PURE AZIMUTHAL SHEAR OF COMPRESSIBLE NONLINEARLY ELASTIC CIRCULAR TUBES

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Abstract. The azimuthal (or circular) shear problem for a hollow circular cylinder, composed of homogeneous isotropic *compressible* nonlinearly elastic material, is described. The inner surface of the tube is bonded to a rigid cylinder. The deformation may be achieved either by applying a uniformly distributed azimuthal shear traction on the outer surface together with zero radial traction (Problem 1) *or* by subjecting the outer surface to a prescribed angular displacement, with zero radial displacement (Problem 2). For an arbitrary compressible material, the cylinder will undergo both a radial and angular deformation. These axisymmetric fields are governed by a coupled pair of nonlinear ordinary differential equations, one of which is second-order and the other first-order. The class of materials for which *pure azimuthal shear* (i.e., a deformation with zero radial displacement) is possible is described. The corresponding angular displacement and stresses are determined explicitly. Specific material models are used to illustrate the results.

1. Introduction. In two recent papers [1, 2], the authors have examined a special issue for certain large deformations of homogeneous isotropic *compressible* nonlinearly elastic solids. For such materials, in contrast to the incompressible case, volume changes usually occur. The resulting boundary-value problems are generally much more difficult to solve than their counterparts for incompressible materials. The question addressed in [1, 2] is to determine the class of *compressible* materials for which *isochoric* (volume preserving) deformations *can* occur. Such an identification is of value in the design of experiments and in the development of constitutive models for nonlinear elastic solids.

In [1] this issue was examined for the problem of *torsion* of a circular cylinder subjected to prescribed twisting moments at its ends. The torsion problem for a circular cylinder composed of an *incompressible* material was one of the classic problems solved by Rivlin. The resulting isochoric nonhomogeneous deformation describes *pure torsion*, that is, there is no radial displacement and the cross section of the cylinder remains circular. This is a universal (or controllable) deformation, sustainable, in the absence of body force, in *all* homogeneous isotropic incompressible

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materials. For *compressible* materials, the deformation cannot be universal by virtue of Ericksen's theorem [3]. The class of *compressible* materials for which pure torsion is possible was described in [1], building on earlier work in [4]. It was shown in [1] that a wide variety of compressible materials belong to this class, including the special Blatz-Ko [5] and Hadamard materials [6]. The analogous issue was investigated in [2] for axial shear deformations of a hollow circular cylinder. The inner surface of the tube is bonded to a rigid cylinder while the outer surface is subjected to a uniformly distributed axial shear traction, and the radial traction is zero. For an arbitrary compressible material, the cylinder will undergo both a radial and axial deformation. The class of materials for which *axisymmetric antiplane shear* (i.e., a deformation with zero radial displacement) is possible is described in [2]. Again, a wide variety of such admissible materials is obtained, including special Hadamard materials and a subclass of the generalized Blatz-Ko materials [5]. As noted in [1, 2], the results obtained in these papers complement one another in the sense that many of the materials for which pure torsion is possible do not sustain axisymmetric antiplane shear and vice versa.

The purpose of the present paper is to examine the issues described above for a third deformation, namely, that of *azimuthal* (or circular) *shear* of a hollow circular cylinder. The inner surface of the tube is bonded to a rigid cylinder. The deformation may be achieved either by applying a uniformly distributed azimuthal shear traction on the outer surface together with zero radial traction (Problem 1) or by subjecting the outer surface to a prescribed uniform angular displacement, with zero radial displacement (Problem 2). For *incompressible* materials, the azimuthal shear problem was another of the classic problems solved by Rivlin [7] (see also [8–10], which include treatments of the combined axial and azimuthal shear problems). The resulting isochoric nonhomogeneous deformation describes *pure azimuthal shear* so that there is no radial displacement. Denoting the radial components of a cylindrical polar coordinate system in the undeformed and deformed configurations by R and r , respectively, the solutions described in [7–10] are such that

$$r = R. \quad (1.1)$$

The deformed configuration is again a hollow circular cylinder.

For *compressible* materials, deformations of the form (1.1) will *not* be possible for *all* materials. In the next section, we provide a formulation of the *azimuthal shear* problem described above. For an arbitrary compressible material, the cylinder will undergo both a radial *and* an angular deformation. The radial deformation $r(R)$ and the angular displacement $g(R)$ are shown to be governed by a coupled pair of nonlinear ordinary differential equations, one of which is second-order and the other first-order (see Eqs. (2.26), (2.24)). In Sec. 3 we establish necessary conditions on the strain-energy density W for *pure azimuthal shear* to be possible by seeking solutions of the governing equations for which $r = R$ so that Eq. (1.1) holds. We obtain two conditions (see Eqs. (3.3), (3.4)) on W in the form of a second-order and first-order nonlinear ordinary differential equation for $g(R)$ whose solutions must be compatible. In Sec. 4 a subclass of materials, whose dependence on the first and second invariants of the Cauchy-Green deformation tensor is *linear*, is considered.

For this subclass, the conditions reduce to two *algebraic* restrictions on constitutive functions appearing in the representation for W . For Problem 1, it is shown that these conditions are *incompatible*, so that, subject to the traction boundary conditions of Problem 1, pure azimuthal shear is *never possible* for this subclass of materials. However, under the displacement boundary conditions of Problem 2, a wide variety of admissible strain-energies W can be found. These include a subclass of the generalized Blatz-Ko materials though *not* the Hadamard materials. The angular displacement and corresponding stresses are obtained explicitly. In Sec. 5 a wider range of materials is examined.

It will be seen in the sequel that the analysis contained in this paper parallels quite closely that described in [2]. However, unlike [2], where either displacement or traction boundary conditions on the outer surface of the tube give rise to admissible strain-energies, such is the case here only for Problem 2, where displacement boundary conditions are prescribed on the outer surface. The results obtained here will also be seen to complement those of [1, 2]. Thus, for example, the class of Hadamard materials which do *not* depend on the second invariant (see Eq. (4.15)) cannot sustain pure torsion or pure azimuthal shear but do sustain axisymmetric antiplane shear. Several examples of such materials are given in [2]. Another interesting connection between [1], [2], and the present paper arises in consideration of the generalized Blatz-Ko material (see Eq. (4.17)). The parameter f appearing in Eq. (4.17) is related to the volume fraction of voids in the foam-rubber material being modeled [5]. The admissible materials in [1], in the present paper, and in [2] are such that $f = 0, \frac{3}{4}, 1$ respectively. An entirely analogous result holds for the polynomial material of Levinson and Burgess (see Eq. (4.18)).

2. Formulation of the azimuthal shear problem. We are concerned with finite *azimuthal shear* deformation of an isotropic *compressible* nonlinearly elastic hollow circular cylinder. Thus the deformation, which takes the point with cylindrical polar coordinates (R, Θ, Z) in the undeformed region to the point (r, θ, z) in the deformed region has the form

$$r = r(R), \quad \theta = \Theta + g(R), \quad z = Z, \quad (2.1)$$

with $\dot{r} \equiv dr/dR > 0$, where g is the angular displacement. Since the deformation is assumed to be axisymmetric, we have $g = g(R)$. As regards boundary conditions, we suppose that the inner surface $R = A$ is bonded to a rigid cylinder so that

$$r(A) = A, \quad g(A) = 0. \quad (2.2)$$

The deformation may be achieved either by applying a uniformly distributed azimuthal shear traction to the outer surface $R = B$ of the cylinder and assuming that the radial traction is zero there or by prescribing $g(R)$ on the outer surface and assuming zero radial displacement there. The cylinder is assumed sufficiently long so that end effects are negligible.

For a cylinder composed of an *arbitrary incompressible* material, it was shown by Rivlin [7] (see also [8–10]) that *pure azimuthal shear* is possible so that

$$r = R \quad (2.3)$$

in Eq. (2.1). Thus the deformed configuration is again a hollow circular cylinder, which undergoes no volume change. For a cylinder composed of a *compressible* material, there exists *no* global solution describing pure azimuthal shear deformations. In general, the cylinder will undergo both a radial deformation $r(R)$ and an angular deformation with displacement field $g(R)$. These quantities are governed by two coupled nonlinear ordinary differential equations, one of which is second-order and the other first-order (see Eqs. (2.26), (2.24)). These equations also depend on the form of the strain-energy density function, W . Early work on azimuthal shear for *compressible* materials is that of Adkins [11]. A complex variable treatment of these plane strain problems is described in [12] (see also [10, pp. 278–279]) and solutions obtained for harmonic materials. Solutions for the special Blatz-Ko material are briefly described in [13]. A recent paper [14] discusses solutions for other classes of compressible materials, including some anisotropic materials.

The strain-energy density per unit undeformed volume for isotropic elastic compressible materials is given by

$$W = W(I_1, I_2, I_3) \quad (2.4)$$

where I_1, I_2, I_3 are the principal invariants of the left Cauchy-Green deformation tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$, where $\mathbf{F}$ is the deformation gradient tensor. We have

$$\begin{aligned} I_1 &= \text{tr } \mathbf{B} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \\ I_2 &= \text{tr } \mathbf{B}^* = \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2, \\ I_3 &= \det \mathbf{B} = \lambda_1^2 \lambda_2^2 \lambda_3^2, \end{aligned} \quad (2.5)$$

where $\mathbf{B}^* = (\det \mathbf{B})\mathbf{B}^{-1}$ is the adjoint of $\mathbf{B}$ and λ_i ($i = 1, 2, 3$) are the principal stretches. The corresponding response equation for the Cauchy stress tensor $\mathbf{T}$ is (see, e.g., [15])

$$\mathbf{T} = \beta_0 \mathbf{1} + \beta_1 \mathbf{B} + \beta_{-1} \mathbf{B}^{-1}, \quad (2.6)$$

where $\mathbf{1}$ is the unit tensor. Here, the elastic response functions $\beta_s = \beta_s(I_1, I_2, I_3)$ ($s = 0, 1, -1$) are given by

$$\begin{aligned} \beta_0 &= 2I_3^{-1/2} [I_2 W_2 + I_3 W_3], \\ \beta_1 &= 2I_3^{-1/2} W_1, \\ \beta_{-1} &= -2I_3^{1/2} W_2, \end{aligned} \quad (2.7)$$

where $W_i = \partial W / \partial I_i$ ($i = 1, 2, 3$). We shall assume that the strain-energy W is sufficiently smooth and that W and the stress $\mathbf{T}$ vanish in the undeformed state where $I_1 = I_2 = 3$, $I_3 = 1$, and so

$$W(3, 3, 1) = 0, \quad W_1(3, 3, 1) + 2W_2(3, 3, 1) + W_3(3, 3, 1) = 0. \quad (2.8)$$

Corresponding to the deformation field (2.1), we have

$$\mathbf{F} = \begin{pmatrix} \dot{r} & 0 & 0 \\ r\dot{g} & r/R & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (2.9)$$

$$\mathbf{B} = \begin{pmatrix} \dot{r}^2 & r\dot{r}\dot{g} & 0 \\ r\dot{r}\dot{g} & r^2/R^2 + r^2\dot{g}^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (2.10)$$

$$\mathbf{B}^{-1} = \begin{pmatrix} (r^2\dot{g}^2 + 1)/\dot{r}^2 & -R^2\dot{g}/r\dot{r} & 0 \\ -R^2\dot{g}/r\dot{r} & R^2/r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (2.11)$$

$$I_1 = 1 + \frac{r^2}{R^2} + \dot{r}^2 + r^2\dot{g}^2, \quad (2.12)$$

$$I_2 = \frac{r^2}{R^2} + \dot{r}^2 + \dot{r}^2 \frac{r^2}{R^2} + \frac{r^4\dot{g}^2}{R^2}, \quad (2.13)$$

$$I_3 = \dot{r}^2 \frac{r^2}{R^2}, \quad (2.14)$$

where $\dot{g} \equiv dg/dR$. Substitution from Eqs. (2.10), (2.11) in Eq. (2.6) yields the physical components of the Cauchy stress $\mathbf{T}$ as

$$T_{rr} = \beta_0 + \beta_1 \dot{r}^2 + \beta_{-1} \frac{(r^2\dot{g}^2 + 1)}{\dot{r}^2}, \quad (2.15)$$

$$T_{\theta\theta} = \beta_0 + \beta_1 \left(\frac{r^2}{R^2} + r^2\dot{g}^2 \right) + \beta_{-1} \frac{R^2}{r^2}, \quad (2.16)$$

$$T_{zz} = \beta_0 + \beta_1 + \beta_{-1}, \quad (2.17)$$

$$T_{r\theta} = \beta_1 r\dot{r}\dot{g} - \beta_{-1} \frac{R^2\dot{g}}{r\dot{r}}, \quad (2.18)$$

$$T_{rz} = 0, \quad T_{\theta z} = 0, \quad (2.19)$$

where the β_s ($s = 0, 1, -1$) given by Eqs. (2.7) are evaluated at the values of the invariants given by Eqs. (2.12)–(2.14). Note that, since $r = r(R)$, it is convenient to consider $\mathbf{T} = \mathbf{T}(R)$ rather than the more conventional $\mathbf{T} = \mathbf{T}(r)$.

The equilibrium equations, in the absence of body forces, are

$$\text{div } \mathbf{T} = \mathbf{0}, \quad (2.20)$$

which, for this deformation, reduce to the following two equations:

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) = 0, \quad (2.21)$$

$$\frac{\partial T_{r\theta}}{\partial r} + 2\frac{T_{r\theta}}{r} = 0. \quad (2.22)$$

We observe that Eq. (2.22) can be written in the form

$$\frac{d}{dr}(r^2 T_{r\theta}) = 0, \quad (2.23)$$

so that, on integrating and using Eqs. (2.18), (2.14), (2.7), we arrive at the first-order nonlinear ordinary differential equation

$$2\dot{g}R \left[r^2 W_1 + \frac{R^2}{\dot{r}^2} W_2 \right] = K, \quad (2.24)$$

where K is a constant, and W_1, W_2 are evaluated at the values of the invariants given by Eqs. (2.12)–(2.14).

Using the chain-rule, we write Eq. (2.21) as

$$\frac{\partial T_{rr}}{\partial R} + \frac{\dot{r}}{r}(T_{rr} - T_{\theta\theta}) = 0. \quad (2.25)$$

On substitution from Eqs. (2.15), (2.16), (2.7) into Eq. (2.25), we obtain the second-order nonlinear ordinary differential equation

$$\begin{aligned} \frac{d}{dR} \left[\frac{R\dot{r}}{r} W_1 + \left(\frac{R\dot{r}}{r} + \frac{r\dot{r}}{R} + \frac{r}{\dot{r}R} + \frac{Rr\dot{g}^2}{\dot{r}} - \frac{Rr\dot{g}^2}{\dot{r}^3} - \frac{R}{r\dot{r}^3} \right) W_2 + \frac{r\dot{r}}{R} W_3 \right] \\ + W_1 \left(\frac{R\dot{r}^2}{r^2} - \frac{1}{R} - R\dot{g}^2 \right) + W_2 \left(\frac{R^3}{r^4} - \frac{R}{r^2\dot{r}^2} - \frac{R^3\dot{g}^2}{r^2\dot{r}^2} \right) = 0, \end{aligned} \quad (2.26)$$

where again W_i ($i = 1, 2, 3$) are evaluated at the values (2.12)–(2.14) for the invariants, and the ordinary derivative symbol is used since $r = r(R)$, $g = g(R)$. Equations (2.24), (2.26) are a coupled pair of (highly) nonlinear ordinary differential equations for the unknown functions $r(R)$ and $g(R)$. The task of obtaining analytical solutions to the system (2.24), (2.26), even for simple forms of the strain-energy, is formidable. The recent papers [16, 17] describe a numerical approach to solving an equivalent pair of equations, for the case of particular strain-energy densities W .

To complete the formulation of the boundary-value problem, we consider first the former of the previously described boundary conditions at $R = B$. In the sequel, we shall refer to this as Problem 1. Therefore, in addition to Eq. (2.2), we also require

$$T_{rr}(B) = 0, \quad T_{r\theta}(B) = T_0, \quad (2.27)$$

where $T_0 > 0$ is a given constant. The constant K appearing in Eq. (2.24) can now be expressed in terms of T_0 . Thus, on evaluating the left-hand side of Eq. (2.24) at $R = B$ and using Eqs. (2.18), (2.7), (2.27)₂, one finds that

$$K = r(B)^2 T_0. \quad (2.28)$$

When K and T_0 are related as in Eq. (2.28), the boundary condition (2.27)₂ is *automatically* satisfied. Thus the boundary-value problem that arises in this case is to solve the system (2.24), (2.26) for $g(R), r(R)$ on $A < R < B$ subject to the conditions (2.2), (2.27)₁.

Turning now to the latter of the boundary conditions discussed briefly after Eq. (2.2), we require, in addition to Eq. (2.2),

$$r(B) = B, \quad g(B) = g_0, \quad (2.29)$$

where $g_0 > 0$ is a given constant. In this case, the constant K appearing in Eq. (2.24) will be related to g_0 . The boundary-value problem that now arises is to solve the system (2.24), (2.26) for $g(R), r(R)$ on $A < R < B$ subject to Eqs. (2.2), (2.29). This will be called Problem 2.

3. Pure azimuthal shear: a necessary condition. We now proceed to determine a necessary condition on W for pure azimuthal shear to be possible. On setting $r = R$ in Eqs. (2.12)–(2.14), we find that, for pure azimuthal shear deformations

$$I_1 = I_2 = 3 + R^2 \dot{g}^2, \quad I_3 = 1, \quad (3.1)$$

and so the deformation is isochoric. Also, on setting $r = R$ in Eq. (2.26) we obtain

$$\frac{d}{dR}[W_1 + 2W_2 + W_3] - R\dot{g}^2(W_1 + W_2) = 0. \quad (3.2)$$

On employing the chain-rule and noting that $\dot{g} \neq 0$ for a nontrivial solution, Eq. (3.2) can be written as

$$2\left(\ddot{g} + \frac{\dot{g}}{R}\right)(\widehat{W}_{11} + 2\widehat{W}_{22} + 3\widehat{W}_{12} + \widehat{W}_{13} + \widehat{W}_{23}) - \frac{\dot{g}}{R}(\widehat{W}_1 + \widehat{W}_2) = 0. \quad (3.3)$$

Furthermore, on setting $r = R$ in Eq. (2.24), and using Eq. (2.28) with $r = R$, we find that

$$\dot{g}(\widehat{W}_1 + \widehat{W}_2) = \frac{K}{2R^3}. \quad (3.4)$$

It will be assumed henceforth that $\dot{g} \neq \text{const}$ and so $\dot{g} \neq 0$. (Clearly $\dot{g} = \text{constant}$ would not be consistent with Eq. (3.4).) In Eqs. (3.3), (3.4) we have introduced the notation $W_{ij} = \partial^2 W / \partial I_i \partial I_j$, and the superposed hat is used to indicate that the corresponding quantities are evaluated at the values (3.1) for the invariants I_i ($i = 1, 2, 3$). The conditions (3.3), (3.4) are *necessary conditions on W for pure azimuthal shear to be possible*. Condition (3.3) is a second-order nonlinear ordinary differential equation for $g(R)$ on $A < R < B$, and its solution must be compatible with that of the first-order nonlinear ordinary differential equation (3.4). The implications of Eqs. (3.3), (3.4) will be our concern in the remainder of this paper.

When $r = R$, the corresponding stresses follow from Eqs. (2.15)–(2.19) as

$$T_{rr} = \hat{\beta}_0 + \hat{\beta}_1 + \hat{\beta}_{-1}(R^2 \dot{g}^2 + 1), \quad (3.5)$$

$$T_{\theta\theta} = \hat{\beta}_0 + \hat{\beta}_1(R^2 \dot{g}^2 + 1) + \hat{\beta}_{-1}, \quad (3.6)$$

$$T_{zz} = \hat{\beta}_0 + \hat{\beta}_1 + \hat{\beta}_{-1}, \quad (3.7)$$

$$T_{r\theta} = R\dot{g}(\hat{\beta}_1 - \hat{\beta}_{-1}), \quad (3.8)$$

$$T_{rz} = 0, \quad T_{\theta z} = 0, \quad (3.9)$$

where $\hat{\beta}_s = \hat{\beta}_s(3 + R^2 \dot{g}^2, 3 + R^2 \dot{g}^2, 1)$ are given by Eqs. (2.7).

Observe that, when $r = R$, the first of the boundary conditions (2.2) at the inner radius $R = A$ is automatically satisfied and so the remaining boundary condition is

$$g(A) = 0. \quad (3.10)$$

Turning to the outer boundary $R = B$, it is clear that the first of the boundary conditions (2.29) is automatically satisfied when $r = R$. Thus for Problem 2 the only remaining boundary conditions are (3.10), $(2.29)_2$, namely,

$$g(A) = 0, \quad g(B) = g_0. \quad (3.11)$$

On the other hand, for Problem 1, the only remaining boundary conditions are Eqs. (2.27)₁ and (3.10).

In what follows, it will be convenient for us to consider the implications of differentiating both sides of Eq. (3.4) with respect to R . Performing this operation and using the chain-rule, we obtain

$$(\widehat{W}_1 + \widehat{W}_2)\ddot{g} = -\frac{3K}{2R^4} - 2R\dot{g}^2(\dot{g} + R\ddot{g})(\widehat{W}_{11} + 2\widehat{W}_{12} + \widehat{W}_{22}). \quad (3.12)$$

Furthermore, we will write Eq. (3.4) as

$$\dot{g} = \frac{K}{2R^3(\widehat{W}_1 + \widehat{W}_2)}, \quad (3.13)$$

provided that

$$\widehat{W}_1 + \widehat{W}_2 \neq 0 \quad \text{on } A < R < B. \quad (3.14)$$

Since W is assumed sufficiently smooth, it follows from Eq. (3.14) that $\widehat{W}_1 + \widehat{W}_2$ does not change sign on $A < R < B$ and so, without loss of generality, we take

$$\widehat{W}_1 + \widehat{W}_2 > 0 \quad \text{on } A < R < B. \quad (3.15)$$

Note that Eq. (3.15) would also follow as a consequence of the Baker-Ericksen inequality.

To conclude this section, we write the stresses (3.5)–(3.9) in a simplified form. On substitution from Eq. (2.7) into Eqs. (3.5)–(3.8), and then using Eq. (3.13) in Eq. (3.8), we find that

$$T_{rr} = 2[\widehat{W}_1 + 2\widehat{W}_2 + \widehat{W}_3], \quad (3.16)$$

$$T_{\theta\theta} = 2[(1 + R^2\dot{g}^2)\widehat{W}_1 + (2 + R^2\dot{g}^2)\widehat{W}_2 + \widehat{W}_3], \quad (3.17)$$

$$T_{zz} = 2[\widehat{W}_1 + (2 + R^2\dot{g}^2)\widehat{W}_2 + \widehat{W}_3], \quad (3.18)$$

$$T_{r\theta} = K/R^2, \quad T_{rz} = 0, \quad T_{\theta z} = 0. \quad (3.19)$$

4. A subclass of materials. We begin our analysis of Eqs. (3.3), (3.4) by considering materials for which the dependence of W on I_1 and I_2 is linear so that

$$\widehat{W}_{11} = 0, \quad \widehat{W}_{22} = 0, \quad \widehat{W}_{12} = 0. \quad (4.1)$$

On using Eq. (4.1) in Eq. (3.12) we obtain

$$\ddot{g} = -\frac{3K}{2R^4(\widehat{W}_1 + \widehat{W}_2)}, \quad (4.2)$$

again provided that Eq. (3.15) holds. Employing Eqs. (4.1), (3.13), and (4.2) in Eq. (3.3) we find that

$$4(\widehat{W}_{13} + \widehat{W}_{23}) + \widehat{W}_1 + \widehat{W}_2 = 0. \quad (4.3)$$

Equation (4.3) is a nonlinear ordinary differential equation for $g(R)$ on $A < R < B$, and its solution must be compatible with that of Eq. (3.13), subject to Eq. (3.15).

Since W is linear in I_1 and I_2 , we may write W as

$$W = \frac{\mu}{2}[H_1(I_3)(I_1 - 3) + H_2(I_3)(I_2 - 3) + H_3(I_3)], \quad (4.4)$$

where $\mu > 0$ is constant. Thus

$$W_1 = \frac{\mu}{2} H_1(I_3), \quad W_2 = \frac{\mu}{2} H_2(I_3), \quad (4.5)$$

$$W_3 = \frac{\mu}{2} [H'_1(I_3)(I_1 - 3) + H'_2(I_3)(I_2 - 3) + H'_3(I_3)].$$

The prime notation denotes differentiation with respect to argument. To satisfy the normalization conditions (2.8), we see that the functions $H_i(I_3)$ ($i = 1, 2, 3$) must be such that

$$H_3(1) = 0, \quad H_1(1) + 2H_2(1) + H'_3(1) = 0. \quad (4.6)$$

On substitution from Eqs. (4.4), (4.5) in Eqs. (4.3) and (3.15), and evaluating at the values (3.1), we find that

$$4[H'_1(1) + H'_2(1)] + H_1(1) + H_2(1) = 0 \quad (4.7)$$

and

$$H_1(1) + H_2(1) > 0. \quad (4.8)$$

Thus, Eqs. (4.7), (4.8) are *necessary conditions* on $H_\alpha(I_3)$ ($\alpha = 1, 2$) for pure azimuthal shear to be possible for the class of materials (4.4). Observe that we now have two *algebraic conditions* that must be satisfied instead of the differential equations (3.3), (3.4).

Now Eq. (3.13) and the boundary condition $(3.11)_1$ determine the angular displacement $g(R)$ for any W of the form (4.4) as

$$g(R) = \frac{K(A^{-2} - R^{-2})}{2\mu[H_1(1) + H_2(1)]}. \quad (4.9)$$

We consider first Problem 1. On using Eqs. (3.1), (3.16), (4.5), and $(4.6)_2$, one finds that Eq. $(2.27)_1$ is satisfied if

$$H'_1(1) + H'_2(1) = 0. \quad (4.10)$$

On using Eq. (4.10) in Eq. (4.7) we obtain

$$H_1(1) + H_2(1) = 0. \quad (4.11)$$

Equation (4.11) is clearly *incompatible* with Eq. (4.8). Thus, for Problem 1 we conclude that pure azimuthal shear is not possible for any W of the form (4.4).

Turning now to Problem 2, we first evaluate the constant K appearing in Eq. (4.9) in terms of the prescribed angular displacement g_0 at the outer boundary. Using Eqs. (4.9) and $(3.11)_2$ one finds immediately that

$$K = \frac{2\mu g_0 [H_1(1) + H_2(1)]}{A^{-2} - B^{-2}} \quad (4.12)$$

and so Eq. (4.9) simplifies to read

$$g(R) = g_0 \frac{(A^{-2} - R^{-2})}{A^{-2} - B^{-2}}. \quad (4.13)$$

Since $g_0 > 0$, we have $g(R) > 0$ as expected on physical grounds. Thus, for Problem 2, provided that Eqs. (4.7), (4.8) hold, any W of the form (4.4) will allow

pure azimuthal shear deformations, with the angular displacement $g(R)$ given by Eq. (4.13).

The corresponding stresses may then be found from Eqs. (3.16)–(3.19). On using Eqs. (4.10)–(4.12), we obtain

$$\begin{aligned} T_{rr} &= \frac{-\mu[H_1(1) + H_2(1)]g_0^2 R^{-4}}{(A^{-2} - B^{-2})^2}, & T_{rz} &= 0, \\ T_{\theta\theta} &= \frac{3\mu[H_1(1) + H_2(1)]g_0^2 R^{-4}}{(A^{-2} - B^{-2})^2}, & T_{\theta z} &= 0, \\ T_{zz} &= \frac{\mu[3H_2(1) - H_1(1)]g_0^2 R^{-4}}{(A^{-2} - B^{-2})^2}, & T_{r\theta} &= \frac{2\mu[H_1(1) + H_2(1)]g_0 R^{-2}}{A^{-2} - B^{-2}}. \end{aligned} \quad (4.14)$$

We observe that the normal stresses, depending on the prescribed angular displacement g_0 in a quadratic manner, represent a *normal stress effect* characteristic of finite elasticity theory. By virtue of Eq. (4.8), the hoop stress is tensile while the radial stress is compressive. Henceforth in this section we confine attention to Problem 2.

Before considering examples of particular materials of the form (4.4), we observe that, whenever $H_1(I_3)$ and $H_2(I_3)$ are *both constant*, so that Eq. (4.4) describes a Hadamard material, the conditions (4.7), (4.8) will not be consistent. Thus pure azimuthal shear is *never possible* for a Hadamard material. In particular, pure azimuthal shear is *never possible* for the special Hadamard material

$$W = \frac{\mu}{2}[k(I_1 - 3) + H(I_3)] \quad (k > 0, \text{ a constant}), \quad (4.15)$$

where

$$H(1) = 0, \quad H'(1) + k = 0, \quad (4.16)$$

so that the normalization conditions (4.6) hold. A completely analogous result was obtained in [1] for pure torsion. However, for *axisymmetric anti-plane shear deformations*, the exact opposite is true [2]. A variety of constitutive laws of the form (4.15) have been proposed in the literature; some examples are provided in [2].

We turn next to two other material models of the form (4.4) which have also received much attention in the literature. These are the generalized Blatz-Ko material [5],

$$\begin{aligned} W &= \frac{\mu}{2}f \left\{ I_1 - 1 - \frac{1}{\nu} + \frac{(1-2\nu)}{\nu} I_3^{-\nu/(1-2\nu)} \right\} \\ &\quad + \frac{\mu}{2}(1-f) \left\{ \frac{I_2}{I_3} - 1 - \frac{1}{\nu} + \frac{(1-2\nu)}{\nu} I_3^{\nu/(1-2\nu)} \right\}, \end{aligned} \quad (4.17)$$

and the polynomial material proposed by Levinson and Burgess [18],

$$\begin{aligned} W &= \frac{\mu}{2} \left[f(I_1 - 3) + (1-f) \left(\frac{I_2}{I_3} - 3 \right) + 2(1-2f)(I_3^{1/2} - 1) \right. \\ &\quad \left. + \left(2f + \frac{4\nu-1}{1-2\nu} \right) (I_3^{1/2} - 1)^2 \right], \end{aligned} \quad (4.18)$$

respectively. In Eqs. (4.17), (4.18), the constants μ, ν are the shear modulus and Poisson's ratio for infinitesimal deformations so that $\mu > 0$, $0 < \nu < \frac{1}{2}$. In Eq.

(4.17), f , where $0 \leq f \leq 1$, is a parameter related to the volume fraction of voids in the foam-rubber material being modeled. As shown in [1], the material (4.17) can be written in the form (4.4) if one takes

$$H_1(I_3) = f, \quad H_2(I_3) = (1 - f)I_3^{-1} \quad (4.19)$$

$$H_3(I_3) = f \left[-\frac{1}{\nu} + 2 + \frac{(1 - 2\nu)}{\nu} I_3^{-\nu/(1-2\nu)} \right] + (1 - f) \left[3I_3^{-1} - 1 - \frac{1}{\nu} + \frac{(1 - 2\nu)}{\nu} I_3^{\nu/(1-2\nu)} \right]. \quad (4.20)$$

It is readily verified that the normalization conditions (4.6) are satisfied. The necessary conditions (4.7), (4.8) for pure azimuthal shear are easily shown to be satisfied by Eqs. (4.19) if and only if

$$f = \frac{3}{4}. \quad (4.21)$$

Thus, for the generalized Blatz-Ko material (4.17), pure azimuthal shear is possible provided Eq. (4.21) holds, regardless of the value of ν . It is interesting to recall from [1], [2] respectively, that for Eq. (4.17), *pure torsion* is possible if and only if $f = 0$ while *axisymmetric antiplane shear* is possible if and only if $f = 1$. The material (4.18) can also be written in the form (4.4) with the identical choices for $H_\alpha(I_3)$ ($\alpha = 1, 2$) as in Eqs. (4.19). Thus, for this material, completely analogous results to those just described for Eq. (4.17) are obtained.

For both materials (4.17), (4.18), the angular displacement is given by Eq. (4.13). The corresponding stresses follow from Eqs. (4.14), (4.19) as

$$\begin{aligned} T_{rr} &= \frac{-\mu g_0^2 R^{-4}}{(A^{-2} - B^{-2})^2}, & T_{rz} &= 0, \\ T_{\theta\theta} &= \frac{3\mu g_0^2 R^{-4}}{(A^{-2} - B^{-2})^2}, & T_{\theta z} &= 0, \\ T_{zz} &= 0, & T_{r\theta} &= \frac{2\mu g_0 R^{-2}}{A^{-2} - B^{-2}}, \end{aligned} \quad (4.22)$$

where Eq. (4.21) has also been used to evaluate T_{zz} .

For the materials (4.17), (4.18), the function $H_1(I_3)$ given in Eqs. (4.19) is constant. We consider now a material of the form (4.4) where $H_1(I_3)$ is *not* constant. Thus suppose that

$$W = \frac{\mu}{2} [H_1(I_3)(I_1 - 3) + H_3(I_3)], \quad (4.23)$$

with

$$H_3(1) = 0, \quad H_1(1) + H_3'(1) = 0, \quad (4.24)$$

so that the normalization conditions (4.6) hold. In this case, the conditions (4.7), (4.8) reduce to

$$4H_1'(1) + H_1(1) = 0, \quad (4.25)$$

$$H_1(1) > 0. \quad (4.26)$$

One choice for $H_1(I_3)$ satisfying Eqs. (4.25), (4.26) is

$$H_1(I_3) = kI_3^{-1/4} \quad (k > 0, \text{ a constant}). \quad (4.27)$$

Then Eqs. (4.24) reduce to

$$H_3(1) = 0, \quad H_3'(1) = -k. \quad (4.28)$$

A simple choice for $H_3(I_3)$ satisfying Eqs. (4.28) is

$$H_3(I_3) = -k(I_3 - 1). \quad (4.29)$$

Thus, for an arbitrary constant $k > 0$, the material

$$W = \frac{\mu k}{2} [I_3^{-1/4}(I_1 - 3) - (I_3 - 1)] \quad (4.30)$$

can sustain pure azimuthal shear, with angular displacement (4.13). The corresponding stresses follow from Eqs. (4.14), (4.27) as

$$\begin{aligned} T_{rr} &= \frac{-\mu k g_0^2 R^{-4}}{(A^{-2} - B^{-2})^2}, & T_{rz} &= 0, \\ T_{\theta\theta} &= \frac{3\mu k g_0^2 R^{-4}}{(A^{-2} - B^{-2})^2}, & T_{\theta z} &= 0, \\ T_{zz} &= \frac{-\mu k g_0^2 R^{-4}}{(A^{-2} - B^{-2})^2}, & T_{r\theta} &= \frac{2\mu k g_0 R^{-2}}{A^{-2} - B^{-2}}. \end{aligned} \quad (4.31)$$

An alternative choice for $H_1(I_3)$ satisfying Eqs. (4.25), (4.26) is the exponential form

$$H_1(I_3) = \gamma e^{-(I_3-1)/4}, \quad (4.32)$$

($\gamma > 0$, a constant). One still obtains Eqs. (4.28), (4.29) with k replaced by γ . Thus, for an arbitrary constant $\gamma > 0$, the form

$$W = \frac{\mu}{2} \gamma [e^{-(I_3-1)/4}(I_1 - 3) - (I_3 - 1)] \quad (4.33)$$

is also admissible, with stresses given by Eqs. (4.31) with k replaced by γ . Strain-energies with exponential terms of the form (4.33) have been successfully used in biomechanical modeling (see, e.g., [19, 20] and the references cited therein).

To conclude this section, we show that a generalization of Eq. (4.15) will also give rise to a class of admissible materials. Thus, suppose that, for an arbitrary constant k_1 , we have

$$W = \frac{\mu}{2} [k_1(I_1 - 3) + H_2(I_3)(I_2 - 3) + H_3(I_3)], \quad (4.34)$$

with

$$H_3(1) = 0, \quad k_1 + 2H_2(1) + H_3'(1) = 0, \quad (4.35)$$

so that the normalization conditions (4.6) hold, and $H_2(I_3)$ is *not* constant. The conditions (4.7), (4.8) now become

$$4H_2'(1) + H_2(1) + k_1 = 0 \quad (4.36)$$

and

$$k_1 + H_2(1) > 0. \quad (4.37)$$

One choice for $H_2(I_3)$ satisfying Eqs. (4.36), (4.37) is

$$H_2(I_3) = \gamma_1 e^{-(I_3-1)/4} - \frac{k_1}{4}(I_3 - 1), \quad (4.38)$$

(γ_1 a constant), where

$$k_1 + \gamma_1 > 0. \quad (4.39)$$

Then Eqs. (4.35) reduce to

$$H_3(1) = 0, \quad H_3'(1) = -(k_1 + 2\gamma_1). \quad (4.40)$$

A simple choice for $H_3(I_3)$ satisfying Eqs. (4.40) is

$$H_3(I_3) = -(k_1 + 2\gamma_1)(I_3 - 1). \quad (4.41)$$

Thus, for arbitrary constants k_1, γ_1 such that Eq. (4.39) holds, an admissible choice for W is given by Eq. (4.34) with $H_2(I_3), H_3(I_3)$ given by Eq. (4.38), (4.41) respectively. The angular displacement is given by Eq. (4.13), while the corresponding stresses follow from Eqs. (4.14), (4.34), (4.38) as

$$\begin{aligned} T_{rr} &= \frac{-\mu(k_1 + \gamma_1)g_0^2 R^{-4}}{(A^{-2} - B^{-2})^2}, & T_{rz} &= 0, \\ T_{\theta\theta} &= \frac{3\mu(k_1 + \gamma_1)g_0^2 R^{-4}}{(A^{-2} - B^{-2})^2}, & T_{\theta z} &= 0, \\ T_{zz} &= \frac{\mu(3\gamma_1 - k_1)g_0^2 R^{-4}}{(A^{-2} - B^{-2})^2}, & T_{r\theta} &= \frac{2\mu(k_1 + \gamma_1)g_0 R^{-2}}{A^{-2} - B^{-2}}. \end{aligned} \quad (4.42)$$

5. A more general class of materials. The task of seeking a representation for the most general form of W for which Eqs. (3.3), (3.4) are satisfied is formidable. Here we shall pursue a more modest objective, namely, that of broadening the subclass of materials discussed in Sec. 4.

Simplification of Eq. (3.3) to the more tractable Eq. (4.3), which contains no explicit dependence on derivatives of g , relied upon deducing the result Eq. (4.2) from Eq. (3.12). The assumption (4.1) that W depended linearly on I_1 and I_2 made this possible. However, the result (4.2) can be obtained from Eq. (3.12) by merely requiring that

$$\widehat{W}_{11} + 2\widehat{W}_{12} + \widehat{W}_{22} = 0 \quad (5.1)$$

instead of the more restrictive conditions $\widehat{W}_{\alpha\beta} = 0$ ($\alpha, \beta = 1, 2$). On employing Eqs. (5.1), (4.2), and (3.13) in Eq. (3.3) we find that

$$4(\widehat{W}_{22} + \widehat{W}_{12} + \widehat{W}_{13} + \widehat{W}_{23}) + \widehat{W}_1 + \widehat{W}_2 = 0. \quad (5.2)$$

Derived under the assumption that Eq. (5.1) holds, condition (5.2) is the analog of Eq. (4.3). The restriction (3.14) is still required to hold since this was used in deriving Eqs. (3.13), (4.2). As pointed out in Sec. 3, we may, without loss of generality, replace

Eq. (3.14) by Eq. (3.15). With this motivation, we thus consider the class of materials for which W can be written as

$$W = \frac{\mu}{2}[P(I_1 - I_2, I_3)(I_1 - 3) + Q(I_1 - I_2, I_3)(I_2 - 3) + R(I_1 - I_2, I_3)], \quad (5.3)$$

where $\mu > 0$ and P, Q, R are sufficiently smooth functions of the arguments indicated in Eq. (5.3). It is readily verified that W as given in Eq. (5.3) satisfies

$$W_{11} + 2W_{12} + W_{22} = 0, \quad (5.4)$$

so that, in particular, Eq. (5.1) holds for such W . The normalization conditions (2.8) will be satisfied by Eq. (5.3) if

$$R(0, 1) = 0, \quad (5.5)$$

and

$$P(0, 1) + 2Q(0, 1) - R_1(0, 1) + R_2(0, 1) = 0, \quad (5.6)$$

where, for any arbitrary function $F(I_1 - I_2, I_3)$, the notation

$$F_1(x, y) = \frac{\partial}{\partial x} F(x, y), \quad F_2(x, y) = \frac{\partial}{\partial y} F(x, y) \quad (5.7)$$

is used to denote the derivative of $F(I_1 - I_2, I_3)$ with respect to its arguments.

Our task now is to find conditions on the functions P, Q, R appearing in Eq. (5.3) which will ensure that Eqs. (5.2) and (3.15) are satisfied. On substitution from Eq. (5.3) into Eq. (5.2), one finds that

$$4\{P_2(0, 1) + Q_2(0, 1) - [P_1(0, 1) + Q_1(0, 1)]\} + P(0, 1) + Q(0, 1) = 0. \quad (5.8)$$

On substitution from Eq. (5.3) into Eq. (3.15), one also finds that

$$P(0, 1) + Q(0, 1) > 0. \quad (5.9)$$

Thus a cylindrical tube composed of a material described by a strain-energy function W of the form (5.3), with P, Q, R chosen so that Eqs. (5.5), (5.6) hold, can sustain pure azimuthal shear deformations provided Eqs. (5.8), (5.9) hold.

It remains to determine the angular displacement $g(R)$ and to examine the possibility of satisfying the boundary conditions for Problems 1, 2 respectively. From Eqs. (3.13), (3.10), we obtain

$$g(R) = \frac{K(A^{-2} - R^{-2})}{2\mu[P(0, 1) + Q(0, 1)]} \quad (5.10)$$

for any W of the form (5.3). In Eq. (5.10), K is an arbitrary constant of integration. We consider first Problem 1. On using Eqs. (3.1), (3.16), (5.3), and (5.6) one finds that Eq. (2.27)₁ is satisfied if

$$P_2(0, 1) + Q_2(0, 1) - [P_1(0, 1) + Q_1(0, 1)] = 0. \quad (5.11)$$

On using Eq. (5.11) in Eq. (5.8), the latter condition reduces to

$$P(0, 1) + Q(0, 1) = 0. \quad (5.12)$$

Equation (5.12) is clearly incompatible with Eq. (5.9). Thus, for Problem 1, we conclude that pure azimuthal shear is not possible for any W of the form (5.3). This

result is analogous to that found in Sec. 4 for the more restricted class of materials (4.4).

Turning now to Problem 2, we first evaluate the constant K appearing in Eq. (5.10). Using Eqs. (5.10) and (3.11)₂ one finds immediately that

$$K = \frac{2\mu g_0[P(0, 1) + Q(0, 1)]}{A^{-2} - B^{-2}}, \quad (5.13)$$

and then Eq. (5.10) simplifies to read

$$g(R) = \frac{g_0(A^{-2} - R^{-2})}{A^{-2} - B^{-2}}, \quad (5.14)$$

exactly as in Eq. (4.13). Thus, in Problem 2, provided that Eqs. (5.8), (5.9) hold, any W of the form (5.3) is admissible and the angular displacement is given by Eq. (5.14). The corresponding stresses follow from Eqs. (3.16)–(3.19), (5.3), and (5.14) as

$$\begin{aligned} T_{rr} &= \frac{-\mu[P(0, 1) + Q(0, 1)]g_0^2 R^{-4}}{(A^{-2} - B^{-2})^2}, & T_{rz} &= 0, \\ T_{\theta\theta} &= \frac{3\mu[P(0, 1) + Q(0, 1)]g_0^2 R^{-4}}{(A^{-2} - B^{-2})^2}, & T_{\theta z} &= 0, \\ T_{zz} &= \frac{\mu[3Q(0, 1) - P(0, 1) - 4R_1(0, 1)]g_0^2 R^{-4}}{(A^{-2} - B^{-2})^2} - \frac{16\mu[P_1(0, 1) + Q_1(0, 1)]g_0^4 R^{-8}}{(A^{-2} - B^{-2})^4}, \\ T_{r\theta} &= \frac{2\mu[P(0, 1) + Q(0, 1)]g_0 R^{-2}}{A^{-2} - B^{-2}}. \end{aligned} \quad (5.15)$$

We note that if P, Q, R in Eqs. (5.3) are functions of I_3 only, we recover Eq. (4.4). In this case, $P_1 = Q_1 = R_1 \equiv 0$, and all equations in this section are seen to reduce accordingly to the corresponding equations in Sec. 4.

We shall not attempt to describe the most general class of functions P, Q, R for which Eqs. (5.8), (5.9) are satisfied but rather indicate some of the possibilities. It will be seen that a wide variety of admissible strain-energies are included in the form (5.3). Suppose, for example, that we seek P, Q in the separable form

$$P(I_1 - I_2, I_3) = F(I_1 - I_2)U(I_3), \quad (5.16)$$

$$Q(I_1 - I_2, I_3) = G(I_1 - I_2)V(I_3). \quad (5.17)$$

Then, on substitution into Eqs. (5.8), (5.9), we find that F, G, U, V must satisfy

$$4[F(0)U'(1) + G(0)V'(1) - F'(0)U(1) - G'(0)V(1)] + F(0)U(1) + G(0)V(1) = 0, \quad (5.18)$$

$$F(0)U(1) + G(0)V(1) > 0. \quad (5.19)$$

To reduce Eqs. (5.18), (5.19) to restrictions on U, V, U', V' only, we choose

$$F(I_1 - I_2) = e^{\alpha(I_1 - I_2)}, \quad (5.20)$$

$$G(I_1 - I_2) = e^{-\alpha(I_1 - I_2)}, \quad (5.21)$$

where $\alpha \neq 0$ is an arbitrary constant. With this choice for F and G , we find from Eqs. (5.18), (5.19) that

$$4U'(1) + (1 - 4\alpha)U(1) + 4V'(1) + (1 + 4\alpha)V(1) = 0, \quad (5.22)$$

$$U(1) + V(1) > 0. \quad (5.23)$$

The condition (5.22) can be satisfied by choosing U, V to be such that

$$U'(1) + \frac{1}{4}(1 - 4\alpha)U(1) = \beta \quad (5.24)$$

and

$$V'(1) + \frac{1}{4}(1 + 4\alpha)V(1) = -\beta, \quad (5.25)$$

where β is an arbitrary constant. The choice

$$U(I_3) = \gamma e^{-(1-4\alpha)(I_3-1)/4} + \beta(I_3 - 1), \quad (5.26)$$

$$V(I_3) = \gamma e^{-(1+4\alpha)(I_3-1)/4} - \beta(I_3 - 1), \quad (5.27)$$

where $\gamma > 0$ is an arbitrary constant, is seen to satisfy Eqs. (5.24), (5.25), (5.23). Thus the functional forms for P, Q in Eqs. (5.16), (5.17) are determined on using Eqs. (5.20), (5.21), (5.26), and (5.27).

It remains to find $R(I_1 - I_2, I_3)$ satisfying the normalization conditions (5.5), (5.6). With the choices for P, Q made above, the conditions read

$$R(0, 1) = 0, \quad R_2(0, 1) - R_1(0, 1) = -3\gamma. \quad (5.28)$$

A choice for R satisfying Eq. (5.28) is

$$R(I_1 - I_2, I_3) = \gamma[e^{(I_1-I_2)}e^{-2(I_3-1)} - 1]. \quad (5.29)$$

Combining Eqs. (5.20), (5.21), (5.26), (5.27), and (5.29) in Eq. (5.3), we conclude that an admissible form for W is

$$\begin{aligned} W = \frac{\mu}{2} [& e^{\alpha(I_1-I_2)} \{ \gamma e^{-(1-4\alpha)(I_3-1)/4} + \beta(I_3 - 1) \} (I_1 - 3) \\ & + e^{-\alpha(I_1-I_2)} \{ \gamma e^{-(1+4\alpha)(I_3-1)/4} - \beta(I_3 - 1) \} (I_2 - 3) + \gamma \{ e^{(I_1-I_2)} e^{-2(I_3-1)} - 1 \}], \end{aligned} \quad (5.30)$$

with $\alpha \neq 0, \beta, \gamma > 0$ arbitrary constants. The angular displacement is given by Eq. (5.14), while the stresses (5.15) now read

$$\begin{aligned} T_{rr} &= \frac{-2\gamma\mu g_0^2 R^{-4}}{(A^{-2} - B^{-2})^2}, & T_{rz} &= 0, \\ T_{\theta\theta} &= \frac{6\gamma\mu g_0^2 R^{-4}}{(A^{-2} - B^{-2})^2}, & T_{\theta z} &= 0, \\ T_{zz} &= \frac{-2\gamma\mu g_0^2 R^{-4}}{(A^{-2} - B^{-2})^2}, & T_{r\theta} &= \frac{4\gamma\mu g_0 R^{-2}}{A^{-2} - B^{-2}}. \end{aligned} \quad (5.31)$$

Returning to Eqs. (5.18), (5.19), we examine another possibility where F, G are chosen as polynomials rather than as exponentials. For example, suppose that

$$F(I_1 - I_2) = c(I_1 - I_2)^r, \quad (5.32)$$

where c, r are arbitrary constants with $c \neq 0, r > 1$, so that $F(0) = 0, F'(0) = 0$. Then Eqs. (5.18), (5.19) yield

$$4[G(0)V'(1) - G'(0)V(1)] + G(0)V(1) = 0, \quad (5.33)$$

$$G(0)V(1) > 0. \quad (5.34)$$

On dividing the left-hand side of Eq. (5.33) by the positive quantity $G(0)V(1)$, we write Eq. (5.33) as

$$\left[\frac{V'(1)}{V(1)} - \frac{G'(0)}{G(0)} \right] = -\frac{1}{4}. \quad (5.35)$$

The condition (5.35) may be satisfied by choosing

$$\frac{V'(1)}{V(1)} = -c_1, \quad (5.36)$$

$$\frac{G'(0)}{G(0)} = c_2, \quad (5.37)$$

where the constants c_1, c_2 are such that

$$c_1 + c_2 = \frac{1}{4}. \quad (5.38)$$

Suppose, for example, that

$$c_1 = \frac{1}{4}, \quad c_2 = 0, \quad (5.39)$$

so that

$$\frac{V'(1)}{V(1)} = -\frac{1}{4}, \quad (5.40)$$

$$\frac{G'(0)}{G(0)} = 0. \quad (5.41)$$

A simple choice for $V(I_3), G(I_1 - I_2)$ is then

$$V(I_3) = k_1 I_3^{-1/4}, \quad (5.42)$$

$$G(I_1 - I_2) = k_2 (I_1 - I_2)^s + k_3, \quad (5.43)$$

where $k_1, k_2, k_3, s > 1$ are arbitrary constants satisfying

$$k_1 k_3 > 0 \quad (5.44)$$

so that Eq. (5.34) holds. Thus from Eqs. (5.42), (5.43), (5.32), (5.16), (5.17) we obtain

$$P(I_1 - I_2, I_3) = c(I_1 - I_2)' U(I_3), \quad (5.45)$$

$$Q(I_1 - I_2, I_3) = [k_2 (I_1 - I_2)^s + k_3] k_1 I_3^{-1/4}, \quad (5.46)$$

where $U(I_3)$ is still an arbitrary function.

Again the remaining function $R(I_1 - I_2, I_3)$ is to be determined from the normalization conditions (5.5), (5.6). On using Eqs. (5.45), (5.46), one finds that R must satisfy

$$R(0, 1) = 0, \quad R_2(0, 1) - R_1(0, 1) = -2k_1 k_3. \quad (5.47)$$

A choice for $R(I_1 - I_2, I_3)$ in terms of exponentials is again easily determined. Alternatively, it is readily verified that

$$R(I_1 - I_2, I_3) = k_1 k_3 (I_1 - I_2) I_3 - k_1 k_3 (I_3 - 1) \quad (5.48)$$

satisfies Eqs. (5.47). On combining Eqs. (5.45), (5.46), (5.48) in Eq. (5.3) we conclude that an admissible choice for W is

$$W = \frac{\mu}{2} \{ c(I_1 - I_2)^r U(I_3)(I_1 - 3) + [k_2(I_1 - I_2)^s + k_3] k_1 I_3^{-1/4} (I_2 - 3) + k_1 k_3 (I_1 - I_2) I_3 - k_1 k_3 (I_3 - 1) \}, \quad (5.49)$$

where c, k_1, k_2, k_3 are arbitrary nonzero constants such that $k_1 k_3 > 0$, $r > 1$, $s > 1$ are arbitrary constants, and $U(I_3)$ is an arbitrary function. The angular displacement is given by Eq. (5.14) while the stresses (5.15) now read

$$\begin{aligned} T_{rr} &= \frac{\mu k_1 k_3 g_0^2 R^{-4}}{(A^{-2} - B^{-2})^2}, & T_{rz} &= 0, \\ T_{\theta\theta} &= \frac{3\mu k_1 k_3 g_0^2 R^{-4}}{(A^{-2} - B^{-2})^2}, & T_{\theta z} &= 0, \\ T_{zz} &= -\frac{\mu k_1 k_3 g_0^2 R^{-4}}{(A^{-2} - B^{-2})^2}, & T_{r\theta} &= \frac{2\mu k_1 k_3 g_0 R^{-2}}{A^{-2} - B^{-2}}. \end{aligned} \quad (5.50)$$

A similar argument in which the roles of F, G and U, V are interchanged in the preceding leads to another W of the form (5.3) with

$$P(I_1 - I_2, I_3) = [k_2(I_1 - I_2)^s + k_3] k_1 I_3^{-1/4}, \quad (5.51)$$

$$Q(I_1 - I_2, I_3) = c(I_1 - I_2)^r V(I_3), \quad (5.52)$$

where c, k_1, k_2, k_3 are arbitrary nonzero constants such that $k_1 k_3 > 0$, $r > 1$, $s > 1$ are arbitrary constants, and $V(I_3)$ is arbitrary. In this case, one finds from Eqs. (5.5), (5.6) that the remaining function $R(I_1 - I_2, I_3)$ must satisfy

$$R(0, 1) = 0, \quad R_2(0, 1) - R_1(0, 1) = -k_1 k_3. \quad (5.53)$$

A choice for R satisfying Eqs. (5.53) is

$$R(I_1 - I_2, I_3) = \frac{k_1 k_3}{2} (I_1 - I_2) I_3 - \frac{k_1 k_3}{2} (I_3 - 1). \quad (5.54)$$

The explicit form for W is then found on substitution from Eqs. (5.51), (5.52), (5.54) in Eq. (5.3). The angular displacement is still given by Eq. (5.14) while the stresses are formally the same as those given by Eqs. (5.50) except for T_{zz} , which is now equal to $-T_{\theta\theta}$.

Returning to Eq. (5.38), it is clear that the choice (5.39) for c_1, c_2 is special. There are an infinite number of choices for $c_1 \neq 0, c_2 \neq 0$ satisfying Eq. (5.38). The resulting conditions (5.36), (5.37) can then be satisfied by choosing exponentials for V, G in addition to powers as in Eqs. (5.42), (5.43). We omit the details.

Thus we have seen in Sec. 5 that the form (5.3) encompasses a wide variety of strain-energy density functions W for which pure azimuthal shear deformation of the tube is possible.

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Note added in proof. Since this paper was written, the authors have received a preprint [21] in which some issues similar to those addressed in the present paper are examined. Some extensions of the results of [16, 17] are also obtained in [21].

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