a solution of the characteristic equation

$$\lambda^2 + E + \int_{-r}^0 \frac{\alpha e^{\beta \theta}}{(-\theta)^{3/2}} (e^{\lambda \theta} - 1) d\theta = 0, \tag{4.3}$$

then $u(t) = e^{at} \sin bt$ is a solution of Eq. (4.2) with $u_0 = 0$, $u_1 = b$, and $h(s) = e^{as} \sin bs$. In this example we applied our approximation scheme to Eq. (4.2) with this initial data to generate an approximate solution $u^M(t)$, which is then compared

FIG. 4.2

to the true solution $u(t)$. We used the values $E = 96.72238$, $\alpha = 1.56867$, and $\beta = 5$. Then $\lambda = -0.5 + 10i$ is a solution of Eq. (4.3), so that $u(t) = e^{-0.5t} \sin 10t$ is a true solution of Eq. (4.2). In Fig. 4.2 we graph $u(t)$ and $u^M(t)$ on $0 \leq t \leq 2.5$ for various values of M . Once again, we can observe convergence as $M \rightarrow \infty$.

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