NONLINEAR STABILITY PROBLEM OF A ROTATING POROUS LAYER

By

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1. Introduction. In hydrodynamic stability theory there are two useful methods, namely, the linear stability method and the energy method. Both of these methods have been used extensively, and, in fact, complement each other. While the linear stability method determines a critical dimensionless bound above which the disturbances of a basic flow are unstable, the energy method predicts a critical bound below which a basic flow is asymptotically stable (in the Liapunov sense). However, in some cases when the linearized system of governing equations happens to be selfadjoint, the linear stability and energy stability predictions coincide (cf. Davis, 1969, Galdi and Straughan, 1985a).

The energy method is essentially due to Orr (1907), but its recent revival has been inspired by the creative work of Serrin (1959) and Joseph (1965, 1966). Despite the unquestionable success of the energy method in several stability problems, there is some skepticism about its ongoing indiscriminate use. Situations have been encountered, for example, in the magnetic Bénard problem (Rionero 1968, Galdi 1985) and in the Bénard problem between rotating parallel, horizontal planes (Galdi and Straughan 1985b) where the classical energy theory of Serrin and Joseph did not produce expected results. Rapid improvements of the classical energy theory, however, have been made by Galdi and Straughan (1985b), Mulone and Rionero (1989), and others by proposing generalized energy functionals. This new approach to energy theory is summarized and further analysed by Galdi and Padula (1990). In this remarkable paper, the authors have singled out systems for which the new theory furnishes necessary and sufficient conditions for stability, and have provided an algorithm to construct the appropriate generalized energy functionals for a wide class of physical problems. With regard to the rotating Bénard problem, we note that an energy function using only the L^2 norm of the disturbance variables may not be good enough to obtain the desired results. The generalized energy functionals, with stronger norm, usually $W^{1,2}$ or an equivalent one, would be useful in the study of the nonlinear stability of the rotating Bénard problem.

The stability problems of convective flow in porous media using the energy method have been investigated by many authors: Westbrook (1969) applied the energy method to discuss the stability of convective flow in a porous medium, and Homsy and

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Sherwood (1976) extended this work when there is a mass flow. Joseph (1976), using the energy method, considered the problem of finding the critical Rayleigh number for a fluid-saturated porous layer which is subjected to an internal heat source. His results have been extended by Rionero and Straughan (1990) by allowing variable gravity effects in their calculations. In all the above papers the porous medium is represented by Darcy's model. When the porosity of the material is large and when the boundary layer effects need to be taken into consideration, it is now believed that the Brinkman model is superior to Darcy's model. Moreover, use of this model allows us to recover, in the limiting cases, results for flow of pure viscous fluid as well as for flow governed by Darcy's law. Qin and Kaloni (1992) have carried out a theoretical study of the problem of steady convection through porous media based upon the Brinkman model.

In this work we study the nonlinear stability of the rotating Bénard problem in a porous medium. We employ the generalized Brinkman model as a suitable prototype for high porosity porous media, and, based on it, we are able to apply the energy theory of Galdi and Padula (1990) in studying this problem.

In this paper, after presenting the basic perturbation equations, the evolution equation of an energy functional is derived. Adopting the same energy functional as constructed by Galdi and Padula for the viscous rotating Bénard problem, the a priori estimate of the energy functional is then given which ensures the nonlinear stability under some sufficient conditions. We then solve the variational problem and carry out numerical calculations to determine the critical energy bounds.

2. Perturbation equations. Let us consider an infinite horizontal porous layer saturated with a homogeneous fluid under the action of a vertical gravity field $\mathbf{g} = -g\mathbf{k}$ in which an adverse temperature gradient $\beta > 0$ is maintained; $\mathbf{k} = (0, 0, 1)$ is the vertical unit vector. We also assume that the fluid is rotating about the vertical axis z with a constant angular velocity Ω . A perturbation of velocity $\mathbf{u} = (u, v, w)$, temperature θ , and pressure field p is taken about the nonconvecting stationary solution $\mathbf{u} = 0$, $T = -\beta z + T_0$. The fluid is assumed to be contained in a porous medium between two planes $z = 0$ and $z = h$ with assigned temperatures $T = T_0$ at $z = 0$ and $T = T_1$ at $z = h$, where $T_0 > T_1$ and the temperature gradient across the fluid layer is $\beta = (T_0 - T_1)/h$.

The momentum, continuity, and energy equations for the porous fluid layer in the Boussinesq approximation are, respectively (Walker and Homsy, 1977)

$$\mathbf{u}_t + \frac{1}{\phi} \mathbf{u} \cdot \nabla \mathbf{u} = \Delta \mathbf{u} - P_0 \mathbf{u} + R\theta \mathbf{k} + T \mathbf{u} \times \mathbf{k} - \nabla p, \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2)$$

$$P_r(\theta_t + \mathbf{u} \cdot \nabla \theta) = \Delta \theta + R w, \quad (3)$$

where $(\mathbf{x}, t, \mathbf{u}, p, T)$ have been made dimensionless with respect to $(h, h^2/\nu, \nu/h, \rho_0 \nu^2/h^2, (\frac{\beta \nu^3}{\alpha g \kappa h^2})^{1/2})$ respectively, h is the fluid depth, ν is the kinematic viscosity, ρ_0 is the density, κ is the thermal diffusivity, α is the coefficient of

thermal expansion, ϕ is the porosity, $\beta = (T_0 - T_1)/h$ is the temperature gradient, and the following dimensionless numbers are defined:

$$\begin{aligned} R^2 &= \alpha \beta g h^2 / (\kappa \nu) & (\text{Rayleigh number}), \\ T &= 2 h^2 \Omega / \nu & (\text{Taylor number}), \\ P_r &= \nu / \kappa & (\text{Prandtl number}), \\ P_0 &= h^2 / k & (\text{Darcy number}). \end{aligned}$$

Equation (1) is called the Darcy-Brinkman-Boussinesq equation involving convective inertia, the usual viscous force, as well as Darcy resistance force, and Boussinesq approximation in the body force.

For the boundary conditions for the functions $\mathbf{u}$, θ , we suppose that $\mathbf{u}$, θ , p are periodic in x , y with period $2a_1$, $2a_2$, respectively, and are stress-free at the surfaces $z = 0, 1$, so that

$$u_{,z} = v_{,z} = w = \theta = 0, \quad z = 0, 1, \quad (4)$$

where $u_{,z} = \partial u / \partial z$.

To exclude rigid motions we assume that the mean values of u , v are zero, i.e., we require

$$\int_{\Omega} u \, dx = \int_{\Omega} v \, dx = 0, \quad (5)$$

where $\Omega = (0, 2a_1] \times (0, 2a_2] \times (0, 1)$.

3. An energy functional and related evolution equation. Let u be any of the four components of the vector of the perturbation $(\mathbf{u}, \theta)$. The Hilbert space H coincides with $\mathbf{J}(\Omega) \times L^2(\Omega)$, and $\mathbf{J}(\Omega)$ is the subspace of $\mathbf{L}^2(\Omega)$ consisting of solenoidal vector functions $\mathbf{u}$ with $w = 0$ at $z = 0, 1$.

The problem described by (1) – (5) can be written as an evolution equation of the form

$$B u_t = A u + R S u + T M u + N u$$

in the Hilbert space H , in which B is a positive diagonal matrix, A is a linear differential operator, S is a bounded symmetric operator, M is a skew-symmetric operator, N is a nonlinear operator, and R and T are two dimensionless parameters. Following Galdi and Padula (1990), it can be shown that this system is weakly coupled. Besides the usual energy term, an appropriate energy functional is then proposed by considering the interaction between S and M . Rather than verifying all the hypotheses for weakly coupled systems, here we prefer to derive the evolution equation more directly. The procedure followed below, however, is patterned after the work of Galdi and Padula (1990).

Let Π be a projector operator from $\mathbf{L}^2(\Omega)$ onto $\mathbf{J}(\Omega)$; the following properties hold:

$$\begin{aligned} 1) \quad \Pi \mathbf{v} &= \mathbf{v}, & \text{if } \mathbf{v} \in \mathbf{J}(\Omega), \\ 2) \quad \Pi \mathbf{v} &= \mathbf{v} - \nabla \psi, & \text{if } \mathbf{v} \in \mathbf{L}^2(\Omega) \setminus \mathbf{J}(\Omega), \end{aligned} \quad (6)$$

with ψ (periodic in x, y , depending on $\mathbf{v}$) satisfying the problem

$$\begin{aligned}\Delta\psi &= \nabla \cdot \mathbf{v}, \\ \psi_{,z} &= \mathbf{v} \cdot \mathbf{k} \quad \text{at } z = 1, 0.\end{aligned}\tag{7}$$

For $(\mathbf{u}, \theta) \in (C^2(\bar{\Omega}) \times C^2(\bar{\Omega})) \cap (\mathbf{J}(\Omega) \times L^2(\Omega))$, multiplying (1) by $\mathbf{u}$, (3) by θ , integrating over Ω , and using (4), we have

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}\|^2 = -\|\nabla \mathbf{u}\|^2 - P_0 \|u\|^2 + R \langle \theta w \rangle, \tag{8}$$

$$\frac{1}{2} P_r \frac{d}{dt} \|\theta\|^2 = -\|\nabla \theta\|^2 + R \langle \theta w \rangle, \tag{9}$$

where $\|\cdot\|$ denotes the $L^2(\Omega)$ (or $L^2(\Omega)$) norm, and $\langle \cdot \rangle$ denotes the integral over Ω .

Multiplying (1) by $\Pi(\mathbf{k} \times \Pi(\mathbf{u} \times \mathbf{k}))$, and then by $\Pi(\mathbf{k} \times \Pi(\theta \mathbf{k}))$ respectively, integrating over Ω , we obtain

$$\begin{aligned}\frac{1}{2} \frac{d}{dt} \|\Pi(\mathbf{u} \times \mathbf{k})\|^2 &= -\|\mathbf{u}_{,z}\|^2 - P_0 \|\Pi(\mathbf{u} \times \mathbf{k})\|^2 + R \langle \theta \mathbf{k} \cdot \Pi(\mathbf{k} \times \Pi(\mathbf{u} \times \mathbf{k})) \rangle \\ &\quad - \frac{1}{\phi} \langle (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \Pi(\mathbf{k} \times \Pi(\mathbf{u} \times \mathbf{k})) \rangle,\end{aligned}\tag{10}$$

$$\begin{aligned}\langle \Pi(\mathbf{u} \times \mathbf{k})_t \cdot (\theta \mathbf{k}) \rangle &= -\langle \text{curl}(\mathbf{u} \times \mathbf{k}) \cdot \text{curl}(\theta \mathbf{k}) \rangle - P_0 \langle \Pi(\mathbf{u} \times \mathbf{k}) \cdot (\theta \mathbf{k}) \rangle \\ &\quad + T \langle (\mathbf{u} \times \mathbf{k}) \cdot \Pi(\mathbf{k} \times \Pi(\theta \mathbf{k})) \rangle - \frac{1}{\phi} \langle (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \Pi(\mathbf{k} \times \Pi(\theta \mathbf{k})) \rangle,\end{aligned}\tag{11}$$

where the properties in (6) and (7) are used to derive these equations.

Multiplying (3) by $\Pi(\mathbf{u} \times \mathbf{k}) \cdot \mathbf{k}$ and integrating over Ω , we also have

$$\begin{aligned}P_r \langle \theta_t \Pi(\mathbf{u} \times \mathbf{k}) \cdot \mathbf{k} \rangle &= -\langle \text{curl}(\mathbf{u} \times \mathbf{k}) \cdot \text{curl}(\theta \mathbf{k}) \rangle + R \langle \Pi(\mathbf{u} \times \mathbf{k}) \cdot \mathbf{k} w \rangle \\ &\quad - P_r \langle (\mathbf{u} \cdot \nabla \theta) \Pi(\mathbf{u} \times \mathbf{k}) \cdot \mathbf{k} \rangle.\end{aligned}\tag{12}$$

Finally, multiplying (1) by $\Delta \mathbf{u}$, (3) by $\Delta \theta$, and integrating over Ω , we get

$$\frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{u}\|^2 = -\|\Delta \mathbf{u}\|^2 - P_0 \|\nabla \mathbf{u}\|^2 - R \langle \Delta w \theta \rangle + \frac{1}{\phi} \langle (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \Delta \mathbf{u} \rangle, \tag{13}$$

$$\frac{1}{2} P_r \frac{d}{dt} \|\nabla \theta\|^2 = -\|\Delta \theta\|^2 - R \langle \Delta \theta w \rangle - P_r \langle \mathbf{u} \cdot \nabla \theta \Delta \theta \rangle. \tag{14}$$

We now choose the same energy functional as constructed by Galdi and Padula (1990, p. 246)

$$E = \frac{1}{2} (\|\mathbf{u}\|^2 + P_r \|\theta\|^2 + \lambda_2 \|\Pi(\mathbf{u} \times \mathbf{k})\|^2 + 2\lambda P_r^{\frac{1}{2}} \langle \Pi(\mathbf{u} \times \mathbf{k}) \cdot \theta \mathbf{k} \rangle), \tag{15}$$

$$E_1 = \frac{1}{2} (\|\mathbf{u}\|^2 + P_r \|\theta\|^2 + \|\nabla \mathbf{u}\|^2 + P_r \|\nabla \theta\|^2), \tag{16}$$

and write

$$V = E + \eta E_1 \tag{17}$$

where λ, λ_2, η are the positive coupling parameters. To ensure that E is positive-definite, the condition

$$\lambda_2 > \lambda^2 \tag{18}$$

must be imposed. Using (15)–(17), together with all the results in (8)–(14), we arrive at

$$dV/dt = F(u) - D(u) + N(u) + \eta(F_1(u) - D_1(u) + N_1(u)), \quad (19)$$

where

$$\begin{aligned} F(u) = & 2R\langle\theta w\rangle - \lambda P_r^{1/2}(1 + 1/P_r)\langle\text{curl}(\mathbf{u} \times \mathbf{k}) \cdot \text{curl}(\theta \mathbf{k})\rangle \\ & - \lambda P_0 P_r^{1/2}\langle\Pi(\mathbf{u} \times \mathbf{k}) \cdot \theta \mathbf{k}\rangle + \lambda \frac{R}{P_r^{1/2}}\langle\Pi(\mathbf{u} \times \mathbf{k}) \cdot \mathbf{k} w\rangle \end{aligned} \quad (20)$$

$$\begin{aligned} & + \lambda P_r^{1/2} T\langle(\mathbf{u} \times \mathbf{k}) \cdot \Pi(\mathbf{k} \times \Pi(\theta \mathbf{k}))\rangle + \lambda_2 R\langle\theta \mathbf{k} \cdot \Pi(\mathbf{k} \times \Pi(\mathbf{u} \times \mathbf{k}))\rangle, \\ D(u) = & \|\nabla \mathbf{u}\|^2 + \|\nabla \theta\|^2 + \lambda_2 \|\mathbf{u}_{,z}\|^2 + P_0 \|\mathbf{u}\|^2 + \lambda_2 P_0 \|\Pi(\mathbf{u} \times \mathbf{k})\|^2, \end{aligned} \quad (21)$$

$$\begin{aligned} N(u) = & -\frac{\lambda_2}{\phi}\langle(\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \Pi(\mathbf{k} \times \Pi(\mathbf{u} \times \mathbf{k}))\rangle - \frac{\lambda P_r^{1/2}}{\phi}\langle(\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \Pi(\mathbf{k} \times \Pi(\theta \mathbf{k}))\rangle \\ & - \lambda P_r^{1/2}\langle(\mathbf{u} \cdot \nabla \theta)(\Pi(\mathbf{u} \times \mathbf{k}) \cdot \mathbf{k})\rangle, \end{aligned} \quad (22)$$

$$F_1(u) = 2R\langle\nabla \theta \cdot \nabla w\rangle + 2R\langle\theta w\rangle, \quad (23)$$

$$D_1(u) = \|\Delta \mathbf{u}\|^2 + \|\Delta \theta\|^2 + \|\nabla \mathbf{u}\|^2 + \|\nabla \theta\|^2 + P_0 \|\nabla \mathbf{u}\|^2 + P_0 \|\mathbf{u}\|^2, \quad (24)$$

$$N_1(u) = -\frac{1}{\phi}\langle(\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \Delta \mathbf{u}\rangle - P_r\langle(\mathbf{u} \cdot \nabla \theta)\Delta \theta\rangle. \quad (25)$$

On using the properties of operator Π in (6) and (7), we can write

$$\Pi(\mathbf{k} \times \Pi(\mathbf{u} \times \mathbf{k})) = \mathbf{k} \times (\mathbf{u} \times \mathbf{k}) - \mathbf{k} \times \nabla \zeta - \nabla \zeta_1 \quad (26)$$

with

$$\begin{aligned} \Delta \zeta = & \nabla \cdot (\mathbf{u} \times \mathbf{k}) = v_{,x} - u_{,y}, \\ \zeta_{,z} = & 0 \quad \text{at } z = 0, 1, \end{aligned} \quad (27)$$

and

$$\begin{aligned} \Delta \zeta_1 = & \nabla \cdot (\mathbf{k} \times (\mathbf{u} \times \mathbf{k}) - \mathbf{k} \times \nabla \zeta) = u_{,x} + v_{,y}, \\ \zeta_{1,z} = & 0 \quad \text{at } z = 0, 1. \end{aligned} \quad (28)$$

We also write

$$\Pi(\mathbf{k} \times \Pi(\theta \mathbf{k})) = \mathbf{k} \times \Pi(\theta \mathbf{k}) = \mathbf{k} \times (\theta \mathbf{k} - \nabla \tau) = -(\mathbf{k} \times \nabla) \tau \quad (29)$$

with

$$\Delta \tau = \theta_{,z}, \quad \tau_{,z} = 0 \quad \text{at } z = 0, 1. \quad (30)$$

We then observe that

$$\langle\theta \mathbf{k} \cdot \Pi(\mathbf{k} \times \Pi(\mathbf{u} \times \mathbf{k}))\rangle = -\langle\theta \zeta_{1,z}\rangle = \langle\theta_{,z} \zeta_1\rangle, \quad (31)$$

and

$$\begin{aligned} \langle(\mathbf{u} \times \mathbf{k}) \cdot \Pi(\mathbf{k} \times \Pi(\theta \mathbf{k}))\rangle & = \langle(v\tau_{,y} + u\tau_{,x})\rangle = -\langle\tau(u_{,x} + v_{,y})\rangle \\ & = -\langle\tau \Delta \zeta_1\rangle = -\langle(\Delta \tau) \zeta_1\rangle = -\langle\theta_{,z} \zeta_1\rangle. \end{aligned} \quad (32)$$

With the choice

$$\lambda_2 = \lambda T P_r^{1/2} / R \quad (33)$$

and using (31) and (32), we eliminate the last two terms in (20), which in general are not of one sign and may destabilize the solution. In view of the condition (18), we are led to choose

$$\lambda = \xi T P_r^{1/2} / R, \quad \xi \in (0, 1). \quad (34)$$

4. A priori estimation of $\dot{V}$. We now need an a priori estimate of $\dot{V}$ to find the sufficient conditions which ensure that the energy V decays monotonically to zero.

We make the following observations:

$$\begin{aligned} N(u) &\leq \frac{\lambda_2}{\phi} \sup_{\Omega} |u| \|\nabla \mathbf{u}\| \|\Pi(\mathbf{u} \times \mathbf{k})\| + \frac{\lambda P_r^{1/2}}{\phi} \sup_{\Omega} |u| \|\nabla \mathbf{u}\| \|\Pi(\theta \mathbf{k})\| \\ &\quad + \lambda P_r^{1/2} \sup_{\Omega} |u| \|\nabla \theta\| \|\Pi(\mathbf{u} \times \mathbf{k})\| \\ &\leq \frac{\lambda_2 c}{\phi} \|\Delta \mathbf{u}\| \|\nabla \mathbf{u}\| \|\Pi(\mathbf{u} \times \mathbf{k})\| + \frac{\lambda c}{\phi} P_r^{1/2} \|\Delta \mathbf{u}\| \|\nabla \mathbf{u}\| \|\theta\| \\ &\quad + \lambda c P_r^{1/2} \|\Delta \mathbf{u}\| \|\nabla \theta\| \|\Pi(\mathbf{u} \times \mathbf{k})\| \\ &\leq \left(\frac{c \lambda_2}{\phi} + \frac{c \lambda \lambda_2^{1/2}}{\phi} + c \lambda p_r^{1/2} \right) \frac{1}{(2\lambda_2 - 2\lambda^2)^{1/2}} D_1 E^{1/2} \\ &= b D_1 E^{1/2}, \end{aligned} \quad (35)$$

where

$$b = \frac{c \lambda_2}{\phi (2\lambda_2 - 2\lambda^2)^{1/2}} + \frac{c \lambda (\lambda_2)^{1/2}}{\phi (2\lambda_2 - 2\lambda^2)^{1/2}} + \frac{c \lambda P_r^{1/2}}{(2\lambda_2 - 2\lambda^2)^{1/2}} \quad (36)$$

and

$$\begin{aligned} \|\Pi(\mathbf{u} \times \mathbf{k})\| &\leq \left(\frac{2}{\lambda_2 - \lambda^2} \right)^{1/2} E^{1/2}, \quad \|\theta\| \leq \left(\frac{2\lambda_2}{(\lambda_2 - \lambda^2) P_r} \right)^{1/2} E^{1/2}, \\ \sup_{\Omega} |u| &\leq c \|\Delta \mathbf{u}\| \end{aligned} \quad (37)$$

have been used with

$$c = \frac{3^{1/2}}{2^{1/4} [\pi h^3 (\sqrt{2} - 1)]^{1/2}} + \frac{40h^{3/5}}{3}, \quad h = \min \{a_1, a_2, 1\} \quad (38)$$

(cf. Galdi and Straughan (1985, p. 267)). We also observe that

$$\begin{aligned} N_1(u) &\leq \sup_{\Omega} |u| \left(\frac{1}{\phi} \|\nabla \mathbf{u}\| \|\Delta \mathbf{u}\| + P_r \|\nabla \theta\| \|\Delta \theta\| \right) \\ &\leq \sqrt{2} c \left(\frac{1}{\phi} + \frac{1}{2} P_r^{1/2} \right) E_1^{1/2} D_1, \end{aligned} \quad (39)$$

and

$$\begin{aligned}
 F_1(u) &\leq 2R\|w\|\|\Delta\theta\| + 2R\|w\|\|\theta\| \\
 &\leq \frac{2}{\pi}R\|w\|_{,z} \left(\|\Delta\theta\| + \frac{1}{\pi}\|\nabla\theta\| \right) \\
 &\leq \frac{1}{2}(\|\nabla\theta\|^2 + \|\Delta\theta\|^2) + 2R^2\frac{\pi^2+1}{\pi^4}\|u_{,z}\|^2 \\
 &\leq \frac{1}{2}D_1 + \frac{2R^2(\pi^2+1)}{\lambda_2\pi^4}D,
 \end{aligned} \tag{40}$$

or

$$\begin{aligned}
 F_1(u) &\leq \frac{2R}{\pi}\|\nabla w\| \left(\|\Delta\theta\| + \frac{1}{\pi}\|\nabla\theta\| \right) \\
 &\leq \frac{1}{2}D_1 + \frac{2R^2(1+\pi^2)}{\pi^4}D,
 \end{aligned} \tag{41}$$

where the Poincaré inequalities have been used in the derivation of the above results.

We now define

$$m = \max_{\varphi} \frac{F}{D}, \tag{42}$$

where φ is a set of the admissible functions,

$$\varphi = \{u, \theta \text{ which are periodic in } x \text{ and } y \text{ and satisfy (1)–(4)}\}. \tag{43}$$

We suppose first that $T > 1$; using (35)–(40) and (42) in (19) yields

$$\begin{aligned}
 \dot{V} &\leq \left\{ (m-1) + \frac{2\eta R^2(\pi^2+1)}{\lambda_2\pi^4} \right\} D - \frac{1}{2}\eta D_1 \\
 &\quad + bD_1E^{1/2} + \sqrt{2}\eta c \left(\frac{1}{\phi} + \frac{1}{2}P_r^{1/2} \right) E^{1/2}D_1.
 \end{aligned} \tag{44}$$

On choosing

$$\eta = \frac{\lambda_2\pi^4(1-m)}{4R^2(\pi^2+1)} \tag{45}$$

and setting

$$\mathcal{D} = \frac{1}{2}[(1-m)D + \eta D_1], \tag{46}$$

we conclude that

$$\dot{V} \leq -\mathcal{D}(1 - AV^{1/2}), \tag{47}$$

where

$$A = 2\frac{b}{\eta} + \frac{2\sqrt{2}c(\frac{1}{\phi} + \frac{1}{2}P_r^{1/2})}{\eta^{1/2}}. \tag{48}$$

By virtue of the Poincaré (Wirtinger) inequality (Galdi and Straughan, 1985), we also have

$$\mathcal{D} \geq \frac{(1-m)\pi^2}{2P^*}V \tag{49}$$

where

$$P^* = \begin{cases} 1 & \text{if } P_r \leq 1, \\ P_r & \text{if } P_r > 1. \end{cases} \tag{50}$$

Then from (47) we may show that

$$V(t) \leq V(0) \exp \left\{ -\frac{\pi^2(1-m)}{2P^*} [1 - AV^{1/2}(0)]t \right\}, \quad (51)$$

which means that $V(t) \rightarrow 0$ in a monotonic fashion as $t \rightarrow \infty$, whenever

$$V(0) < A^{-2}. \quad (52)$$

For $T \leq 1$ we use (41) to replace (40), then get the estimation in the form

$$\begin{aligned} \dot{V} \leq & \left\{ (m-1) + \frac{2\eta R^2(\pi^2+1)}{\pi^4} \right\} D - \frac{1}{2}\eta D_1 \\ & + b_1 D_1 E^{1/2} + \sqrt{2}\eta c \left(\frac{1}{\phi} + \frac{1}{2}P_r^{1/2} \right) E_1^{1/2} D_1, \end{aligned} \quad (53)$$

where b_1 coincides with the value of b for $T = 1$. From (53), it follows that (51) holds with

$$A = \frac{2b_1}{\eta} + \frac{2\sqrt{2}c(\frac{1}{\phi} + \frac{1}{2}P_r^{1/2})}{\eta^{1/2}} \quad (54)$$

and

$$\eta = \frac{\pi^4(1-m)}{4R^2(1+\pi^2)}. \quad (55)$$

When $V(0) < A^{-2}$, we again obtain monotonic stability.

5. Variational problem. We now solve the variational problem associated with the maximum problem (42). The corresponding Euler-Lagrange equations are

$$\begin{aligned} m\Delta \mathbf{u} - mP_0 \mathbf{u} + m\lambda_2 \mathbf{u}_{,zz} - m\lambda_2 P_0 (\mathbf{k} \times \Pi(\mathbf{u} \times \mathbf{k})) + R\theta \mathbf{k} \\ - \frac{\lambda(1+P_r)}{2\sqrt{P_r}} \mathbf{k} \times \text{curl curl}(\theta \mathbf{k}) - \frac{\lambda P_0 \sqrt{P_r}}{2} \mathbf{k} \times \Pi(\theta \mathbf{k}) \\ + \frac{\lambda R}{2\sqrt{P_r}} (\Pi(\mathbf{u} \times \mathbf{k}) \cdot \mathbf{k}) \mathbf{k} + \mathbf{k} \times \Pi(\mathbf{k}w) = \nabla p, \end{aligned} \quad (56)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (57)$$

$$m\Delta \theta + Rw - \frac{\lambda(1+P_r)}{2\sqrt{P_r}} \text{curl curl}(\mathbf{u} \times \mathbf{k}) \cdot \mathbf{k} - \frac{\lambda}{2} P_0 \sqrt{P_r} \Pi(\mathbf{u} \times \mathbf{k}) \cdot \mathbf{k} = 0. \quad (58)$$

Using the variables defined by (27) and (30) for ζ , τ respectively, and a variable ψ which is associated with $\Pi(\mathbf{k}w)$ in the form of

$$\Pi(\mathbf{k}w) = \mathbf{k}w - \nabla \psi$$

with

$$\Delta \psi = w_{,z}, \quad \psi_{,z} = 0 \quad \text{at } z = 0, 1, \quad (59)$$

we can write (56) and (58) as

$$\begin{aligned}
 m\Delta \mathbf{u} - mP_0 \mathbf{u} + m\lambda \frac{T\sqrt{P_r}}{R} \mathbf{u}_{,zz} - m\lambda \frac{T\sqrt{P_r}}{R} P_0 (\mathbf{u} - w\mathbf{k} - (\mathbf{k} \times \nabla)\zeta) \\
 + R\theta \mathbf{k} + \lambda \frac{(1+P_r)}{2\sqrt{P_r}} (\nabla \theta_{,z} \times \mathbf{k}) + \frac{\lambda}{2} P_0 \sqrt{P_r} (\mathbf{k} \times \nabla \tau) \\
 + \frac{\lambda R}{2\sqrt{P_r}} (-\zeta_{,z} \mathbf{k} - \mathbf{k} \times \nabla \psi) = \nabla p,
 \end{aligned} \quad (60)$$

$$m\Delta \theta + R w - \frac{\lambda(1+P_r)}{2\sqrt{P_r}} \omega_{,z} + \frac{\lambda}{2} P_0 \sqrt{P_r} \zeta_{,z} = 0, \quad (61)$$

where $\omega = (\nabla \times \mathbf{u}) \cdot \mathbf{k}$. Operating on (60) with $\mathbf{k} \cdot \Delta \text{curl}$ and then with $\mathbf{k} \cdot \Delta \text{curl curl}$, operating on (61) with Δ , and using (33), we arrive at

$$\begin{aligned}
 m\Delta^2 \omega - mP_0 \Delta \omega + m \frac{\lambda T \sqrt{P_r}}{R} \Delta \omega_{,zz} - m \frac{\lambda T \sqrt{P_r} P_0}{R} \omega_{,zz} \\
 - \lambda \frac{(1+P_r)}{2\sqrt{P_r}} \Delta \Delta_1 \theta_{,z} + \frac{\lambda P_0 \sqrt{P_r}}{2} \Delta_1 \theta_{,z} - \frac{\lambda R}{2\sqrt{P_r}} \Delta_1 w_{,z} = 0,
 \end{aligned} \quad (62)$$

$$\begin{aligned}
 m\Delta^3 w - mP_0 \Delta^2 w + m \frac{\lambda T \sqrt{P_r}}{R} \Delta^2 w_{,zz} - m \frac{\lambda T \sqrt{P_r} P_0}{R} \Delta w_{,zz} \\
 + R \Delta \Delta_1 \theta + \frac{\lambda R}{2\sqrt{P_r}} \Delta_1 w_{,z} = 0,
 \end{aligned} \quad (63)$$

$$m\Delta^2 \theta + R \Delta w - \frac{\lambda(1+P_r)}{2\sqrt{P_r}} \Delta \omega_{,z} + \frac{\lambda}{2} P_0 \sqrt{P_r} \omega_{,z} = 0. \quad (64)$$

We remark that if we set $P_0 = 0$ in the above system of equations (62)–(64), these reduce to the same set of equations as studied by Galdi and Padula (1990). Eliminating variables θ and ω yields

$$\begin{aligned}
 \Delta(\Delta - P_0)[\Delta + \lambda T \sqrt{P_r}/R(D^2 - P_0)] \\
 \cdot \left\{ \left(\frac{\lambda(1+P_r)}{2\sqrt{P_r}} \Delta - \frac{\lambda P_0 \sqrt{P_r}}{2} \right)^2 \Delta_1 D^2 - m^2 \Delta^2 (\Delta - P_0) \left(\Delta + \frac{\lambda T \sqrt{P_r}}{R} D^2 \right) \right\} w \\
 + \left\{ -\frac{\lambda^2 R^2}{4P_r} \Delta^2 \Delta_1 D^2 + R^2 \Delta^2 \Delta_1 (\Delta - P_0) \left(\Delta + \frac{\lambda T \sqrt{P_r}}{R} D^2 \right) \right\} w = 0,
 \end{aligned} \quad (65)$$

where $D = \partial/\partial z$ and $\Delta_1 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

On employing standard normal mode analysis we may write $w = W(z)e^{2i(a_1x+a_2y)}$. The boundary conditions (4) together with the equations (1)–(3) can produce $D^{(2l)}W = 0$ at $z = 0, 1$ for $l = 1, 2, \dots$. This indicates that $W(z)$ can be assumed to have the form $W(z) = H \sin r\pi z$, for $r = 1, 2, \dots$ and H an arbitrary constant. Substituting these functions into equation (65) and noting (34), we obtain

$$\begin{aligned} & - (r^2\pi^2 + a^2)(r^2\pi^2 + a^2 + P_0) \left(r^2\pi^2 + a^2 + \frac{\xi T^2 P_r}{R^2} r^2\pi^2 \right) \\ & \times \left\{ \frac{\xi^2 T^2}{4R^2} [(1 + P_r)(r^2\pi^2 + a^2) + P_0 P_r]^2 r^2\pi^2 a^2 \right. \\ & \left. - m^2 (r^2\pi^2 + a^2)^2 (r^2\pi^2 + a^2 + P_0) \left(r^2\pi^2 + a^2 + \frac{\xi T^2 P_r}{R^2} r^2\pi^2 \right) \right\} \\ & + \left\{ \frac{\xi^2 T^2}{4} (r^2\pi^2 + a^2)^2 r^2\pi^2 a^4 - R^2 a^2 (r^2\pi^2 + a^2)^2 (r^2\pi^2 + a^2 + P_0) \right. \\ & \left. \times \left(r^2\pi^2 + a^2 + \frac{\xi T^2 P_r}{R^2} r^2\pi^2 \right) \right\} = 0, \end{aligned} \quad (66)$$

where $a^2 = 4(a_1^2 + a_2^2)$.

For convenience, we rewrite the above equation as a polynomial equation in R^2 in the form

$$p(R^2, m) = R^6 + A_1(m)R^4 + B_1(m)R^2 + C_1(m) = 0, \quad (67)$$

where

$$\begin{aligned} A_1(m) &= -m^2 r^4 \pi^4 \frac{(1 + \alpha)^2 (1 + \alpha + \beta)}{\alpha} + \frac{\xi T^2 P_r}{(1 + \alpha)} \\ &\quad - \frac{\xi^2 T^2}{4} \frac{\alpha}{(1 + \alpha)(1 + \alpha + \beta)}, \\ B_1(m) &= r^4 \pi^4 \left\{ -2m^2 \xi T^2 P_r \frac{(1 + \alpha)(1 + \alpha + \beta)}{\alpha} \right. \\ &\quad \left. + \frac{\xi^2 T^2}{4} \frac{[(1 + P_r)(1 + \alpha) + \beta P_r]^2}{(1 + \alpha)} \right\}, \\ C_1(m) &= r^4 \pi^4 \left\{ -m^2 \xi^2 T^4 P_r^2 \frac{(1 + \alpha + \beta)}{\alpha} \right. \\ &\quad \left. + \frac{\xi^3 T^4 P_r}{4} \frac{[(1 + P_r)(1 + \alpha) + \beta P_r]^2}{(1 + \alpha)^2} \right\}, \end{aligned} \quad (68)$$

with

$$\alpha = a^2/(r^2\pi^2), \quad \beta = P_0/(r^2\pi^2). \quad (69)$$

Choosing

$$\xi < \frac{4P_r(1+\alpha)^2(1+\alpha+\beta)}{\alpha[(1+P_r)(1+\alpha)+\beta P_r]^2}, \quad (70)$$

and denoting $A = A_1(1)$, $B = B_1(1)$, $C = C_1(1)$, we observe that there is one and only one positive number R_c^2 satisfying the equation

$$P(R^2, 1) = R^6 + AR^4 + BR^2 + C = 0, \quad (71)$$

and

$$P(R^2, 1) < 0 \quad \text{for any } R^2 < R_c^2, \quad (72)$$

because both B and C are negative under the assumption of (70).

Moreover, a straightforward calculation shows that

$$\frac{dA_1(m)}{dm} < A, \quad \frac{dB_1(m)}{dm} < B, \quad \frac{dC_1(m)}{dm} < C, \quad (73)$$

for any $m \geq 1$, where the restriction in (70) was used to derive the first inequality in (73).

Therefore, a comparison shows for any $R^2 < R_c^2$

$$\frac{dP(R^2, m)}{dm} < P(R^2, 1) < 0, \quad (74)$$

for any $m \geq 1$. This implies that $P(R^2, m) < 0$ if $m \geq 1$ for any $R^2 < R_c^2$, while R_c^2 is a positive root of equation (71) at the critical argument $m = 1$. We thus conclude that $R < R_c$ implies $m < 1$. We also remark here that (70) will certainly be satisfied if the further choice

$$\xi < \frac{4P_r(1+\alpha)}{\alpha(1+P_r)^2(1+\beta)} \quad (75)$$

is required.

With the restriction in (75), it is also straightforward to verify that $r = 1$ minimizes R_c . Actually taking differentiation with respect to r^2 on both sides of (71), we reduce the equation to

$$(3R_c^4 + 2AR_c^2 + B)R_c^{2'} = -A'R_c^4 - B'R_c^2 - C', \quad (76)$$

where the prime denotes differentiation with respect to r^2 . An inspection of expres-

sions for A , B , C and the observations

$$\begin{aligned}
 A' &= \frac{\partial A}{\partial r^2} + \frac{\partial A}{\partial \beta} \frac{\partial \beta}{\partial r^2} = \frac{\partial A}{\partial r^2} + \frac{\partial A}{\partial \beta} \left(-\frac{\beta}{r^2} \right) \\
 &= -2r^2 \pi^4 \frac{(1+\alpha)^2(1+\alpha+\beta)}{\alpha} + r^2 \pi^4 \frac{(1+\alpha)^2 \beta}{\alpha} - \frac{\xi^2 T^2 \alpha \beta}{(1+\alpha)(1+\alpha+\beta)^2 r^2} \\
 &= -r^2 \pi^4 \frac{(1+\alpha)^2(1+\alpha+\beta)}{\alpha} - r^2 \pi^4 \frac{(1+\alpha)^3}{\alpha} - \frac{\xi^2 T^2 \alpha \beta}{(1+\alpha)(1+\alpha+\beta)^2 r^2} < 0, \\
 B' &= \frac{\partial B}{\partial r^2} + \frac{\partial B}{\partial \beta} \left(-\frac{\beta}{r^2} \right) \\
 &= r^2 \pi^4 \left\{ -2\xi T^2 P_r \frac{(1+\alpha)(1+\alpha+\beta)}{\alpha} + \xi^2 T^2 \frac{[(1+P_r)(1+\alpha) + \beta P_r]^2}{4(1+\alpha)} \right\} \\
 &\quad + r^2 \pi^4 \left\{ -2\xi T^2 P_r \frac{(1+\alpha)^2}{\alpha} + \frac{\xi^2 T^2}{4} (1+P_r)^2 (1+\alpha) - \frac{\gamma^2 \pi^2 \xi^2 T^2}{4(1+\alpha)} \beta^2 P_r^2 \right\}, \\
 C' &= \frac{\partial C}{\partial r^2} + \frac{\partial C}{\partial \beta} \left(-\frac{\beta}{r^2} \right) \\
 &= r^2 \pi^4 \left\{ -\xi^2 T^4 P_r^2 \frac{(1+\alpha+\beta)}{\alpha} + \frac{\xi^3 T^4 P_r [(1+P_r)(1+\alpha) + \beta P_r]^2}{4(1+\alpha)^2} \right\} \\
 &\quad + r^2 \pi^4 \left\{ -\xi^2 T^4 P_r^2 \frac{(1+\alpha)}{\alpha} + \frac{\xi^3 T^4 P_r}{4} (1+P_r)^2 \right\} - r^2 \pi^4 \frac{\xi^3 T^4 P_r}{4} \frac{\beta^2 P_r^2}{(1+\alpha)^2} \quad (77)
 \end{aligned}$$

together with the restriction of the choice in (75) show that the right-hand side of (76) is positive while the term in the brackets on the left side is positive. We thus conclude that $\partial R_c^2 / \partial (r^2) > 0$ and the minimum is attained at $r = 1$. The critical energy bound will be defined by

$$R_e^2 = \min_{\alpha} \max_{\xi} R_c^2(\alpha, \xi, P_0, P_r, T^2). \quad (78)$$

Equation (78) has been solved numerically, and Tables 1 and 2 give several values of R_e^2 as a function of T and P_0 . The values of P_r are, however, fixed and, for comparison purposes, are taken to be the same as those of Galdi and Padula (1990). Since we are concerned with high porosity systems only, comparatively small values for P_0 are considered. Except with a minor difference in one of the values, our calculations completely agree with the results given by Galdi and Padula (1990) when the parameter $P_0 = 0$. We point out that for the low porosity system, P_0 becomes very large and the coupling parameter ξ has to be very small due to the choice of (75). In this case R_e^2 tends to $P_0 \pi^2 / 4$, which is the critical value without rotation.

From the tables we note that as T increases, the critical energy bound also increases. For the range of values of the Taylor number in the tables, the effect of porosity is to stabilize the system; that is, as P_0 increases so does the value of R_e^2 .

Table 1. Energy Critical Rayleigh Numbers

$$P_r = 6.8$$

T^2	$P_0 = 0.0$ R_e^2	$P_0 = 0.1$ R_e^2	$P_0 = 1.0$ R_e^2	R_{G-P}^2
0	657.511	661.950	701.690	657.511
10	672.381	676.697	715.453	672.609
50	723.235	727.254	763.453	723.236
100	775.105	778.921	813.305	775.104
200	858.523	862.115	894.579	858.514
300	926.967	930.251	961.702	926.750
400	985.907	989.332	1020.11	1001.92
1000	1249.95	1253.02	1282.44	1249.83

R_{G-P}^2 -results obtained by Galdi and Padula

Table 2. Energy Critical Rayleigh Numbers

$$P_r = 200$$

T^2	$P_0 = 0.0$ R_e^2	$P_0 = 0.1$ R_e^2	$P_0 = 1.0$ R_e^2	$P_0 = 1.0$ R_{G-P}^2
0	657.511	661.950	701.690	657.511
10	676.37	680.637	718.966	676.75
50	738.95	742.805	777.592	738.94
100	800.99	804.574	836.988	800.99
250	940.09	943.304	972.407	
500	1106.76	1109.74	1136.60	
1000	1347.78	1350.54	1375.55	1347.78
2000	1690.67	1693.28	1716.90	
5000	2369.71	2371.79	2394.14	

R_{G-P}^2 -results obtained by Galdi and Padula

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