AN AGE-DEPENDENT POPULATION MODEL WITH NONLINEAR DIFFUSION IN R^n

BY

CHAOCHENG HUANG

School of Mathematics, University of Minnesota, Minneapolis, Minnesota

1. Introduction. In this paper we study a population model suggested by Gurtin and MacCamy in their papers [8-10]. This model describes a single species growing in a multi-dimensional space. Let $\rho(x, t, a)$ designate the population density of age a at time t and location x . Then the total population at time t and location x is given by

$$u(x, t) = \int_0^\infty \rho(x, t, a) da. \quad (1.1)$$

The balance law of the population which includes diffusion and age-dependence is assumed to be

$$\rho_t + \rho_a = \operatorname{div}(\rho \nabla u) - \mu(a, u)\rho, \quad (1.2)$$

where $\mu(a, u)$ is the death modulus. The notation div and ∇ mean, respectively, divergence and gradient with respect to the space variable x only. The birth law is assumed to be

$$\rho(x, t, 0) = \int_0^\infty \beta(a, u)\rho(x, t, a) da, \quad (1.3)$$

where $\beta(a, u)$ is the birth modulus. We are also given the initial distribution of the population density:

$$\rho(x, 0, a) = \rho_0(x, a). \quad (1.4)$$

Gurtin and MacCamy proposed the additional assumptions:

$$\beta(a, u) = \beta(u)e^{-\alpha a}, \quad \mu(a, u) = \mu(u), \quad (1.5)$$

for some $\alpha \geq 0$. The first assumption corresponds to the case in which the individuals are more fertile at a young age, and the second one says that age is not a significant cause of death.

The model (1.1)–(1.5) is of integral-differential type and can be converted into a system of differential equations as follows.

Integrating (1.2) with respect to a , we get

$$u_t = \operatorname{div}(u \nabla u) + \beta(u)G - \mu(u)u,$$

Received May 18, 1992.

1991 *Mathematics Subject Classification*. Primary 35, 92.

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where

$$G(x, t) = \int_0^\infty e^{-\alpha a} \rho(x, t, a) da.$$

If we multiply (1.2) by $e^{-\alpha a}$ and then integrate with respect to a , then we get

$$G_t = \operatorname{div}(G \nabla u) + [\beta(u) - \alpha - \mu(u)]G.$$

Introducing a function $p(x, t)$ by

$$p(x, t) = \frac{G(x, t)}{u(x, t)}$$

and initial data by

$$u_o(x) = \int_0^\infty \rho_o(x, a) da,$$

$$p_o(x) = \begin{cases} (u_o(x))^{-1} \int_0^\infty e^{-\alpha a} \rho_o(x, a) da & \text{if } u_o(x) \neq 0, \\ p_o(x) = 0 & \text{if } u_o(x) = 0, \end{cases}$$

we arrive at the following system:

$$u_t = \operatorname{div}(u \nabla u) + [\beta(u)p - \mu(u)]u, \quad (1.6)$$

$$p_t = \nabla p \cdot \nabla u + [\beta(u) - \alpha]p - \beta(u)p^2, \quad (1.7)$$

$$u(x, 0) = u_o(x), \quad p(x, 0) = p_o(x). \quad (1.8)$$

Note that the initial data u_o and p_o both come from the initial distribution ρ_o ; they generally depend on each other.

For any fixed function $p(x, t)$, equation (1.6) becomes a porous medium equation of the form

$$u_t = \operatorname{div}(u \nabla u) + g(u). \quad (1.9)$$

It is well known (cf. [1]) that if $u_o(x) \not\equiv 0$ has compact support, then in general the porous medium equation (1.9) with $g(u) = 0$ has no classical solutions. This motivates us to look for weak solutions for system (1.6)–(1.8).

To define a weak solution, we multiply (1.6) by p and (1.7) by u and then add them. This gives the following equation:

$$(up)_t = \operatorname{div}(up \nabla u) + [\beta(u) - \alpha - \mu(u)]up. \quad (1.10)$$

For any $T > 0$, let $\Omega_T = R^n \times (0, T)$. Following [11], we define a weak solution of (1.6)–(1.8) as follows:

DEFINITION. A pair of functions (u, p) is called a weak solution of (1.6)–(1.8) in Ω_T if $p \in L^\infty(\Omega_T)$, $u \in C(\overline{\Omega_T})$, $\nabla(u^2) \in L^2_{\text{loc}}(\Omega_T)$, and for any $\phi, \psi \in C^\infty(\overline{\Omega_T})$ with compact supports in $R^n \times [0, T)$, the following two equations hold:

$$\iint_{\Omega_T} \left[\frac{1}{2} \nabla(u^2) \cdot \nabla \phi - u \phi_t \right] dx dt$$

$$= \iint_{\Omega_T} [\beta(u)p - \mu(u)]u \phi dx dt + \int_{R^n} u_o(x) \phi(x, 0) dx, \quad (1.11)$$

$$\begin{aligned}
& \iint_{\Omega_T} \left[\frac{1}{2} p \nabla(u^2) \cdot \nabla \psi - p u \psi_t \right] dx dt \\
&= \iint_{\Omega_T} [\beta(u) - \alpha - \mu(u)] p u \psi dx dt + \int_{R^n} u_0 p_0 \psi(x, 0) dx.
\end{aligned} \tag{1.12}$$

Note that (1.11) is the weak formulation of (1.6) and (1.12) is the weak formulation of (1.10).

MacCamy [15] studied the above model in the one-dimensional space R^1 under the assumption that $\beta(u)$ and $\mu(u)$ are constants. Among other things he proved that there exists a weak solution for problem (1.6)–(1.8). Hernandez [11, 13] extended this existence result to R^n for $n \geq 3$ with more general $\beta(u)$ and $\mu(u)$. He assumed however that the initial data u_0 and p_0 are radially symmetric (and therefore the problem is essentially one dimensional). Hernandez also studied the regularity of solutions and the behavior of the interfaces for the radially symmetric case. The proof of Hernandez's existence theorem strongly depends on the fact that any radially symmetric solution of a porous medium equation such as (1.9) is γ -Hölder continuous for any $0 < \gamma < 1$. Since this is no longer true for non-radially symmetric solutions (cf. [17]), his proof cannot be generalized to arbitrary initial data.

In the present paper we study problem (1.6)–(1.8) in R^n with general initial data (not necessary radially symmetric). We show that under the assumptions (A1)–(A5) (see the next section) for $\beta(u)$, $\mu(u)$, and the initial data, there exists a weak solution for problem (1.6)–(1.8) that is Hölder continuous. We also show that the interface of the weak solution can be represented as the graph of a continuous function $t = S(x)$. Finally we shall study the asymptotic behavior of weak solutions as $t \rightarrow \infty$.

2. Existence and uniqueness of ϵ -approximate solution. Throughout the paper, $C(\overline{\Omega_T})$ denotes the space of continuous functions $u(x, t)$ in $\overline{\Omega_T}$ that have finite norms $\|u\|_0 = \sup |u(x, t)|$. For $0 \leq \gamma < 1$, $C^\gamma(\overline{\Omega_T})$ denotes the Banach space of all functions $u(x, t)$, for which the γ -norm

$$\|u\|_\gamma = \sup_{\Omega_T} |u| + \sup_{\Omega_T} \frac{|u(x, t) - u(y, s)|}{d((x, t), (y, s))^\gamma}$$

is finite, where $d((x, t), (y, s)) = (|x - y|^2 + |t - s|)^\frac{1}{2}$ is the parabolic distance between two points (x, t) and (y, s) in Ω_T . $C^{2+\gamma}(\overline{\Omega_T})$ is the space of all functions $u(x, t)$ for which the $C^{2+\gamma}$ -norm

$$\|u\|_{2+\gamma} = \|u\|_\gamma + \|u_t\|_\gamma + \sum_{i=1}^n \|u_{x_i}\|_\gamma + \sum_{i,j=1}^n \|u_{x_i x_j}\|_\gamma$$

is finite. We assume:

$$0 \leq u_0(x), \quad 0 \leq p_0(x) \leq 1, \quad \alpha \geq 0; \tag{A1}$$

$$u_0 \text{ has compact support}; \tag{A2}$$

$$p_0 \in C(R^n), \quad u_0 \in C^{2+\gamma}(R^n) \text{ for some } 0 < \gamma < 1; \tag{A3}$$

$\mu(s)$, $\beta(s)$, $-\mu'(s)$, $\beta'(s)$ are positive, continuous, and bounded in R^1 . (A4)

We also assume that there exist some constants $\eta_0 > 0$, $K_0 > 0$ such that

$$u_0(x) \geq K_0 \text{dist}(x, \Gamma_0)^{2-\eta_0} \quad \text{for } x \in \{u_0 > 0\}, \quad (\text{A5})$$

where $\Gamma_0 = \partial\{u_0 > 0\}$ is the interface of the initial datum u_0 . The assumptions (A1), (A2), and (A4) follow from the original model; on the other hand, assumptions (A3) and (A5) are made for technical reasons. We note that if $\alpha = 0$, then $G = u$ and $p \equiv 1$. Hence system (1.6)–(1.8) reduces to the Cauchy problem (1.6), (1.8) for u . Since this Cauchy problem has already been extensively studied in the literature, we shall only consider the case $\alpha > 0$.

Since $u_0(x)$ has compact support, equation (1.6) is a degenerate parabolic equation. We first “regularize” the problem by modifying the equations and the initial conditions. Let $F_1(x)$ be a nonnegative function in $C^\infty(R^n)$ with support in the unit ball and $\int_{R^n} F_1(x) dx = 1$; let $F_2(t)$ be a nonnegative function in $C^\infty(R^1)$ with support in the interval $[0, 1]$ and $\int_{R^1} F_2(t) dt = 1$. Set $F(x, t) = F_1(x)F_2(t)$. For any $\epsilon > 0$ and $p \in L^2(\Omega_T)$, set

$$(F_\epsilon * p)(x, t) = \frac{1}{\epsilon^{n+1}} \iint_{\Omega_T} F\left(\frac{x-y}{\epsilon}, \frac{t-s}{\epsilon}\right) p(y, s) dy ds.$$

Note that $F_\epsilon * p \in C^\infty(R^{n+1})$. For fixed $\epsilon > 0$, if $p \in C(\overline{\Omega}_T)$, then the derivatives of $F_\epsilon * p$ are bounded by $K(\epsilon)\|p\|_0$, where $K(\epsilon)$ is a constant depending only on ϵ . We now consider the following ϵ -approximate problem to problem (1.6)–(1.8):

$$u_t = \text{div}(u \nabla u) + [\beta(u)F_\epsilon * p - \mu(u)]u, \quad (2.1)$$

$$p_t = \nabla u \cdot \nabla p + [\beta(u) - \alpha]p - \beta(u)(F_\epsilon * p)p, \quad (2.2)$$

$$u(x, 0) = u_0(x) + \epsilon, \quad (2.3)$$

$$p(x, 0) = p_0^\epsilon(x), \quad (2.4)$$

where $\epsilon > 0$, $p_0^\epsilon(x) \in C^\infty(R^n)$, $0 \leq p_0^\epsilon \leq 1$, and

$$p_0^\epsilon(x) \longrightarrow p_0(x) \quad \text{in } C(R^n) \quad \text{as } \epsilon \longrightarrow 0.$$

In this section we shall prove existence and uniqueness of the ϵ -approximate problem (2.1)–(2.4).

LEMMA 2.1. Let $u \in C^2(\overline{\Omega}_T)$. Then for any $(\tilde{x}, \tilde{t}) \in \Omega_T$, there exists a unique solution in $[0, T)$ for

$$\frac{dx_i}{dt} = -u_{x_i}(x, t), \quad i = 1, 2, \dots, n, \quad (2.5)$$

$$x_i(\tilde{t}; \tilde{x}, \tilde{t}) = \tilde{x}_i, \quad i = 1, 2, \dots, n. \quad (2.6)$$

Moreover, the characteristic curves $x(t; \tilde{x}, \tilde{t})$ defined by (2.5), (2.6) are differentiable with respect to all their arguments and satisfy

$$\frac{\partial x_i}{\partial \tilde{t}} = \sum_{j=1}^n u_{x_j}(\tilde{x}, \tilde{t}) \frac{\partial x_i}{\partial \tilde{x}_j}, \quad i = 1, 2, \dots, n, \quad (2.7)$$

$$\left| \frac{\partial x_i}{\partial \tilde{x}_j} \right| \leq \exp(nT\|u\|_2). \quad (2.8)$$

Proof. Existence and uniqueness of the characteristics (2.5), (2.6) follow from standard ODE theory. To prove (2.7), set

$$f_i(t) = \frac{\partial x_i}{\partial \tilde{t}} - \sum_{j=1}^n u_{x_j}(\tilde{x}, \tilde{t}) \frac{\partial x_i}{\partial \tilde{x}_j}.$$

Then $f_i(\tilde{t}) = 0$ and

$$\begin{aligned} \frac{d}{dt} f_i(t) &= \frac{\partial}{\partial \tilde{t}} \frac{dx_i}{dt} - \sum_{j=1}^n u_{x_j}(\tilde{x}, \tilde{t}) \frac{\partial}{\partial \tilde{x}_j} \frac{dx_i}{dt} \\ &= \frac{\partial}{\partial \tilde{t}} (-u_{x_i}(x(t); \tilde{x}, \tilde{t}), t) + \sum_{j=1}^n u_{x_j}(\tilde{x}, \tilde{t}) \frac{\partial}{\partial \tilde{x}_j} u_{x_i}(x(t); \tilde{x}, \tilde{t}), t) \\ &= - \sum_{j=1}^n u_{x_i x_j}(x(t), t) \frac{\partial x_i}{\partial \tilde{t}} + \sum_{j,k=1}^n u_{x_j}(\tilde{x}, \tilde{t}) u_{x_i x_k}(x(t), t) \frac{\partial x_k}{\partial \tilde{x}_j} \\ &= \sum_{j=1}^n u_{x_i x_j}(x(t), t) f_j(t). \end{aligned}$$

By a uniqueness theorem for ODE we conclude that $f_i = 0$, so that (2.7) holds. Next, taking the derivative with respect to $\tilde{x}_j$ in (2.5), we find that $\partial x_i / \partial \tilde{x}_j$ satisfies

$$\frac{d}{dt} \frac{\partial x_i}{\partial \tilde{x}_j} = \sum_{k=1}^n u_{x_i x_k}(x(t), t) \frac{\partial x_k}{\partial \tilde{x}_j},$$

from which we derive inequality (2.8) by using Gronwall's inequality.

LEMMA 2.2. For any $\epsilon > 0$, $u \in C^2(\overline{\Omega}_T)$, and $p \in C(\overline{\Omega}_T)$, there exists a unique solution $\hat{p}$ for (2.4) and

$$\hat{p}_t = \nabla u \cdot \nabla \hat{p} + [\beta(u) - \alpha] \hat{p} - \beta(u)(F_\epsilon * p) \hat{p}. \quad (2.9)$$

Moreover, if $p \geq 0$, then

$$0 \leq \hat{p}(x, t) \leq e^{TM_1},$$

where $M_1 = \sup |\beta(s) + \mu(s)|$.

Proof. Let $x(t; \tilde{x}, \tilde{t})$ be the characteristics defined by (2.5), (2.6). Suppose $\hat{p}$ is a solution for (2.4), (2.9). Along each characteristic curve we have

$$\frac{d\hat{p}}{dt} = [\beta(u) - \alpha - \beta(u)(F_\epsilon * p)] \hat{p}.$$

Hence we obtain the expression for $\hat{p}$:

$$\hat{p}(\tilde{x}, \tilde{t}) = p_\epsilon(x(0; \tilde{x}, \tilde{t})) \exp \left\{ \int_0^{\tilde{t}} [\beta(u) - \alpha - \beta(u)(F_\epsilon * p)] dt \right\}, \quad (2.10)$$

where the integration is along the characteristic curve. This proves uniqueness. Next we use (2.10) to show existence. For the given u and p , define a function $\hat{p}$ by the

right-hand side of (2.10). Since the characteristics $x(t; \tilde{x}, \tilde{t})$ are differentiable, $\hat{p}$ is differentiable. By directly computing the derivatives of $\hat{p}$ and using (2.7), we find that $\hat{p}$ satisfies equation (2.9). Since $p_o^\epsilon \geq 0$, it follows from (2.10) that $\hat{p}(\tilde{x}, \tilde{t}) \geq 0$. If $p \geq 0$, then obviously $F_\epsilon * p \geq 0$ and hence from (2.10)

$$\begin{aligned} \hat{p}(\tilde{x}, \tilde{t}) &\leq p_o^\epsilon(x(0; \tilde{x}, \tilde{t})) \exp \left\{ \int_0^{\tilde{t}} \beta(u) dt \right\} \\ &\leq e^{M_1 \tilde{t}} \leq e^{TM_1}. \end{aligned}$$

For any positive constants T and B , set

$$\mathcal{K}_T^B = \{(u, p) : u \in C^2(\overline{\Omega}_T), p \in C(\overline{\Omega}_T), \|u\|_2 \leq B, 0 \leq p \leq B\}.$$

We define a mapping $\mathcal{A} : \mathcal{K}_T^B \rightarrow C^2(\overline{\Omega}_T) \times C(\overline{\Omega}_T)$ by $\mathcal{A}(u, p) = (\hat{u}, \hat{p})$, where $(\hat{u}, \hat{p})$ is the unique solution for (2.3), (2.4), (2.9) and

$$\hat{u}_t = \operatorname{div}(\hat{u} \nabla \hat{u}) + [\beta(\hat{u}) F_\epsilon * \hat{p} - \mu(\hat{u})] \hat{u}. \quad (2.11)$$

By Lemma 2.2 and the fact that the Cauchy problem (2.3), (2.11) is uniquely solvable, $\mathcal{A}$ is well defined.

LEMMA 2.3. The mapping $\mathcal{A}$ is continuous.

Proof. Let $\mathcal{A}(u, p) = (\hat{u}, \hat{p})$. By (2.10), $\hat{p}$ can be expressed as

$$\hat{p}(\tilde{x}, \tilde{t}) = p_o^\epsilon(x(0; \tilde{x}, \tilde{t})) \exp \left\{ \int_0^{\tilde{t}} [\beta(u) - \alpha - \beta(u)(F_\epsilon * p)] dt \right\},$$

where $x(t; \tilde{x}, \tilde{t})$ are the characteristics. For any $u, w \in C^2(\overline{\Omega}_T)$, $\|u\|_2, \|w\|_2 \leq K$, $\|u - w\|_2 \leq \delta$, let x, y be the characteristic curves corresponding to u, w , respectively. From (2.5) we get

$$\begin{aligned} |x(t; \tilde{x}, \tilde{t}) - y(t; \tilde{x}, \tilde{t})| &\leq \left| \int_t^{\tilde{t}} |\nabla u(x(t; \tilde{x}, \tilde{t}), t) - \nabla w(y(t; \tilde{x}, \tilde{t}), t)| dt \right| \\ &\leq \left| \int_t^{\tilde{t}} |\nabla u(x(t; \tilde{x}, \tilde{t}), t) - \nabla u(y(t; \tilde{x}, \tilde{t}), t)| dt \right| \\ &\quad + \left| \int_t^{\tilde{t}} |\nabla u(y(t; \tilde{x}, \tilde{t}), t) - \nabla w(y(t; \tilde{x}, \tilde{t}), t)| dt \right| \\ &\leq \delta T + K \left| \int_t^{\tilde{t}} |x(t; \tilde{x}, \tilde{t}) - y(t; \tilde{x}, \tilde{t})| dt \right|. \end{aligned}$$

Using Gronwall's inequality, we find that $|x(t; \tilde{x}, \tilde{t}) - y(t; \tilde{x}, \tilde{t})| \leq \delta T e^{KT}$, for $0 \leq t, \tilde{t} \leq T$. Hence, the characteristics $x(\cdot; \tilde{x}, \tilde{t})$ continuously depend on u . It is easy to see that $F_\epsilon * p$ and its derivatives depend continuously on p . From the expression (2.10) and the standard theory for parabolic equations it follows that $\mathcal{A}$ is continuous.

THEOREM 2.4. For the ϵ -approximate problem (2.1)–(2.4), there exists a classical solution (u_ϵ, p^ϵ) in Ω_T that satisfies

$$0 \leq p^\epsilon(x, t) \leq e^{TM_1}, \quad \epsilon \exp(-TM_1 e^{TM_1}) \leq u_\epsilon(x, t) \leq M \exp(TM_1 e^{TM_1}), \quad (2.12)$$

for any $(x, t) \in \Omega_T$, where $M = \|u_o\|_{2+\gamma}$.

Proof. From Lemma 2.2 we know that for any $(u, p) \in \mathcal{K}_T^B$, $0 \leq \hat{p} \leq e^{TM_1}$, where $(\hat{u}, \hat{p}) = \mathcal{A}(u, p)$. Hence $|\beta(\hat{u})F_\epsilon * \hat{p} - \mu(\hat{u})| \leq M_1 e^{TM_1}$. By the maximum principle we get $\epsilon \exp(-TM_1 e^{TM_1}) \leq \hat{u} \leq M \exp(TM_1 e^{TM_1})$. Since $F_\epsilon * \hat{p}$ is bounded by $\|\hat{p}\|_0$, from L^k -estimates for parabolic equations (cf. [14]) and the Sobolev inequalities it follows that $\|\hat{u}\|_{1+\gamma}$ is bounded by a constant depending only on M_1, T , and ϵ . We now apply the Schauder-type estimates for parabolic equations to (2.11) to obtain

$$\|\hat{u}\|_{2+\gamma} \leq K(\epsilon)(\|F_\epsilon * \hat{p}\|_\gamma + \|u_o\|_{2+\gamma}) \leq K_1(\epsilon), \quad (2.13)$$

where $K_1(\epsilon)$ is a constant which depends only on M, M_1, T , and ϵ . We choose $B = \max\{K_1(\epsilon), e^{TM_1}\}$. Then $\mathcal{A}$ maps $\mathcal{K}_T^B$ into itself. By Lemma 2.3, $\mathcal{A}$ is continuous. To apply the Schauder fixed-point theorem we need to show that the image of $\mathcal{A}$ is precompact in $\mathcal{K}_T^B$. Since any bounded subset in $C^{2+\gamma}$ is precompact in $C^2(\overline{\Omega}_T)$, it follows from (2.13) that the set of the first components of all elements in $\mathcal{A}(\mathcal{K}_T^B)$ is precompact in $C^2(\overline{\Omega}_T)$. It remains to show that the set of the second components of the image of $\mathcal{A}$ is precompact in $C(\overline{\Omega}_T)$. To this end we compute the derivatives of $\hat{p}$ with respect to $\tilde{x}_j$. From (2.10), we get

$$\begin{aligned} \frac{\partial \hat{p}}{\partial \tilde{x}_j} &= \sum_{i=1}^n \frac{\partial p_o^\epsilon}{\partial x_i} \frac{\partial x_i}{\partial \tilde{x}_j} \exp \left\{ \int_0^t [\beta(u) - \alpha - \beta(u)(F_\epsilon * p)] dt \right\} \\ &\quad + p_o^\epsilon \exp \left\{ \int_0^t [\beta(u) - \alpha - \beta(u)(F_\epsilon * p)] dt \right\} \\ &\quad \times \int_0^t \left[\beta' \sum_{i=1}^n \frac{\partial u}{\partial x_i} \frac{\partial x_i}{\partial \tilde{x}_j} (1 - F_\epsilon * p) - \beta \sum_{i=1}^n \frac{\partial}{\partial x_i} (F_\epsilon * p) \frac{\partial x_i}{\partial \tilde{x}_j} \right] dt. \end{aligned}$$

Using (2.8) and the fact that

$$\begin{aligned} \frac{\partial}{\partial x_i} (F_\epsilon * p)(x, t) &= \frac{1}{\epsilon^{n+1}} \iint_{\Omega} \frac{\partial}{\partial x_i} F \left(\frac{x-y}{\epsilon}, \frac{t-s}{\epsilon} \right) p(y, s) dy ds \\ &= \frac{1}{\epsilon} \left(\frac{\partial F}{\partial x_i} \right)_\epsilon * p \end{aligned}$$

is bounded by $\|p\|_0$, multiplying by a constant we arrive at

$$|\nabla \hat{p}| \leq K(\epsilon)$$

and, by (2.9),

$$\left| \frac{\partial}{\partial t} \hat{p} \right| \leq K(\epsilon),$$

where $K(\epsilon)$ stands for another constant depending only on ϵ , B , and T . Hence the derivatives of $\hat{p}$ are bounded uniformly for any $(u, p) \in \mathcal{K}_T^B$. It follows that the set $\{\hat{p} : (\hat{u}, \hat{p}) = \mathcal{A}(u, p), \text{ for some } (u, p) \in \mathcal{K}_T^B \text{ and } \hat{u} \in C^2\}$ is precompact. Now we can apply the Schauder fixed-point theorem to conclude that there exists a pair of functions (u_ϵ, p^ϵ) such that $\mathcal{A}(u_\epsilon, p^\epsilon) = (u_\epsilon, p^\epsilon)$, which is clearly a solution of the ϵ -approximate problem (2.1)–(2.4) satisfying (2.12).

Next we prove uniqueness.

THEOREM 2.5. For any $T > 0$, the classical solution for the ϵ -approximate problem (2.1)–(2.4) is unique in Ω_T .

Proof. Suppose we have two solutions (u, p) and $(\hat{u}, \hat{p})$ in Ω_T . Since both satisfy (2.1), (2.2), by subtraction we obtain the equations

$$(u - \hat{u})_t = \frac{1}{2} \Delta(u^2 - \hat{u}^2) + h_1(x, t)(u - \hat{u}) + h_2(x, t)F_\epsilon * (p - \hat{p}), \quad (2.14)$$

$$(p - \hat{p})_t = \nabla(p - \hat{p}) \cdot \nabla u + h_3(x, t)(p - \hat{p}) + h_4(x, t)F_\epsilon * (p - \hat{p}) + \nabla \hat{p} \cdot \nabla(u - \hat{u}) + h_5(x, t)(u - \hat{u}), \quad (2.15)$$

where

$$h_1(x, t) = -\frac{\mu(u)u - \mu(\hat{u})\hat{u}}{u - \hat{u}} + \frac{\beta(u)u - \beta(\hat{u})\hat{u}}{u - \hat{u}}(F_\epsilon * p),$$

$$h_2(x, t) = \beta(\hat{u})\hat{u},$$

$$h_3(x, t) = \beta(\hat{u}) - \alpha - \beta(\hat{u})(F_\epsilon * \hat{p}),$$

$$h_4(x, t) = -p\beta(\hat{u}),$$

$$h_5(x, t) = (p - (F_\epsilon * p)p) \frac{\beta(u) - \beta(\hat{u})}{u - \hat{u}}.$$

From assumption (A4) and the fact that $u, \hat{u} \in C^{2+\gamma}$, $p, \hat{p} \in C^1$, one can see that all h_i are bounded by a constant $K(\epsilon, T)$. Integrating (2.15) along the characteristic curve (2.6), (2.7) corresponding to (u, p) , we derive that for any $0 < \tau \leq T$,

$$\sup_{\Omega_\tau} |p - \hat{p}| \leq \tau K(\epsilon, T) \left(\sup_{\Omega_\tau} |u - \hat{u}| + \sup_{\Omega_\tau} |\nabla(u - \hat{u})| + \sup_{\Omega_\tau} |F_\epsilon * (p - \hat{p})| \right).$$

Since $F_\epsilon(x, t) = 0$ for $t < 0$, we have

$$\sup_{\Omega_\tau} |F_\epsilon * (p - \hat{p})| \leq \sup_{\Omega_\tau} |p - \hat{p}|.$$

Therefore,

$$\sup_{\Omega_\tau} |p - \hat{p}| \leq \tau K(\epsilon, T) \left(\sup_{\Omega_\tau} |u - \hat{u}| + \sup_{\Omega_\tau} |\nabla(u - \hat{u})| + \sup_{\Omega_\tau} |p - \hat{p}| \right). \quad (2.16)$$

By the L^k -estimates for the parabolic equation (2.14) and the Sobolev inequalities (cf. [6, 14]), we obtain estimates for both $|u - \hat{u}|$ and $|\nabla u - \nabla \hat{u}|$. Substituting these into (2.16) yields

$$\sup_{\Omega_\tau} |p - \hat{p}| \leq 2\tau K_1(\epsilon, T) \sup_{\Omega_\tau} |p - \hat{p}|, \quad (2.17)$$

for some constants $K_1(\epsilon, T)$. Choosing τ such that $2\tau K(\epsilon, T) < 1$, inequality (2.17) leads to

$$p(x, t) = \hat{p}(x, t) \quad \text{in } \Omega_\tau.$$

By a uniqueness theorem for the parabolic equation (2.14), it follows that $u \equiv \hat{u}$ in Ω_τ . We can repeat the above procedure step-by-step to get that $p \equiv \hat{p}$ and $u \equiv \hat{u}$ in Ω_T .

COROLLARY 2.6. The solution of the ϵ -approximate problem (2.1)–(2.4) can be extended uniquely to all time $t > 0$ and satisfies (2.12) for any $T > 0$.

REMARK 2.7. If $\beta(s) \geq \alpha$ for all $s \geq 0$, then similarly we can show that for any $T > 0$, there exists a unique solution for problem (2.1), (2.3), (2.4), and

$$p_t = \nabla u \cdot \nabla p + [\beta(u) - \alpha]F_\epsilon * p - \beta(u)(F_\epsilon * p)p. \quad (2.18)$$

Indeed, to see existence, for any $(u, p) \in \mathcal{X}_T^B$, we define $\mathcal{A}_1(u, p) = (\hat{u}, \hat{p})$ by (2.3), (2.4), (2.11), and

$$\hat{p}_t = \nabla u \cdot \nabla \hat{p} + [\beta(u) - \alpha]F_\epsilon * \hat{p} - \beta(u)(F_\epsilon * \hat{p})\hat{p}.$$

As before, we can show that for some $B > 0$, $\mathcal{A}_1$ maps $\mathcal{X}_T^B$ into itself and is compact. Hence there exists a fixed point, which gives a solution for (2.1), (2.3), (2.4), and (2.18). Uniqueness follows from the same arguments we used to prove Theorem 2.5. We further observe that if (u_ϵ, p^ϵ) is a solution for (2.1), (2.3), (2.4), and (2.18), then

$$p_t^\epsilon \leq \nabla u_\epsilon \cdot \nabla p^\epsilon + \beta(u_\epsilon)F_\epsilon * p^\epsilon (1 - p^\epsilon).$$

Hence along the characteristics we have

$$\frac{d}{dt}(p^\epsilon - 1) \leq -\beta(u_\epsilon)(F_\epsilon * p^\epsilon)(p^\epsilon - 1).$$

Since $p_0^\epsilon(x) \leq 1$, it follows that $p^\epsilon(x, t) \leq 1$.

REMARK 2.8. If we introduce a new function $q = e^{\alpha t} p$, then problem (1.6)–(1.8) becomes

$$u_t = \operatorname{div}(u \nabla u) + [\beta(u)e^{-\alpha t} q - \mu(u)]u, \quad (2.19)$$

$$q_t = \nabla u \cdot \nabla q + \beta(u)q - \beta(u)e^{-\alpha t} q^2, \quad (2.20)$$

$$u(x, 0) = u_0, \quad q(x, 0) = p_0. \quad (2.21)$$

Analogously we can show that there exists a unique positive solution (u_ϵ, q^ϵ) of the corresponding ϵ -approximate problem

$$u_t = \operatorname{div}(u \nabla u) + [\beta(u)e^{-\alpha t} F_\epsilon * q - \mu(u)]u, \quad (2.22)$$

$$q_t = \nabla u \cdot \nabla q + \beta(u)F_\epsilon * q - \beta(u)e^{-\alpha t}(F_\epsilon * q)q, \quad (2.23)$$

with initial data (2.3), (2.4). Furthermore, we have $0 \leq q^\epsilon \leq e^{\alpha t}$.

System (2.1), (2.3), (2.4), (2.18) and system (2.3), (2.4), (2.22), (2.23) can also be viewed as the approximations to problem (1.6)–(1.8). The advantage of these approximations is that here p^ϵ is bounded by one. All of the results in the next section will remain true if we use these two systems to approximate problem (1.6)–(1.8).

3. Existence of a weak solution. In this section we show that there exists a sequence of ϵ -approximate solutions that converges to a weak solution. Let (u_ϵ, p^ϵ) be the solution for (2.1)–(2.4). Then u_ϵ satisfies the following porous medium equation:

$$u_t = \operatorname{div}(u \nabla u) + g(u)u \quad \text{in } \Omega_T, \quad (3.1)$$

$$u(x, 0) = u_0(x) + \epsilon, \quad (3.2)$$

where $g(u) = \beta(u)F_\epsilon * p^\epsilon - \mu(u)$ and $|g(u)| \leq M_2$, $0 \leq u_\epsilon \leq M_3$, for some constants M_2, M_3 independent of ϵ .

LEMMA 3.1. Let u be a solution of (3.1)–(3.2). Then $\nabla u^2 \in L^2_{\text{loc}}(\Omega_T)$. Moreover, for any $R, \delta > 0$

$$|\nabla u^2|_{L^2(B_R \times (0, T-\delta))} \leq K(R, \delta),$$

where $B_R = \{x \in \mathbb{R}^n : |x| < R\}$ and the constant on the right-hand side depends only on R, δ .

Proof. We choose cut-off functions $\xi_1(t) \in C^\infty(-\infty, +\infty)$, $\xi_2(x) \in C^\infty_0(\mathbb{R}^n)$ such that

$$\xi_1(t) = 0 \quad \text{for } t \geq T, \quad \xi_1(t) = 1 \quad \text{for } 0 \leq t \leq T - \delta;$$

$$0 \leq \xi_1(t) \leq 1, \quad |\xi_1'(t)| \leq c_0 \delta^{-1};$$

$$\xi_2(x) = 0 \quad \text{for } |x| > 2R, \quad \xi_2(x) = 1 \quad \text{for } x \in B_R;$$

$$0 \leq \xi_2(x) \leq 1, \quad |\nabla \xi_2(x)| \leq c_0 R^{-1}.$$

Multiplying (3.1) by $u^2 \xi_1^2 \xi_2^2$ and integrating over Ω_T , we get

$$\begin{aligned} & \frac{1}{3} \iint_{\Omega_T} \xi_1^2(t) \xi_2^2(x) (u^3(x, t))_t dx dt \\ &= \frac{1}{2} \iint_{\Omega_T} \xi_1^2 \xi_2^2 u^2 \operatorname{div}(\nabla u^2) dx dt + \iint_{\Omega_T} g(u) \xi_1^2 \xi_2^2 u^3 dx dt. \end{aligned} \quad (3.3)$$

By integration by parts, the left-hand side of (3.3) is

$$\begin{aligned} & \frac{1}{3} \iint_{\Omega_T} \xi_1^2(t) \xi_2^2(x) (u^3(x, t))_t dx dt \\ &= \frac{1}{3} \int_{\mathbb{R}^n} \xi_2^2(x) dx \int_0^T \xi_1^2(t) (u^3(x, t))_t dt \\ &= -\frac{2}{3} \iint_{\Omega_T} \xi_2^2(x) \xi_1(t) \xi_1'(t) u^3(x, t) dx dt - \frac{1}{3} \int_{\mathbb{R}^n} \xi_2^2(x) u^3(x, 0) dx, \end{aligned}$$

which is obviously bounded by a constant K . The first term on the right-hand side of (3.3) is

$$-\frac{1}{2} \iint_{\Omega_T} \xi_1^2 \xi_2^2 |\nabla u^2|^2 dx dt - \iint_{\Omega_T} \xi_1^2 \xi_2 u^2 \nabla u^2 \cdot \nabla \xi_2 dx dt,$$

and the last integral is bounded by

$$\left[\iint_{\Omega_T} \xi_1^2(t) \xi_2^2(x) |\nabla u^2(x, t)|^2 dx dt \right]^{\frac{1}{2}} \left[\iint_{\Omega_T} |\xi_1(t) u^2(x, t) \nabla \xi_2|^2 dx dt \right]^{\frac{1}{2}}.$$

Hence from (3.3) we deduce

$$\begin{aligned} & \iint_{\Omega_T} \xi_1^2(t) \xi_2^2(x) |\nabla u^2(x, t)|^2 dx dt \\ & \leq K \left[\iint_{\Omega_T} \xi_1^2(t) \xi_2^2(x) |\nabla u^2(x, t)|^2 dx dt \right]^{\frac{1}{2}} + K. \end{aligned}$$

It follows that

$$\iint_{B_R \times (0, T-\delta)} |\nabla u^2|^2 dx dt \leq \iint_{\Omega_T} \xi_1^2 \xi_2^2 |\nabla u^2|^2 dx dt \leq K(R, \delta). \quad (3.4)$$

The next lemma is an application of DiBenedetto and Friedman's result in [4].

LEMMA 3.2. Let u be a solution of (3.1)–(3.2). Then for any positive constants δ , R , there exist constants γ , K that depend only on δ , R such that

$$|u(x^1, t^1) - u(x^2, t^2)| \leq K(|x^1 - x^2| + |t^1 - t^2|^{\frac{1}{2}})^\gamma, \quad (3.5)$$

for any $(x^i, t^i) \in B_R \times (\delta, T)$, $i = 1, 2$.

Proof. Since $|u(x, t)| \leq M_3$ and $|\nabla u^2|_{L^2(B_{2R} \times (0, T))} \leq K$ (by Lemma 3.1), the assertion of Lemma 3.2 follows immediately by applying Theorem 1.2 in [4] to the solution u in $B_{2R} \times (0, T)$.

Let (u_ϵ, p^ϵ) be an ϵ -approximate solution of (2.1)–(2.4). By Lemma 3.2 we have

$$|u_\epsilon(x^1, t^1) - u_\epsilon(x^2, t^2)| \leq K(|x^1 - x^2| + |t^1 - t^2|^{\frac{1}{2}})^\gamma \quad (3.6)$$

in $B_R \times (\delta, T)$, for some K and γ that are independent of ϵ . Hence we can select a subsequence $\{u_{\epsilon_m}\}$ that converges uniformly in $B_R \times (\delta, T)$. The limit function is also Hölder continuous. In fact we have:

THEOREM 3.3. For any $\epsilon > 0$, let (u_ϵ, p^ϵ) be the solution of (2.1)–(2.4). Then there exists a subsequence $\{(u_{\epsilon_m}, p^{\epsilon_m})\}$ and a pair of functions (u, p) defined in $R^n \times [0, T)$ such that

- (1) $u(x, t)$ is continuous in $R^n \times [0, T)$, and locally Hölder continuous in $R^n \times (0, T)$;
- (2) $0 \leq p(x, t) \leq e^{TM_1}$, $0 \leq u(x, t) \leq M \exp(TM_1 e^{TM_1})$;
- (3) for any $(x, t) \in R^n \times [0, T)$, $u_{\epsilon_m}(x, t) \rightarrow u(x, t)$, and the convergence is uniform in any compact subset of $R^n \times (0, T)$;
- (4) $p^{\epsilon_m}(x, t)$ converges to $p(x, t)$ weakly in $L^\infty(\Omega_T)$;
- (5) $\nabla u_{\epsilon_m}^2$ converges to ∇u^2 weakly in $L^2(\mathcal{O})$, for any bounded open set $\mathcal{O} \subset \Omega_T$.

Proof. By (3.6), for any integer k , there exists a subsequence $\{u_{\epsilon_m^k}\}_{m=1}^\infty$ that converges uniformly to a function $u_k(x, t)$ in any compact subset of $B_k \times (\frac{1}{k}, T)$, where B_k is the ball in R^n centered at the origin with radius k . The limit u_k is Hölder continuous with constant K and exponent γ depending on k . By the diagonal argument, we can select a subsequence $\{u_{\epsilon_m}\}$ that converges uniformly to

a function $u(x, t)$ in any compact subset of $R^n \times (0, T)$. Since $0 \leq p^{\epsilon_m}(x, t) \leq e^{TM_1}$, there exists a subsequence that converges weakly in L^∞ to a function $p(x, t)$. Thus we have proved (2)–(4) and the second part of (1). Assertion (5) follows from Lemma 3.1 by selecting a subsequence from $\{u_{\epsilon_m}\}$. It remains to show that u is continuous at $t = 0$.

Consider the Cauchy problem for the porous medium equation

$$w_t = \operatorname{div}(w \nabla w), \quad (3.7)$$

$$w(x, 0) = u_o(x) + \epsilon. \quad (3.8)$$

As shown by MacCamy [15], if $w_\epsilon(x, t)$ solves (3.7)–(3.8), then the functions $v_\epsilon, \tilde{v}_\epsilon$ defined by

$$v_\epsilon(x, t) = e^{-M_2 t} w_\epsilon(x, M_2^{-1}(1 - e^{-M_2 t})), \quad (3.9)$$

$$\tilde{v}_\epsilon(x, t) = e^{M_2 t} w_\epsilon(x, M_2^{-1}(e^{M_2 t} - 1)) \quad (3.10)$$

solve the Cauchy problems

$$v_t = \operatorname{div}(v \nabla v) - M_2 v, \quad (3.11)$$

$$v(x, 0) = u_o(x) + \epsilon \quad (3.12)$$

and

$$v_t = \operatorname{div}(v \nabla v) + M_2 v, \quad (3.13)$$

$$v(x, 0) = u_o(x) + \epsilon, \quad (3.14)$$

respectively. Using a standard comparison principle, we obtain

$$v_\epsilon(x, t) \leq u_\epsilon(x, t) \leq \tilde{v}_\epsilon(x, t). \quad (3.15)$$

To prove continuity of u at $(x^\circ, 0)$, consider first the case where $u_o(x^\circ) = \eta > 0$. By a known result for the porous medium equation (3.7) with initial data (3.8) (cf. [2, p. 111]), there exist $c, r > 0$ such that

$$w_\epsilon(x, t) \geq c > 0 \quad \text{in } B_r(x^\circ) \times [0, T],$$

where the constants c, r are independent of ϵ . From (3.15) it follows that

$$u_\epsilon(x, t) \geq v_\epsilon(x, t) \geq c \quad \text{in } B_r(x^\circ) \times [0, T],$$

with another (smaller) positive constant c . This shows that the parabolic equation (2.1) for $u_\epsilon(x, t)$ is nondegenerate in $B_r(x^\circ) \times [0, T]$. We can therefore apply the Schauder-type estimates for parabolic equations (cf. [6]) to deduce (3.6) in $B_r(x^\circ) \times [0, T]$ with the constants independent of ϵ . It follows that $u(x, t)$ is Hölder continuous at $(x^\circ, 0)$ if $u_o(x^\circ) > 0$.

Suppose next that $u_o(x^\circ) = 0$. For any $t > 0$ we have

$$u(x, t) = \lim_{\epsilon_m \rightarrow 0} u_{\epsilon_m}(x, t) \leq e^{M_2 t} \lim_{\epsilon_m \rightarrow 0} w_{\epsilon_m}(x, t).$$

From [2, p. 111] it follows that the right-hand side approaches zero when $(x, t) \rightarrow (x^\circ, 0)$. Hence $u(x, t)$ is continuous at $(x^\circ, 0)$ with $u(x^\circ, 0) = 0$, and the proof is complete.

We now wish to show that the pair (u, p) constructed in Theorem 3.3 is a weak solution of (1.6)–(1.8). It is easy to verify that the ϵ -approximate solution $(u_{\epsilon_m}, p^{\epsilon_m})$ satisfies (1.12) and

$$\begin{aligned} & \iint_{\Omega_T} \left[\frac{1}{2} \nabla(u^2) \cdot \nabla \phi - u \phi_t \right] dx dt \\ &= \iint_{\Omega_T} [\beta(u) F_\epsilon * p - \mu(u)] u \phi dx dt + \int_{R^n} (u_0(x) + \epsilon) \phi(x, 0) dx. \end{aligned}$$

Note that if p^ϵ converges to p weakly, then $F_\epsilon * p^\epsilon$ converges weakly to p . Therefore Theorem 3.3 implies that we can pass to the limit through the above integrals as $\epsilon_m \rightarrow 0$. It follows that (1.11) is valid for (u, p) . Hence $u(x, t)$ is a weak solution of (3.1), (3.2) with $\epsilon = 0$, where $g(u) = \beta(u)p - \mu(u)$. To show that (1.12) is also valid, it suffices to show that

$$\nabla u_{\epsilon_m}^2(x, t) \rightarrow \nabla u^2(x, t) \quad \text{a.e.} \quad (3.16)$$

Introduce the region

$$\Omega_T^+ = \{(x, t) : (x, t) \in \Omega_T, u(x, t) > 0\}$$

and its boundary

$$\Gamma = \partial \Omega_T^+.$$

We first show that (3.16) holds in Ω_T^+ .

LEMMA 3.4. There exists a subsequence $\{\nabla u_{\epsilon_m}^2\}$ such that (3.16) holds in Ω_T^+ .

Proof. Since in Ω_T^+ , equation (3.1) is nondegenerate, by L^k -estimates and the Sobolev inequalities we know that $\|u_\epsilon\|_{1+\gamma}$ is bounded. We can therefore select a subsequence that converges to $u(x, t)$ in the $C^{1+\gamma'}$ -norm for any $0 < \gamma' < \gamma$ in any compact subset of Ω_T^+ .

Next we shall show that (3.16) holds in $\Omega_T \setminus \overline{\Omega_T^+}$.

LEMMA 3.5. For any compact set $\mathcal{D} \subset \Omega_T \setminus \overline{\Omega_T^+}$, $\nabla u_{\epsilon_m}^2$ converges to 0 in $L^2(\mathcal{D})$.

Proof. For any $(x^\circ, t^\circ) \in \Omega_T \setminus \overline{\Omega_T^+}$, we take some $\delta, r > 0$ such that

$$B_{2r}(x^\circ) \times (t^\circ - 2\delta, t^\circ + 2\delta) \subset \Omega_T \setminus \overline{\Omega_T^+}.$$

Choose a cut-off function $\xi(x, t) = \xi_1(t)\xi_2(x)$ such that

$$0 \leq \xi_1(t) \leq 1, \quad \xi_1(t) = 1 \quad \text{in } |t - t^\circ| < \delta;$$

$$\xi_1(t) = 0 \quad \text{in } |t - t^\circ| > \delta, \quad |\xi_1'(t)| < \frac{c_\circ}{\delta};$$

$$0 \leq \xi_2(x) \leq 1, \quad \xi_2(x) = 1 \quad \text{in } B_{2r}(x^\circ);$$

$$\xi_2(x) = 0 \quad \text{in } R^n \setminus B_{2r}(x^\circ), \quad \text{and } |\nabla \xi_2(x)| \leq \frac{c_\circ}{r}.$$

As in the proof of Lemma 3.1, we have estimates similar to (3.4):

$$\begin{aligned} & \iint_{B_{2r} \times (t^\circ - \delta, t^\circ + \delta)} |\nabla u_{\epsilon_m}^2(x, t)|^2 dx dt \\ & \leq \iint_{\Omega_T} \xi_1^2(t) \xi_2^2(x) |\nabla u_{\epsilon_m}^2|^2 dx dt \leq A + 2B, \end{aligned}$$

where

$$\begin{aligned} A &= \iint_{\Omega_T} u_{\epsilon_m}^4(x, t) \xi_t(t) |\nabla \xi_2(x)|^2 dx dt \\ &\leq K_\circ \iint_{B_{2r} \times (t^\circ - 2\delta, t^\circ + 2\delta)} u_{\epsilon_m}^4(x, t) dx dt, \\ B &= \frac{2}{3} \iint_{\Omega_T} |\xi_1'(t) \xi_2(x)| u_{\epsilon_m}^3(x, t) dx dt + M_2 \iint_{\Omega_T} \xi_1(t) \xi_2(x) u_{\epsilon_m}^3 dx dt \\ &\leq K_\circ \iint_{B_{2r} \times (t^\circ - 2\delta, t^\circ + 2\delta)} u_{\epsilon_m}^3(x, t) dx dt. \end{aligned}$$

Since $u_{\epsilon_m} \rightarrow 0$ uniformly in $B_{2r} \times (t^\circ - 2\delta, t^\circ + 2\delta)$, we derive from above that $\nabla u_{\epsilon_m}^2 \rightarrow 0$ in $L^2(B_{2r} \times (t^\circ - \delta, t^\circ + \delta))$. This implies Lemma 3.5 (by selecting a subsequence).

The above two lemmas imply that (3.16) holds in $\Omega_T \setminus \Gamma$. To establish (3.16) in Ω_T , it suffices to show that the interface Γ has measure zero. For any $0 \leq t < T$, set

$$\Omega^t = \{x \in R^n : u(x, t) > 0\}.$$

We have

LEMMA 3.6. The region Ω^t is strictly increasing; moreover, there exists a continuous function $t = S(x)$ defined in an open subset of R^n such that $\Gamma = \{(x, S(x)) : x \in R^n\}$. Consequently the Lebesgue measure of the interface Γ is equal to 0.

Proof. It is well known (cf. [3]) that the solution of the porous medium equation (3.7)–(3.8) with $\epsilon = 0$ has compact support in space and $\text{supp } w(\cdot, t)$ increases with t . Hence by (3.15) Ω^t is bounded for any t . We first show that Ω^t is also increasing with t .

Let $(x^\circ, t^\circ) \in \Omega_T^+$, $u(x^\circ, t^\circ) = \eta > 0$. By Hölder continuity we know that

$$u(x, t^\circ) \geq \eta - K|x - x^\circ|^\gamma \geq \frac{1}{2}\eta$$

if $|x - x^\circ| \leq [\eta(2K)^{-1}]^{\frac{1}{\gamma}} = r$. Consider the comparison function

$$V(x, t) = \frac{r^2 - |x|^2}{2nt + B} \geq 0$$

in $B_r(x^\circ) \times [0, T - t^\circ]$, where

$$B = \frac{2r^2}{\eta} = K\eta^{\frac{2}{\gamma}-1}$$

is a positive constant. It has the following properties:

$$V(x, t) = 0, \quad \text{if } |x| = r,$$

$$V(x, 0) = \frac{r^2 - |x|^2}{B} \leq \frac{r^2}{B} = \frac{\eta}{2} \leq u(x, t^\circ),$$

and

$$V_t - \operatorname{div}(V \nabla V) \leq 0.$$

Let $w(x, t)$ solve the following porous medium equation:

$$w_t = \operatorname{div}(w \nabla w), \quad (3.17)$$

with the initial condition

$$w(x, 0) = u(x, t^\circ). \quad (3.18)$$

Since $V \leq w$ on the parabolic boundary of $B_r(x^\circ) \times [0, T - t^\circ]$, we deduce that

$$w(x, t) \geq V(x, t) = \frac{r^2 - |x|^2}{2nt + B} > 0, \quad \text{for } |x - x^\circ| < r, \quad t > 0. \quad (3.19)$$

Let $v(x, t)$ be the solution of (3.11) with the initial data

$$v(x, 0) = u(x, t^\circ).$$

As before (cf. (3.9) and (3.15)), $v(x, t) \leq u(x, t + t^\circ)$ and we can express $v(x, t)$ explicitly by

$$v(x, t) = e^{-M_2 t} w(x, M_2^{-1}(1 - e^{-M_2 t})).$$

Hence by (3.19) we get $u(x, t + t^\circ) > 0$, for $|x - x^\circ| < r$, $t > 0$. This implies that $(x^\circ, t + t^\circ) \in \Omega_T^+$, and hence Ω^t is increasing in t .

Next we want to show that Ω^t is strictly increasing, i.e., $\Omega^t \supset \overline{\Omega^\tau}$ whenever $t > \tau$. Indeed, otherwise Γ will contain a line segment $\{(x^\circ, t) : t' \leq t \leq t''\}$ for some $t'' > t' \geq 0$ and $x^\circ \in R^n$. Since Ω^t is increasing, we may assume that either $t' = 0$ or else $t' > 0$ and $(x^\circ, t' - \varepsilon) \notin \Gamma$ for any $\varepsilon > 0$. We show that the case $t' > 0$ cannot happen.

Suppose that there exist $t'' > t' > 0$ such that $\{(x_\circ, t) : t' \leq t \leq t''\} \subset \Gamma$ and $(x^\circ, t) \notin \Gamma$ for any $0 < t < t'$. Let $a = t'' - t' > 0$. It is easy to see that there exists a positive constant δ satisfying $t' > 2\delta$ and

$$e^{M_2 \delta} + e^{-M_2(\delta+a)} < 2.$$

Let $w(x, t)$ be the solution of the porous medium equation (3.17) with initial data (3.18), where $t^\circ = t' - \delta$. Then (3.15) becomes

$$e^{-M_2 t} w(x, M_2^{-1}(1 - e^{-M_2 t})) \leq u(x, t + t^\circ) \leq e^{M_2 t} w(x, M_2^{-1}(e^{M_2 t} - 1)). \quad (3.20)$$

Since $\{(x^\circ, t) : \delta < t < \delta + a\} \subset \partial\{u(\cdot, t^\circ + \cdot)\}$, (3.20) says that $w(x^\circ, \tau) = 0$ for

$$M_2^{-1}(1 - e^{-M_2 \delta}) < \tau < M_2^{-1}(1 - e^{-M_2(\delta+a)}). \quad (3.21)$$

On the other hand, (3.20) implies that for any τ satisfying

$$M_2^{-1}(e^{M_2 \delta} - 1) < \tau < M_2^{-1}(e^{M_2(\delta+a)} - 1), \quad (3.22)$$

we can find an $x \in R^n$ which could be arbitrarily close to x° such that $w(x, \tau) > 0$. Now by the choice of δ , the intersection of intervals (3.21) and (3.22) is the interval

$$M_2^{-1}(e^{M_2\delta} - 1) < \tau < M_2^{-1}(1 - e^{-M_2(\delta+a)}) \quad (3.23)$$

which is nonempty. Hence the line segment $\{(x^\circ, \tau) : \tau \text{ satisfies (3.23)}\}$ lies in the interface $\partial\{w(x, t) > 0\}$ for w . By the well-known results for porous medium equations (cf. Theorem 4.1 in [2]), we conclude

$$\{(x^\circ, t) : 0 < t < M_2^{-1}(1 - e^{-M_2(\delta+a)})\} \subset \partial\{w(x, t) > 0\}.$$

Using (3.20) and the same arguments as above, we conclude that there exists a $t''' > 0$ such that $\{(x^\circ, t) : 0 < t < t'''\} \subset \Gamma$, a contradiction to the assumption on t' .

We have known that if Γ contains a line segment, then it must contain an interval $\{(x^\circ, t) : 0 < t < t''\}$ for some x°, t'' . For the solution w of the porous medium equation (3.7), (3.8) ($\epsilon = 0$), we know from [3] that $\Omega_w^t \supset \overline{\Omega_w^0}$ for $t > 0$ (here we need to use the assumption (A5)), where Ω_w^t stands for the region $\{x \in R^n : w(x, t) > 0\}$. Therefore, by (3.20) ($t^\circ = 0$), we can easily verify that $\Omega^t \supset \overline{\Omega^0}$, for small $t > 0$. Hence, in fact Γ cannot contain any line segments. We have thus proved that Ω^t is strictly increasing in the sense that $\Omega^t \supset \overline{\Omega^\tau}$ if $t > \tau$.

For any $x \in R^n$, the line segment $\{(x, t) : 0 < t < T\}$ intersects Γ in at most one point. Therefore, there exists a function $S(x)$ such that $\Gamma = \{(x, S(x))\}$. Since Γ contains no vertical line segments and $u(x, t)$ is continuous, the function $S(x)$ is actually continuous.

Having proved (3.16), we can pass to the limit through integral (1.12) to get a weak solution in Ω_T for any $T > 0$. By Corollary 2.6 and the diagonal argument, this solution can be extended to all time $t > 0$. We conclude this section by the following existence theorem:

THEOREM 3.7. There exists a weak solution (u, p) of problem (1.6)–(1.8) in $R^n \times [0, +\infty)$ which has the properties (1) and (2) listed in Theorem 3.3 for any $T > 0$. Furthermore, there exists a continuous, nonnegative function $t = s(x)$ defined in an open subset of R^n such that $\{(x, t) : t > S(x)\}$ is exactly the populated region $\{(x, t) : u(x, t) > 0\}$.

4. Asymptotic behavior of the weak solutions. In this section we study the long-time behavior of the weak solution (u, p) . Set

$$\begin{aligned} \beta_1 &= \inf_{s \geq 0} \beta(s), & \beta_2 &= \sup_{s \geq 0} \beta(s), \\ a_1 &= 1 - \frac{\alpha}{\beta_1}, & a_2 &= 1 - \frac{\alpha}{\beta_2}, \\ b_1 &= \inf_{s \geq 0} \frac{\mu(s)}{\beta(s)}, & b_2 &= \sup_{s \geq 0} \frac{\mu(s)}{\beta(s)}. \end{aligned}$$

Note that $\beta_1 > 0$, $b_1 > 0$ and that $a_1\beta(s) \leq \beta(s) - \alpha \leq a_2\beta(s)$. The main results are the following two theorems:

THEOREM 4.1. If $\beta_2 < \alpha$, then there exist a weak solution (u, p) of (1.6)–(1.8) and a constant $r > 0$ such that for all $t > 0$

$$\Omega^t \subset B_r(0)$$

and

$$u(x, t) \longrightarrow 0 \quad \text{as } t \longrightarrow \infty.$$

THEOREM 4.2. Assume $\beta_1 > \alpha$. Then there exists a weak solution (u, p) of (1.6)–(1.8), for which the following statements are true:

(1) If $a_2 < b_1$, then there exists a constant $r > 0$ such that for all t ,

$$\Omega^t \subset B_r(0)$$

and

$$u(x, t) \longrightarrow 0 \quad \text{as } t \longrightarrow \infty;$$

(2) If $a_1 > b_2$, $u_0(x) \not\equiv 0$, and there exists a positive constant c_0 such that $p_0(x) \geq c_0$, then as $t \longrightarrow \infty$,

$$u(x, t) \longrightarrow \infty,$$

$$\Omega^t \longrightarrow R^n.$$

To interpret these results, we suppose for simplicity that the birth modulus β and the death modulus μ are constants. If the difference $\beta - \mu$ is less than α , then the population will eventually disappear; whereas if this difference $\beta - \mu$ is greater than α (and $p_0(x) \geq c_0 > 0$), then the population will spread over everywhere and go to infinity.

To prove the theorems, we need the following lemma:

LEMMA 4.3. Let $u(x, t)$ be a solution for (3.1) with the initial condition $u(x, 0) = u_0(x)$, where $u_0(x) \geq 0$, $u_0(x) \not\equiv 0$. If $g(u) \geq \delta$ for some positive constants δ , then when $t \longrightarrow \infty$, $u(x, t) \longrightarrow \infty$ uniformly in any compact subset of R^n , and $\Omega^t \longrightarrow R^n$; if $g(u) \leq -\delta$, then there exists an $r > 0$ such that for all $t > 0$, $\Omega^t \subset B_r(0)$ and $u(x, t) \longrightarrow 0$ uniformly as $t \longrightarrow \infty$.

Proof. Consider the solution of Barenblatt-Prattle [17] for the porous medium equation $w_t = \operatorname{div}(w \nabla w)$:

$$w^\circ(x, t) = 2\lambda^{2/(n+2)}(t+1)^{-n/(n+2)} \left(1 - \frac{\lambda^{-n/(n+2)}|x|^2}{4(n+2)(t+1)^{n/(n+2)}} \right)^+,$$

for any $\lambda > 0$. It is easy to check that

$$w^\circ(0, 0) = 2\lambda^{2/(n+2)} \quad (4.1)$$

and

$$\operatorname{supp}\{w^\circ(\cdot, 0)\} = \{x : |x|^2 \leq 4(n+2)\lambda^{n/(n+2)}\}. \quad (4.2)$$

Let $u(x, t)$ be a solution of (3.1). We first assume that $g(u) \geq \delta$, for some positive constants δ . By the comparison principles we know that $u(x, t) \geq V(x, t)$, where

$$V(x, t) = e^{\delta t} w(x, \delta^{-1}(e^{\delta t} - 1)),$$

$w_t = \operatorname{div}(w \nabla w)$, and $w(x, 0) = u_o(x)$. Without loss of generality, we may assume $u_o(0) > 0$. By (4.1), (4.2), we can choose λ small enough such that

$$w^\circ(x, 0) \leq u_o(x) \quad \text{for } x \in R^n.$$

By comparison,

$$V(x, t) \geq e^{\delta t} w^\circ(x, \delta^{-1}(e^{\delta t} - 1)).$$

Hence for any $x \in R^n$, $V(x, t) \rightarrow \infty$ as $t \rightarrow \infty$. Furthermore, the convergence is uniform in any compact subset of R^n . Therefore, if $g(u) \geq \delta > 0$, $u(x, t)$ will approach ∞ at a rate faster than $e^{\delta t}$, and $\Omega^t \rightarrow R^n$ as $t \rightarrow \infty$.

Next we assume $g(u) \leq -\delta < 0$. In this case we can take λ large enough such that $u_o(x) \leq w_o(x, 0)$. Similarly we derive the inequality

$$u(x, t) \leq e^{-\delta t} w^\circ(x, \delta^{-1}(1 - e^{-\delta t})).$$

Hence, $u(x, t) \rightarrow 0$ as $t \rightarrow \infty$, and there exists a ball $B_r(0)$ such that $\Omega^t \subset B_r(0)$ for any $t \geq 0$.

We shall use this lemma to prove Theorem 4.1 and Theorem 4.2.

Proof of Theorem 4.1. Let (u_ϵ, p^ϵ) be the ϵ -approximate solution. From (2.10), we have

$$p^\epsilon(x, t) \leq p_o^\epsilon(x(0; x, t))e^{t(\beta_2 - \alpha)} \leq e^{t(\beta_2 - \alpha)}.$$

Hence for large t , $g(u) = \beta(u)p - \mu(u) \leq -\mu_1/2$, where $\mu_1 = \inf_{s>0} \mu(s) \geq b_1\beta_1 > 0$. The assertion of Theorem 4.1 follows from Lemma 4.3.

Proof of Theorem 4.2. We first prove (2). Let (u_ϵ, p^ϵ) be the approximate solution constructed in Remark 2.7 (here we need the assumption $\beta_1 > \alpha$). For any constant β'_1 satisfying $\beta_1 > \beta'_1 > \alpha$, we get from (2.18)

$$\begin{aligned} p^\epsilon(\tilde{x}, \tilde{t}) &= \exp\left(-\int_0^{\tilde{t}} \beta F_\epsilon * p^\epsilon dt\right) \\ &\quad \times \left[p_o^\epsilon(x(0; \tilde{x}, \tilde{t})) + \int_0^{\tilde{t}} (\beta(u_\epsilon) - \alpha)(F_\epsilon * p^\epsilon) \exp\left(\int_0^t \beta F_\epsilon * p^\epsilon dt\right) dt\right] \\ &\geq \exp\left(-\int_0^{\tilde{t}} \beta F_\epsilon * p^\epsilon dt\right) \\ &\quad \times \left[c_o + \frac{\beta'_1 - \alpha}{\beta_2} \int_0^{\tilde{t}} \beta(u_\epsilon)(F_\epsilon * p^\epsilon) \exp\left(\int_0^t \beta F_\epsilon * p^\epsilon dt\right) dt\right] \\ &= \exp\left(-\int_0^{\tilde{t}} \beta F_\epsilon * p^\epsilon dt\right) \left[c_o + \frac{\beta'_1 - \alpha}{\beta_2} \left(\exp\left(\int_0^{\tilde{t}} \beta F_\epsilon * p^\epsilon dt\right) - 1\right)\right] \\ &= \frac{\beta'_1 - \alpha}{\beta_2} + \left(c_o - \frac{\beta'_1 - \alpha}{\beta_2}\right) \exp\left(-\int_0^{\tilde{t}} \beta F_\epsilon * p^\epsilon dt\right); \end{aligned}$$

all integrations above are along the characteristics (2.5), (2.6). Choosing β'_1 such that $0 < \beta'_1 - \alpha \leq \beta_2 c_o$, we derive the inequality

$$p^\epsilon(x, t) \geq \frac{\beta'_1 - \alpha}{\beta_2} = c'_o > 0.$$

Hence $F_\epsilon * p^\epsilon(x, t) \geq c'_0$. Since $\beta(s) - \alpha \geq a_1 \beta(s)$, we obtain that along the characteristics,

$$\begin{aligned} \frac{d}{dt}(p^\epsilon - a_1) &= (\beta(u_\epsilon) - \alpha)F_\epsilon * p^\epsilon - \beta(u_\epsilon)(F_\epsilon * p^\epsilon)p^\epsilon \\ &\geq -\beta(u_\epsilon)(F_\epsilon * p^\epsilon)(p^\epsilon - a_1). \end{aligned}$$

Therefore, by integration

$$\begin{aligned} (p^\epsilon - a_1)(\tilde{x}, \tilde{t}) &\geq (p^\epsilon_0(x(0; \tilde{x}, \tilde{t})) - a_1) \exp\left(-\int_0^{\tilde{t}} \beta F_\epsilon * p^\epsilon dt\right) \\ &\geq -a_1 \exp\left(-\int_0^{\tilde{t}} \beta_1 c'_0 dt\right) = -a_1 e^{-\beta_1 c'_0 \tilde{t}}. \end{aligned}$$

Choose a constant c_1 such that $a_1 > c_1 > b_2$. Then

$$p^\epsilon(\tilde{x}, \tilde{t}) \geq a_1 - a_1 e^{-\beta_1 c'_0 \tilde{t}} \geq c_1 \quad \text{for } t \geq T = \frac{1}{\beta_1 c'_0} \log \frac{a_1}{a_1 - c_1}.$$

Letting $\epsilon \rightarrow 0$, we get for $t \geq T$,

$$g(u) = \beta(u)p - \mu(u) \geq c_1 \beta - \mu \geq (c_1 - b_2)\beta_1 > 0.$$

The assertion (2) of Theorem 4.2 now follows from Lemma 4.3.

We next prove assertion (1). Choose c_2 such that $a_2 < c_2 < b_1$. Since $p_0(x) \leq 1$, we may assume that $p^\epsilon_0(x)$ was selected such that there exists a positive constant $\eta_1 < 1$ (depending on ϵ) satisfying $p^\epsilon_0 \leq \eta_1$. Consider the following Cauchy problem for the ODE:

$$u'(t) = [\beta(u)e^{-\alpha t} F_\epsilon * q - \mu(u)]u, \quad (4.3)$$

$$q'(t) = \beta(u)(F_\epsilon * q) - \beta(u)e^{-\alpha t}(F_\epsilon * q)q, \quad (4.4)$$

$$u(0) = \sup_{x \in R^n} u_0(x) + 1, \quad q(0) = 1. \quad (4.5)$$

From Remark 2.8 we find that this is actually an ϵ -approximation to problem (1.6)–(1.8) in which the initial data are space-independent. It is easy to see that there exists a unique positive solution $(\tilde{u}_\epsilon, \tilde{q}^\epsilon)$ for problem (4.3)–(4.5). From equation (4.4), we have

$$(\tilde{q}^\epsilon - e^{\alpha t})' \leq -\beta F_\epsilon * \tilde{q}^\epsilon e^{-\alpha t}(\tilde{q}^\epsilon - e^{\alpha t}).$$

Hence $\tilde{q}^\epsilon(t) \leq e^{\alpha t}$. It follows that $(\tilde{q}^\epsilon)' \geq 0$. Therefore, $1 < \tilde{q}^\epsilon(t)$ for any $t > 0$. Since $\tilde{q}^\epsilon(t)$ is increasing and $F_\epsilon(x, t) = 0$ for $t \leq 0$, we find that $(F_\epsilon * \tilde{q}^\epsilon)(t) \leq \tilde{q}^\epsilon(t)$. Hence

$$(\tilde{q}^\epsilon)' \leq \beta(\tilde{u}_\epsilon)(1 - e^{-\alpha t} \tilde{q}^\epsilon) \tilde{q}^\epsilon.$$

To estimate the upper bound of $\tilde{q}^\epsilon$, we make the substitution $r(t) = \tilde{q}^\epsilon(t)e^{-\alpha t}$. Then $r(t)$ satisfies

$$r' \leq [\beta(\tilde{u}_\epsilon) - \alpha]r - \beta(\tilde{u}_\epsilon)r^2.$$

Integrating the above differential inequality, we get

$$r(t) \leq \exp\left(\int_0^t (\beta - \alpha) dt\right) \left[1 + \int_0^t \beta(\tilde{u}_\epsilon) \exp\left(\int_0^t (\beta - \alpha) dt\right) dt\right]^{-1}.$$

Since $(\beta - \alpha)/\beta \leq a_2 < 1$ and $\beta - \alpha \geq \beta_1 - \alpha > 0$, it follows that

$$\begin{aligned} r(t) &\leq \exp\left(\int_0^t (\beta - \alpha) dt\right) \left[1 + a_2^{-1} \int_0^t [\beta(\tilde{u}_\epsilon) - \alpha] \exp\left(\int_0^t (\beta - \alpha) dt\right) dt\right]^{-1} \\ &= \exp\left(\int_0^t (\beta - \alpha) dt\right) \left[1 + a_2^{-1} \left(\exp\left(\int_0^t (\beta - \alpha) dt\right) - 1\right)\right]^{-1} \\ &\leq \frac{a_2}{1 + (a_2 - 1)e^{-(\beta_1 - \alpha)t}}. \end{aligned}$$

Hence for any $t > 0$, $r(t) < 1$. If we take $T = (\beta_1 - \alpha)^{-1}[\log c_2(1 - a_2) - \log(c_2 - a_2)]$, then $r(t) \leq c_2$ when $t \geq T$. Therefore, we obtain $\tilde{q}^\epsilon(t) < e^{\alpha t}$ for $t > 0$ and $\tilde{q}^\epsilon(t) \leq c_2 e^{\alpha t}$ for $t \geq T$.

We now compare $(\tilde{u}_\epsilon, \tilde{q}^\epsilon)$ with (u_ϵ, q^ϵ) , the ϵ -approximate solution constructed in Remark 2.8. By subtraction, we get

$$(u_\epsilon - \tilde{u}_\epsilon)_t = \frac{1}{2} \Delta(u_\epsilon^2 - \tilde{u}_\epsilon^2) + \beta(u_\epsilon) \tilde{u}_\epsilon e^{-\alpha t} F_\epsilon * (q^\epsilon - \tilde{q}^\epsilon) + k_1(x, t)(u_\epsilon - \tilde{u}_\epsilon), \quad (4.6)$$

$$(q^\epsilon - \tilde{q}^\epsilon)_t = \nabla(q^\epsilon - \tilde{q}^\epsilon) \cdot \nabla u_\epsilon + k_2 F_\epsilon * (q^\epsilon - \tilde{q}^\epsilon) + k_3(u_\epsilon - \tilde{u}_\epsilon) + k_4(q^\epsilon - \tilde{q}^\epsilon), \quad (4.7)$$

where

$$k_1(x, t) = \beta(u_\epsilon) e^{-\alpha t} F_\epsilon * q^\epsilon - \mu(u_\epsilon) - \frac{\mu(u_\epsilon) - \mu(\tilde{u}_\epsilon)}{u_\epsilon - \tilde{u}_\epsilon} + \tilde{u}_\epsilon (F_\epsilon * \tilde{q}^\epsilon) e^{-\alpha t} \frac{\beta(u_\epsilon) - \beta(\tilde{u}_\epsilon)}{u_\epsilon - \tilde{u}_\epsilon},$$

$$k_2(x, t) = \beta(u_\epsilon)(1 - e^{-\alpha t} q^\epsilon),$$

$$k_3(x, t) = \frac{\beta(u_\epsilon) - \beta(\tilde{u}_\epsilon)}{u_\epsilon - \tilde{u}_\epsilon} (1 - e^{-\alpha t} \tilde{q}^\epsilon) (F_\epsilon * \tilde{q}^\epsilon),$$

$$k_4(x, t) = -e^{-\alpha t} \beta(u_\epsilon) (F_\epsilon * \tilde{q}^\epsilon).$$

By assumption (A4), all $k_i(x, t)$ are bounded by $Ke^{\alpha t}$ for some positive constants K . Since $1 \leq \tilde{q}^\epsilon(t) \leq e^{\alpha t}$, we can see that $k_2, k_3 \geq 0$, $k_4 \leq -\beta_1 e^{-\alpha t}$. Since $q^\epsilon(x, t)$ solves

$$q_t = \nabla u_\epsilon \cdot \nabla q + \beta F_\epsilon * q - \beta e^{-\alpha t} (F_\epsilon * q) q,$$

by integration along the characteristics we get

$$\begin{aligned} q^\epsilon &= \exp\left(-\int_0^t \beta e^{-\alpha t} F_\epsilon * q^\epsilon dt\right) \\ &\quad \times \left[p_o^\epsilon + \int_0^t \beta(u_\epsilon)(F_\epsilon * q^\epsilon) \exp\left(-\int_0^t \beta e^{-\alpha t} F_\epsilon * q^\epsilon dt\right) dt\right] \\ &\leq \eta_1 + \int_0^t \beta_2 e^{\alpha t + \beta_2 t} dt. \end{aligned}$$

Hence (recall that $\eta_1 < 1$) there exists a $t_o > 0$ such that $q^\epsilon(x, t) \leq 1 \leq \tilde{q}^\epsilon(t)$ for $t \leq t_o$, $x \in R^n$.

We now claim that $q^\epsilon(x, t) \leq \tilde{q}^\epsilon(t)$ for all $t > 0$. Indeed, otherwise there exists a maximal $t_o > 0$ such that the inequality $q^\epsilon(x, t) \leq \tilde{q}^\epsilon(t)$ holds for all $t \leq t_o$ but

not for $t \leq t_0 + \delta$ for any $\delta > 0$. From (4.6) we derive that in $R^n \times (0, t_0)$,

$$(u_\epsilon - \tilde{u}_\epsilon)_t \leq \frac{1}{2}(u_\epsilon + \tilde{u}_\epsilon)\Delta(u_\epsilon - \tilde{u}_\epsilon) + \nabla(u_\epsilon + \tilde{u}_\epsilon) \cdot \nabla(u_\epsilon - \tilde{u}_\epsilon) + k_5(u_\epsilon - \tilde{u}_\epsilon),$$

where $k_5 = k_1 + \Delta(u_\epsilon + \tilde{u}_\epsilon)/2$ is bounded in $R^n \times (0, t_0)$. By the maximum principle, $(u_\epsilon - \tilde{u}_\epsilon) \leq 0$. Let $\eta_2 > 0$ be a constant such that $k_5 \geq -\eta_2$. Then the function $w = (u_\epsilon - \tilde{u}_\epsilon)e^{\eta_2 t} + \frac{1}{2}$ satisfies

$$w_t \leq \frac{1}{2}(u_\epsilon + \tilde{u}_\epsilon)\Delta w + \nabla(u_\epsilon + \tilde{u}_\epsilon) \cdot \nabla w.$$

It follows from the maximum principle that

$$(u_\epsilon - \tilde{u}_\epsilon)(x, t) \leq -\frac{1}{2}e^{-\eta_2 t} \leq -\eta_3 \quad \text{in } R^n \times (0, t_0),$$

where η_3 is another positive constant. Since $k_2, k_3 \geq 0$, along the characteristics we get from (4.7)

$$\frac{d}{dt}(q^\epsilon - \tilde{q}^\epsilon) \leq -\eta_3 k_3 + k_4(q^\epsilon - \tilde{q}^\epsilon).$$

Hence

$$(q^\epsilon - \tilde{q}^\epsilon) \leq -\eta_3 \exp\left(\int_0^t k_4 dt\right) \int_0^t k_3 \exp\left(-\int_0^t k_4 dt\right) dt.$$

Since $\tilde{q}^\epsilon(t) < e^{\alpha t}$ for $t > 0$, we get $k_3 > 0$. It follows that there exists another positive constant η_4 such that

$$q^\epsilon(x, t) - \tilde{q}^\epsilon(t) \leq -\eta_4 \quad \text{in } R^n \times (\tfrac{1}{2}t_0, t_0).$$

Since (from (4.7)) along the characteristic curve $d(q^\epsilon - \tilde{q}^\epsilon)/dt$ is bounded by a constant independent of $x \in R^n$ and $t \leq 2t_0$, there exists a $t_1 > t_0$ such that $q^\epsilon(x, t) \leq \tilde{q}^\epsilon(t)$ for $t \leq t_1$. This is a contradiction to our assumption on t_0 .

Therefore, for all $(x, t) \in R^n \times [T, +\infty)$, we have $q^\epsilon(x, t) \leq \tilde{q}^\epsilon(t) \leq c_2 e^{\alpha t}$. Note that the constant T is independent of ϵ . Letting $\epsilon \rightarrow 0$, we get, for $t > T$,

$$g(u) = \beta(u)q(x, t)e^{-\alpha t} - \mu(u) \leq (c_2 - b_1)\beta_1 < 0.$$

The assertion (1) follows from Lemma 4.3.

Acknowledgment. The author would like to express his appreciation to his thesis advisor, Professor Avner Friedman, for his guidance and many valuable suggestions.

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