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EXISTENCE OF CLASSICAL SOLUTIONS FOR SINGULAR PARABOLIC PROBLEMS

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Abstract. Let $Lu \equiv u_{xx} + bu_x/x - u_t$ with b a constant less than 1. Its Green's function corresponding to first boundary conditions is constructed by eigenfunction expansion. With this, a representation formula is established to obtain existence of a classical solution for the linear first initial-boundary value problem. Uniqueness of a solution follows from the strong maximum principle. Properties of Green's function and of the solution are also investigated.

1. Introduction. Let

$$L \equiv \frac{\partial^2}{\partial x^2} + \frac{b}{x}\frac{\partial}{\partial x} - \frac{\partial}{\partial t}.$$

We are interested in studying existence and uniqueness of classical solutions for linear initial-boundary value problems involving L. This operator arises in many situations, such as degenerate elliptic-parabolic operators (cf. Brezis, Rosenkrantz, and Singer with an appendix by Lax [2]), stochastic processes (cf. Lamperti [14]), and phase change processes (cf. Solomon [18]). When $b = 0$, it is the heat operator. For further discussions of the study and the significance of L, we refer to Chan and Chen [5, 6], Chan and Cobb [7], Chan and Kaper [8], and the references cited there.

Without loss of generality and for simplicity, we take the spatial interval to be $[0, 1]$. Let b (< 1) and Γ (> 0) be constants, $\Omega_\Gamma \equiv (0, 1) \times (0, \Gamma)$, $Q_\Gamma \equiv (0, 1) \times (0, \Gamma]$, $Q_\Gamma^- \equiv (0, 1) \times [0, \Gamma]$, and $\overline{Q}_\Gamma$ denote the closure of Q_Γ. We study the linear singular problem,

$$Lu = -\Psi(x, t) \quad \text{in } Q_\Gamma, \tag{1.1}$$

$$u(x, 0) = g(x) \quad \text{for } 0 \le x \le 1, \qquad u(0, t) = 0 = u(1, t) \quad \text{for } 0 < t \le \Gamma. \tag{1.2}$$

More general linear problems with b a real constant were investigated by Alexiades [1]. Hence, an existence result for the above problem can be deduced from his work [1, Sec. 11]. For $b < 1$, he assumed that $x^{b-1}\Psi(x, t)$ is in $C(\overline{Q}_\Gamma)$; we note that if the solution u were known, the function $x^{b-1}[1-u(x, t)]^{-1}$ would be discontinuous at $x = 0$, and thus would not satisfy his assumption in the case $b < 1$. Hence, his

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(linear) result cannot be used through methods of successive approximations to study semilinear singular problems of the type,

$$v_{xx} - v_t = -(1-v)^{-1} \quad \text{in } \Omega_T,$$
$$v(x,0) = g(x) \quad \text{for } 0 \le x \le 1, \qquad v(0,t) = 0 = v(1,t) \quad \text{for } 0 < t < T \le \infty.$$

This problem with $g(x) \equiv 0$ was studied by Kawarada [12], through which he introduced the concept of quenching. Since then, many scientists have studied quenching problems (cf. Chan [4]).

In Sec. 2, we construct explicitly Green's function corresponding to the problem (1.1) and (1.2). Under appropriate conditions on $g(x)$ and $\Psi(x,t)$ (without assuming $x^{b-1}\Psi(x,t)$ is in $C(\overline{Q}_\Gamma)$), we prove existence of a unique classical solution by establishing its representation formula. We also establish properties of Green's function and of the solution. In Sec. 3, we extend existence of a unique classical solution to nonhomogeneous boundary conditions.

2. Linear problem. Using separation of variables on the homogeneous problem corresponding to the problem (1.1) and (1.2), we obtain the singular Sturm-Liouville problem,

$$(x^b X')' + \lambda x^b X = 0, \qquad X(0) = 0 = X(1),$$

where λ is an eigenvalue. Let $\nu = (1-b)/2$. Since $\nu > 0$, it follows from McLachlan [15, pp. 26 and 116] that the eigenvalues λ are positive and satisfy the equation $J_\nu(\lambda^{1/2}) = 0$, where $J_\nu(z)$ is the Bessel function of the first kind of order ν. For $z > 0$, $J_\nu(z)$ has infinitely many countable zeros; hence, there are infinitely many countable eigenvalues λ_n, which can be arranged as $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ with $\lambda_n \to \infty$ as $n \to \infty$ (cf. Watson [19, pp. 490–492]). The corresponding eigenfunctions,

$$\phi_n(x) = 2^{1/2} x^\nu J_\nu(\lambda_n^{1/2} x)/(|J_{\nu+1}(\lambda_n^{1/2})|),$$

form an orthonormal set with weight function x^b (cf. McLachlan [15, pp. 102–104]).

In the sequel, we let k_j $(j = 1, 2, 3, \ldots, 8)$ denote appropriate constants. For simplicity, we introduce the following notations:

$$E_n(y) \equiv \exp(-\lambda_n y),$$
$$I_n(h) \equiv \int_0^1 x^b h(x) \phi_n(x)\, dx.$$

If instead of $h(x)$, we have $h(x,t)$, then we use the notation $I_n(h)(t)$. Similarly, let

$$I(h) \equiv \int_0^1 x^{b/2} h(x)\, dx,$$
$$I^2(h) \equiv \int_0^1 x^b h^2(x)\, dx,$$

and define $I(h)(t)$ and $I^2(h)(t)$ accordingly.

For convenience, we state the following results.

LEMMA 1.

(a) $|\phi_n(x)| \leq k_1 x^{-b/2}$ for x in $(0, 1]$.

(b) $|\phi_n(x)| \leq k_2 \lambda_n^{1/4}$ for x in $[0, 1]$.

(c) If $I^2(h_1)(t) \leq k_3$ for t in $[0, \Gamma_1]$, then for t in $[0, \Gamma_1]$,

$$\sum_{n=1}^{\infty} [I_n(h_1)(t)]^2 \leq I^2(h_1)(t).$$

(d) $|\phi_n'(x)| \leq k_4 \lambda_n^{1/2}$ for x in $[x_0, 1]$ where $x_0 > 0$ and k_4 depends on x_0.

(e) If $I(h_2)$ exists (and is absolutely convergent in case the integral is improper), and if $h_2(x)$ is continuous and of bounded variation on $[x_1, x_2]$, where $0 < x_1 < x_2 < 1$, then $\sum_{n=1}^{\infty} I_n(h_2)\phi_n(x)$ converges uniformly to $h_2(x)$ on $(x_1 + \varepsilon, x_2 - \varepsilon)$ where ε is any positive number.

For the proofs of Lemma 1(a), (b), (d), and (e), we refer to Lemma 1(i) and (ii), (2.15), and Lemma 3 of Chan and Wong [9]. Lemma 1(c) follows directly from the Bessel inequality (cf. Weinberger [20, p. 73]).

Let us construct Green's function $G(x, t; \xi, \tau)$ corresponding to the problem (1.1) and (1.2). It is determined by the following system: for x and ξ in $(0, 1)$, and t and τ in $(-\infty, \infty)$,

$$\begin{gathered} LG(x, t; \xi, \tau) = -\delta(x - \xi)\delta(t - \tau), \\ G(x, t; \xi, \tau) = 0, \quad t < \tau, \\ G(0, t; \xi, \tau) = 0 = G(1, t; \xi, \tau), \end{gathered}$$

where $\delta(x)$ is the Dirac delta function. By the eigenfunction expansion,

$$G(x, t; \xi, \tau) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x).$$

Since

$$\phi_n''(x) + \frac{b}{x}\phi_n'(x) + \lambda_n \phi_n(x) = 0,$$

it follows that

$$\sum_{n=1}^{\infty} [a_n'(t) + \lambda_n a_n(t)]\phi_n(x) = \delta(x - \xi)\delta(t - \tau).$$

Multiplying both sides by $x^b \phi_n(x)$, and integrating from 0 to 1 with respect to x, we obtain

$$\frac{d}{dt}\{[\exp(\lambda_n t)]a_n(t)\} = \xi^b \phi_n(\xi)[\exp(\lambda_n t)]\delta(t - \tau).$$

By integrating from τ^- to t,

$$[\exp(\lambda_n t)]a_n(t) - [\exp(\lambda_n \tau^-)]a_n(\tau^-) = \xi^b \phi_n(\xi)\exp(\lambda_n \tau).$$

Since $G(x, t; \xi, \tau) = 0$ for $t < \tau$, it follows that $a_n(\tau^-) = 0$ for all n. Thus,

$$a_n(t) = \xi^b \phi_n(\xi) E_n(t - \tau),$$

and hence

$$G(x, t; \xi, \tau) = \sum_{n=1}^{\infty} \xi^b \phi_n(\xi)\phi_n(x)E_n(t-\tau). \tag{2.1}$$

Let $D = \{(x, t; \xi, \tau) : x$ and ξ are in $(0, 1)$, and $t > \tau\}$. By Lemma 1(b) and the fact that $O(\lambda_n) = O(n^2)$ for large n (cf. Watson [19, p. 506]), it follows that the series in (2.1) converges in D. Hence, $G(x, t; \xi, \tau)$ exists.

A function u is said to be a classical solution of the problem (1.1) and (1.2) if

(a) u is in $C(\overline{Q}_\Gamma)$,
(b) u_x, u_{xx}, and u_t are in $C(Q_\Gamma)$,
(c) u satisfies (1.1) and (1.2).

Throughout this paper, by a solution of the problem (1.1) and (1.2), we refer to its classical solution.

Let $\Psi(x, t)$ be defined in Q_Γ^-. We need the following conditions:

(A) $I^2(\Psi)(t) \le k_5$ for t in $[0, \Gamma]$,
(B) $I(|\Psi_t|)(t) \le k_6$ a.e. for t in $[0, \Gamma]$.

THEOREM 2. The problem (1.1) and (1.2) has at most one solution. Suppose $\Psi(x, t)$ is in $C(Q_\Gamma^-)$, absolutely continuous on the interval $0 \le t \le \Gamma$ for each x in $(0, 1)$, and of bounded variation with respect to x on every given closed subinterval of $(0, 1)$. If Conditions (A) and (B) hold, then the problem (1.1) and (1.2) with $g \equiv 0$ has a unique solution u given by

$$u(x, t) = \int_0^t \int_0^1 G(x, t; \xi, \tau)\Psi(\xi, \tau)\, d\xi\, d\tau. \tag{2.2}$$

Proof. Uniqueness of a solution follows from the strong maximum principle (cf. Protter and Weinberger [16, pp. 168–170]).

From (2.1) and (2.2),

$$u(x, t) = \int_0^t \int_0^1 \sum_{n=1}^{\infty} \xi^b \phi_n(\xi)\phi_n(x)E_n(t-\tau)\Psi(\xi, \tau)\, d\xi\, d\tau.$$

By Lemma 1(a) and (b), we have for x in $[0, 1]$ and ξ in $(0, 1]$,

$$|\xi^b \phi_n(\xi)\phi_n(x)\Psi(\xi, \tau)| \le k_1 k_2 \lambda_n^{1/4} \xi^{b/2} |\Psi(\xi, \tau)|. \tag{2.3}$$

For any fixed (x, t) in $\overline{Q}_\Gamma$, let

$$G_m(\xi, \tau) = \begin{cases} \sum_{n=1}^{m} \xi^b \phi_n(\xi)\phi_n(x)E_n(t-\tau) & \text{for } t-\tau > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $G_m(\xi, \tau)\Psi(\xi, \tau)$ converges to $G(x, t; \xi, \tau)\Psi(\xi, \tau)$ a.e. on $\overline{Q}_t$. From (2.3),

$$|G_m(\xi, \tau)\Psi(\xi, \tau)| \le \rho(\xi, \tau)$$

for all positive integers m where

$$\rho(\xi, \tau) = \begin{cases} k_1 k_2 \xi^{b/2} |\Psi(\xi, \tau)| \sum_{n=1}^{\infty} \lambda_n^{1/4} E_n(t-\tau) & \text{for } t-\tau > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\rho_m(\xi, \tau)$ be the mth partial sum of $\rho(\xi, \tau)$. Then, $\{\rho_m\}$ is a sequence of nonnegative measurable functions that converge monotonically to ρ on $\overline{Q}_t$ and $\rho_m \leq \rho$ for all positive integers m. By the Monotone Convergence Theorem and the Fubini Theorem (cf. Royden [17, pp. 84 and 269]),

$$\begin{aligned}\int_{Q_t} \rho(\xi, \tau)\, d\xi\, d\tau &= \lim_{m\to\infty} \int_0^t \int_0^1 \rho_m(\xi, \tau)\, d\xi\, d\tau \\ &= \lim_{m\to\infty} k_1 k_2 \sum_{n=1}^{m} \left[\int_0^t I(|\Psi|)(\tau) \lambda_n^{1/4} E_n(t-\tau)\, d\tau \right].\end{aligned}$$

By the Schwarz inequality and Condition (A),

$$\int_{Q_t} \rho(\xi, \tau)\, d\xi\, d\tau \leq k_1 k_2 k_5^{1/2} \lim_{m\to\infty} \sum_{n=1}^{m} \lambda_n^{-3/4}.$$

Since $O(\lambda_n) = O(n^2)$ for large n, it follows that $\sum_{n=1}^{m} \lambda_n^{-3/4}$ converges. Hence, $\rho(\xi, \tau)$ is integrable, and for each fixed (x, t) in $\overline{Q}_\Gamma$, the integral in (2.2) exists. By the Lebesgue Convergence Theorem (cf. Royden [17, p. 88]) and the Fubini Theorem,

$$u(x, t) = \sum_{n=1}^{\infty} \int_0^t I_n(\Psi)(\tau) E_n(t-\tau)\, d\tau \phi_n(x).$$

By Lemma 1(c) and Condition (A),

$$\left| \int_0^t I_n(\Psi)(\tau) E_n(t-\tau)\, d\tau \right| \leq k_5^{1/2} \lambda_n^{-1}.$$

It follows from Lemma 1(b) that the series representing $u(x, t)$ converges absolutely and uniformly on $\overline{Q}_\Gamma$. Thus, $u(x, t)$ is in $C(\overline{Q}_\Gamma)$, and hence $u(x, t)$ satisfies the homogeneous initial and boundary conditions.

Next, we would like to show the differentiability of the solution $u(x, t)$. Let

$$\begin{aligned}S_m(x, t) &= \sum_{n=1}^{m} \int_0^t I_n(\Psi)(\tau) E_n(t-\tau)\, d\tau \phi_n(x) \\ &= \sum_{n=1}^{m} \int_0^1 \xi^b \phi_n(\xi) \left[\int_0^t \Psi(\xi, \tau) E_n(t-\tau)\, d\tau \right] d\xi \phi_n(x).\end{aligned}$$

Since $\Psi(\xi, \tau)$ is absolutely continuous on the interval $0 \leq \tau \leq \Gamma$ for each ξ in $(0, 1)$, it follows from integration by parts with respect to τ (cf. Chae [3, pp. 227–228]) that

$$\begin{aligned}S_m(x, t) = \sum_{n=1}^{m} \lambda_n^{-1} \Bigg[& I_n(\Psi)(t) - I_n(\Psi)(0) E_n(t) \\ & - \int_0^1 \xi^b \phi_n(\xi) \int_0^t \Psi_\tau(\xi, \tau) E_n(t-\tau)\, d\tau\, d\xi \Bigg] \phi_n(x).\end{aligned} \tag{2.4}$$

For x in $[x_0, 1]$ where x_0 is any positive number in $(0, 1)$, it follows from Lemma 1(d) that for any positive integers p and m with $p > m$,

$$\begin{aligned}\left|\frac{\partial S_p}{\partial x} - \frac{\partial S_m}{\partial x}\right| \le k_4 \sum_{n=m+1}^{p} \lambda_n^{-1/2} |I_n(\Psi)(t)| + k_4 \sum_{n=m+1}^{p} \lambda_n^{-1/2} |I_n(\Psi)(0)| \\ + k_4 \sum_{n=m+1}^{p} \lambda_n^{-1/2} \left| \int_0^1 \xi^b \phi_n(\xi) \int_0^t \Psi_\tau(\xi, \tau) E_n(t-\tau)\, d\tau\, d\xi \right|.\end{aligned} \tag{2.5}$$

From Condition (A) and Lemma 1(c),

$$\left(\sum_{n=m+1}^{p} |I_n(\Psi)(t)|^2 \right)^{1/2} \le k_5^{1/2}.$$

By the Schwarz inequality, the first term on the right-hand side of the inequality (2.5) is bounded by

$$k_4 k_5^{1/2} \left(\sum_{n=m+1}^{p} \lambda_n^{-1} \right)^{1/2},$$

which converges to 0 as p and m tend to infinity since $O(\lambda_n) = O(n^2)$ for large n. Similarly, the second term converges to 0 as p and m tend to infinity. By Lemma 1(a) and Condition (B),

$$\begin{aligned}\left| \int_0^t I_n(\Psi_\tau)(\tau) E_n(t-\tau)\, d\tau \right| &\le k_1 \int_0^t I(|\Psi_\tau|)(\tau) E_n(t-\tau)\, d\tau \\ &\le k_1 k_6 \lambda_n^{-1} [1 - E_n(t)] \\ &\le k_1 k_6 \lambda_n^{-1}.\end{aligned} \tag{2.6}$$

It follows from the Tonelli Theorem (cf. Royden [17, p. 270]) that

$$\xi^b \phi_n(\xi) \Psi_\tau(\xi, \tau) E_n(t-\tau)$$

is integrable on $\overline{Q}_\Gamma$. By the Fubini Theorem,

$$\begin{aligned}\left| \int_0^1 \xi^b \phi_n(\xi) \int_0^t \Psi_\tau(\xi, \tau) E_n(t-\tau)\, d\tau\, d\xi \right| &= \left| \int_0^t I_n(\Psi_\tau)(\tau) E_n(t-\tau)\, d\tau \right| \\ &\le k_1 k_6 \lambda_n^{-1}.\end{aligned}$$

Thus, the third term on the right-hand side of (2.5) is bounded by

$$k_1 k_4 k_6 \sum_{n=m+1}^{p} \lambda_n^{-3/2},$$

which converges to 0 as p and m tend to infinity. Therefore on $[x_0, 1] \times [0, \Gamma]$, $|\partial S_p/\partial x - \partial S_m/\partial x|$ converges to 0 uniformly as p and m tend to infinity. Hence, $\partial S_m/\partial x$ converges uniformly. Since x_0 (> 0) is arbitrarily chosen and each term in the series representing $\partial S_m/\partial x$ is continuous, it follows that $\partial S_m/\partial x$ converges uniformly on every given closed subset of $(0, 1] \times [0, \Gamma]$ to

$$u_x(x, t) = \sum_{n=1}^{\infty} \int_0^t I_n(\Psi)(\tau) E_n(t-\tau)\, d\tau \phi_n'(x),$$

and $u_x(x, t)$ is in $C(\overline{Q}_\Gamma \backslash P_1)$ where $P_1 \equiv \{(0, t) : 0 \le t \le \Gamma\}$.

Let the mth partial sum of $u_x(x, t)$ be denoted by $S_{xm}(x, t)$. Since

$$\phi_n''(x) + \frac{b}{x}\phi_n'(x) + \lambda_n \phi_n(x) = 0,$$

we have from (2.4) that

$$\begin{aligned} \partial S_{xm}(x, t)/\partial x = &-\frac{b}{x} S_{xm}(x, t) - \sum_{n=1}^{m} I_n(\Psi)(t)\phi_n(x) \\ &+ \sum_{n=1}^{m} I_n(\Psi)(0)E_n(t)\phi_n(x) + \sum_{n=1}^{m} \int_0^t I_n(\Psi_\tau)(\tau)E_n(t-\tau)\, d\tau \phi_n(x). \end{aligned} \tag{2.7}$$

Since $S_{xm}(x, t)$ converges uniformly on $[x_0, 1] \times [0, \Gamma]$ for arbitrarily fixed $x_0 > 0$, we have $(b/x)S_{xm}(x, t)$ converges uniformly there. For each fixed $t \ge 0$, it follows from Condition (A) and Lemma 1(e) that the second term on the right-hand side of (2.7) converges uniformly to $-\Psi(x, t)$ on every given closed subinterval of $(0, 1)$. By Lemma 1(e) and the Abel test (cf. Knopp [13, p. 346]), the third term converges uniformly on every given closed subset of Q_Γ^-; because of the term $E_n(t)$, it converges absolutely and uniformly on every given closed subset of $[0, 1] \times (0, \Gamma]$. Hence, the third term converges uniformly on every given closed subset of $\overline{Q}_\Gamma \backslash P_2$ where $P_2 \equiv \{(0, 0)\} \cup \{(1, 0)\}$. From (2.6), the absolute value of the last term is bounded by $\sum_{n=1}^{m} k_1 k_6 \lambda_n^{-1} |\phi_n(x)|$, and hence converges absolutely and uniformly on $\overline{Q}_\Gamma$. Therefore, for each fixed $t \ge 0$, $\partial S_{xm}(x, t)/\partial x$ converges uniformly on every given closed subinterval of $(0, 1)$. Thus from (2.7),

$$\begin{aligned} u_{xx}(x, t) &= \sum_{n=1}^{\infty} \int_0^t I_n(\Psi)(\tau)E_n(t-\tau)\, d\tau \phi_n''(x) \\ &= -\frac{b}{x} u_x(x, t) - \Psi(x, t) + \sum_{n=1}^{\infty} I_n(\Psi)(0)E_n(t)\phi_n(x) \\ &\quad + \sum_{n=1}^{\infty} \int_0^t I_n(\Psi_\tau)(\tau)E_n(t-\tau)\, d\tau \phi_n(x). \end{aligned} \tag{2.8}$$

Since each term on the right-hand side of (2.8) is continuous in Q_Γ^-, it follows that $u_{xx}(x, t)$ is in $C(Q_\Gamma^-)$.

To show that $u(x, t)$ is differentiable with respect to t, it follows from the Leibnitz rule on differentiation that

$$\partial S_m(x, t)/\partial t = \sum_{n=1}^{m} I_n(\Psi)(t)\phi_n(x) - \sum_{n=1}^{m} \lambda_n \int_0^t I_n(\Psi)(\tau)E_n(t-\tau)\, d\tau \phi_n(x).$$

By using integration by parts on $\int_0^t \Psi(\xi, \tau)E_n(t-\tau)\,d\tau$ of the last term, we have

$$\partial S_m(x, t)/\partial t = \sum_{n=1}^{m} I_n(\Psi)(0)E_n(t)\phi_n(x) + \sum_{n=1}^{m} \int_0^t I_n(\Psi_\tau)(\tau)E_n(t-\tau)\,d\tau\phi_n(x),$$

which are equal to the last two terms on the right-hand side of (2.7). Thus, $\partial S_m(x, t)/\partial t$ converges uniformly on every given closed subset of $\overline{Q}_\Gamma \backslash P_2$. Hence,

$$u_t(x, t) = \sum_{n=1}^{\infty} I_n(\Psi)(0)E_n(t)\phi_n(x) + \sum_{n=1}^{\infty} \int_0^t I_n(\Psi_\tau)(\tau)E_n(t-\tau)\,d\tau\phi_n(x); \qquad (2.9)$$

that is,

$$u_t(x, t) = \int_0^1 G(x, t; \xi, 0)\Psi(\xi, 0)\,d\xi + \int_0^t \int_0^1 G(x, t; \xi, \tau)\Psi_\tau(\xi, \tau)\,d\xi\,d\tau. \qquad (2.10)$$

Also, we have $u_t(x, t)$ is in $C(\overline{Q}_\Gamma \backslash P_2)$.

From (2.8) and (2.9), we have

$$Lu(x, t) = -\Psi(x, t) \quad \text{in } Q_\Gamma^-.$$

Therefore, the theorem is proved.

We now use a transformation to deduce the representation formula for the linear problem with nontrivial initial data.

THEOREM 3. Suppose $g(x)$ is in $C[0, 1] \cap C^2(0, 1)$ such that $g(0) = 0 = g(1)$, and both $\Psi(x, t)$ and $Lg(x)$ satisfy the conditions for $\Psi(x, t)$ in Theorem 2. Then, the problem (1.1) and (1.2) has a unique solution.

Proof. Let us consider the problem:

$$Lw = -(\Psi + Lg) \text{ in } Q_\Gamma$$

subject to zero initial and boundary data. Since $\Psi(x, t) + Lg(x)$ satisfies the conditions for $\Psi(x, t)$ in Theorem 2, it follows that $w(x, t)$ exists and is unique. Then, u given by $u(x, t) = w(x, t) + g(x)$ is the unique solution of the problem (1.1) and (1.2).

By the representation formula (2.2) and the above theorem, the solution u of the problem (1.1) and (1.2) is given by

$$u(x, t) = \int_0^t \int_0^1 G(x, t; \xi, \tau)[\Psi(\xi, \tau) + Lg(\xi)]\,d\xi\,d\tau + g(x). \qquad (2.11)$$

Let

$$D_1 \equiv \{(x, t; \xi, \tau) : x \text{ and } \xi \text{ are in } (0, 1),\ t > \tau \geq 0\}.$$

LEMMA 4. (a) For $t > \tau$, $G(x, t; \xi, \tau)$ is continuous for $(x, t; \xi, \tau) \in ([0, 1] \times (0, \Gamma]) \times ((0, 1] \times [0, \Gamma))$.

(b) For each fixed $(\xi, \tau) \in (0, 1] \times [0, \Gamma)$, $G(x, t; \xi, \tau) \in C^\infty((0, 1] \times (\tau, \Gamma])$.

(c) $G(x, t; \xi, \tau)$ is positive in D_1.

Proof. (a) By Lemma 1(b),

$$\sum_{n=1}^{\infty} |\xi^b \phi_n(\xi)\phi_n(x)E_n(t-\tau)| \le \xi^b k_2^2 \sum_{n=1}^{\infty} \lambda_n^{1/2} E_n(t-\tau).$$

Since $O(\lambda_n) = O(n^2)$ for large n, $\sum_{n=1}^{\infty} \lambda_n^{1/2} E_n(t-\tau)$ converges uniformly for $t-\tau \ge \varepsilon$ where ε is any positive number. Hence, $G(x, t; \xi, \tau)$ is continuous for $t-\tau \ge \varepsilon$. Since ε is arbitrarily chosen, our assertion follows.

(b) From Lemma 6 of Chan and Wong [10], the mth derivative of $\phi_n(x)$ satisfies the inequality,

$$|\phi_n^{(m)}(x)| \le K_m \lambda_n^{m/2} x^{\nu-m} / |J_{\nu+1}(\lambda_n^{1/2})|, \quad n = 1, 2, 3, \ldots,$$

where K_m is a constant depending on m. From Lemma 1(a),

$$\sum_{n=1}^{\infty} |\xi^b \phi_n(\xi)\phi_n^{(m)}(x)E_n(t-\tau)| \le k_1 K_m \xi^{b/2} x^{\nu-m} \sum_{n=1}^{\infty} \lambda_n^{m/2} E_n(t-\tau)/|J_{\nu+1}(\lambda_n^{1/2})|.$$

It follows from (2.10) of Chan and Wong [9], and $O(\lambda_n) = O(n^2)$ for large n that

$$\sum_{n=1}^{\infty} \lambda_n^{m/2} E_n(t-\tau)/|J_{\nu+1}(\lambda_n^{1/2})|$$

converges uniformly for $t-\tau \ge \varepsilon$, and hence $\partial^m G/\partial x^m$ is continuous for $t-\tau > 0$ since ε is arbitrarily chosen. Now,

$$\frac{\partial^m}{\partial t^m} E_n(t-\tau) = (-1)^m \lambda_n^m E_n(t-\tau).$$

An argument similar to the above shows that $\partial^m G/\partial t^m$ is continuous for $t-\tau > 0$. Since m is any positive integer, our assertion follows.

(c) Suppose $G(x, t; \xi, \tau) < 0$ at some point $(x_1, t_1; \xi_1, \tau_1)$ in D_1. Since $G(x, t; \xi, \tau)$ is continuous in D_1, we may assume $\tau_1 > 0$. Hence, there exists a positive number ε such that $G(x, t; \xi, \tau) < 0$ in the set

$$W_0 = (x_1-\varepsilon, x_1+\varepsilon) \times (t_1-\varepsilon, t_1+\varepsilon) \times (\xi_1-\varepsilon, \xi_1+\varepsilon) \times (\tau_1-\varepsilon, \tau_1+\varepsilon)$$

contained in D_1. Let

$$W_1 = (\xi_1-\varepsilon, \xi_1+\varepsilon) \times (\tau_1-\varepsilon, \tau_1+\varepsilon),$$
$$W_2 = \left(\xi_1 - \frac{\varepsilon}{2}, \xi_1 + \frac{\varepsilon}{2}\right) \times \left(\tau_1 - \frac{\varepsilon}{2}, \tau_1 + \frac{\varepsilon}{2}\right).$$

There exists (cf. Dunford and Schwartz [11, pp. 1640–1641]) a function $h_3(x, t)$ in $C^\infty(\mathbb{R}^2)$ such that $h_3 \equiv 1$ on $\overline{W}_2$, $h_3 \equiv 0$ outside W_1, and $0 \le h_3 \le 1$ in $W_1 \backslash W_2$. It is clear that $h_3(x, t)$ satisfies the conditions for Ψ in Theorem 2. Hence, the solution of the problem,

$$Lw(x, t) = -h_3(x, t) \quad \text{in } Q_\alpha, \ t_1 < \alpha,$$

with w satisfying zero initial and boundary conditions, is given by

$$w(x, t) = \int_{\tau_1-\varepsilon}^{\tau_1+\varepsilon} \int_{\xi_1-\varepsilon}^{\xi_1+\varepsilon} G(x, t; \xi, \tau) h_3(\xi, \tau)\, d\xi\, d\tau.$$

Since $G(x, t; \xi, \tau) < 0$ in W_0, $h_3(\xi, \tau) \geq 0$ in W_1, and $h_3 \equiv 1$ on $\overline{W}_2$, it follows that

$$w(x, t) < 0 \quad \text{for } (x, t) \text{ in } (x_1 - \varepsilon, x_1 + \varepsilon) \times (t_1 - \varepsilon, t_1 + \varepsilon).$$

On the other hand, $h_3(x, t) \geq 0$ in Q_α implies $w(x, t) \geq 0$ by the weak maximum principle. We have a contradiction. Therefore, $G(x, t; \xi, \tau) \geq 0$ in D_1.

Suppose $G(x, t; \xi, \tau) = 0$ at some point $(x_2, t_2; \xi_2, \tau_2)$ in D_1. Then by the strong maximum principle,

$$G(x, t; \xi_2, \tau_2) = 0 \quad \text{in } D_1 \cap \{(x, t; \xi_2, \tau_2) : 0 < x < 1, t \leq t_2\}.$$

On the other hand,

$$G(\xi_2, t_2; \xi_2, \tau_2) = \sum_{n=1}^{\infty} \xi_2^b \phi_n^2(\xi_2) E_n(t_2 - \tau_2),$$

which is positive. This contradiction implies $G > 0$ in D_1.

We would like to establish some properties of the solution $u(x, t)$. Let

$$\ell \equiv \frac{\partial^2}{\partial x^2} + \frac{b}{x}\frac{\partial}{\partial x}.$$

THEOREM 5. Under the hypotheses of Theorem 3, if $I^2(g)$ exists, then the solution $u(x, t)$ of the problem (1.1) and (1.2) has the following properties:

(a) u_x is in $C(\overline{Q}_\Gamma \backslash P_1)$, u_{xx} is in $C(Q_\Gamma^-)$, and u_t is in $C(\overline{Q}_\Gamma \backslash P_2)$;

(b) $u(x, t)$ is absolutely continuous on the interval $0 \leq t \leq \Gamma$ for each x in $[0, 1]$; furthermore, $I^2(u)(t) \leq k_7$ and $I^2(u_t)(t) \leq k_8$ for t in $[0, \Gamma]$;

(c) $I^2(\ell u)(t) < \infty$ for t in $[0, \Gamma]$.

Proof. (a) This property follows from the hypotheses on $\Psi(x, t)$ and $g(x)$, and a proof as in that of Theorem 2 (with Ψ replaced by $\Psi + Lg$).

(b) It follows from Theorem 5(a) and $u(0, t) = 0 = u(1, t)$ that $u(x, t)$ is absolutely continuous on the interval $0 \leq t \leq \Gamma$ for each x in $[0, 1]$.

By the Schwarz inequality,

$$\begin{aligned} I^2(u)(t) &= I^2(u - g)(t) + I^2(g) + 2\int_0^1 x^b[u(x, t) - g(x)]g(x)\,dx \\ &\leq I^2(u - g)(t) + I^2(g) + 2[I^2(u - g)(t)]^{1/2}[I^2(g)]^{1/2}. \end{aligned} \tag{2.12}$$

From (2.11),

$$u(x, t) = \sum_{n=1}^{\infty} \int_0^t I_n(\Psi + Lg)(\tau) E_n(t - \tau)\,d\tau \phi_n(x) + g(x).$$

From the proof of Theorem 2 (on u with Ψ replaced by $\Psi + Lg$), the above series (on the right-hand side) representing $u(x, t) - g(x)$ is absolutely and uniformly convergent on $\overline{Q}_\Gamma$. By Lemma 1(a) and (c), this is also true for the series representing $x^{b/2}[u(x, t) - g(x)]$. Hence, the series representing $x^b[u(x, t) - g(x)]^2$ is also absolutely and uniformly convergent on $\overline{Q}_\Gamma$ (cf. Knopp [13, pp. 146 and 337]). Since

$\{\phi_n(x)\}$ is an orthonormal set with weight function x^b, it follows that

$$I^2(u-g)(t) = \sum_{n=1}^{\infty} \left[\int_0^t I_n(\Psi + Lg)(\tau) E_n(t-\tau)\, d\tau \right]^2 .$$

By Lemma 1(c),

$$\begin{aligned} I^2(u-g)(t) &\le \left[\sup_{0\le\tau\le\Gamma} I^2(\Psi + Lg)(\tau) \right] \sum_{n=1}^{\infty} \left[\int_0^t E_n(t-\tau)\, d\tau \right]^2 \\ &\le \left[\sup_{0\le\tau\le\Gamma} I^2(\Psi + Lg)(\tau) \right] \sum_{n=1}^{\infty} \lambda_n^{-2}. \end{aligned}$$

From (2.12),

$$\begin{aligned} I^2(u)(t) \le & \left[\sup_{0\le\tau\le\Gamma} I^2(\Psi + Lg)(\tau) \right] \sum_{n=1}^{\infty} \lambda_n^{-2} + I^2(g) \\ & + 2 \left(\left[\sup_{0\le\tau\le\Gamma} I^2(\Psi + Lg)(\tau) \right] \sum_{n=1}^{\infty} \lambda_n^{-2} \right)^{1/2} [I^2(g)]^{1/2}. \end{aligned} \tag{2.13}$$

It follows from the hypotheses on Ψ and Lg that

$$\sup_{0\le\tau\le\Gamma} I^2(\Psi + Lg)(\tau) < \infty.$$

Because $O(\lambda_n) = O(n^2)$ for large n, we have from (2.13) that $I^2(u)(t) \le k_7$ for t in $[0, \Gamma]$.

By (2.9) (with $\Psi(x, t)$ replaced by $\Psi(x, t) + Lg(x)$),

$$u_t(x, t) = \sum_{n=1}^{\infty} I_n(\Psi + Lg)(0) E_n(t) \phi_n(x) + \sum_{n=1}^{\infty} \int_0^t I_n(\Psi_\tau)(\tau) E_n(t-\tau)\, d\tau \phi_n(x). \tag{2.14}$$

Let t_0 in $(0, \Gamma]$ be fixed. By Lemma 1(a) and (c), the right-hand side of (2.14) multiplied by $x^{b/2}$ converges absolutely and uniformly on $[0, 1]$ to $x^{b/2} u_t(x, t_0)$. Hence, the series representing $x^b u_t^2(x, t_0)$ is absolutely and uniformly convergent on $[0, 1]$. Integrating this series representing $x^b u_t^2(x, t_0)$ with respect to x and using the orthonormality of the sequence $\{\phi_n(x)\}$ with weight function x^b, we have

$$\begin{aligned} I^2(u_t)(t_0) = & \sum_{n=1}^{\infty} [I_n(\Psi + Lg)(0) E_n(t_0)]^2 \\ & + \sum_{n=1}^{\infty} \left[\int_0^{t_0} I_n(\Psi_\tau)(\tau) E_n(t_0-\tau)\, d\tau \right]^2 \\ & + 2 \sum_{n=1}^{\infty} \left[\int_0^{t_0} I_n(\Psi_\tau)(\tau) E_n(t_0-\tau)\, d\tau \right] [I_n(\Psi + Lg)(0) E_n(t_0)]. \end{aligned}$$

From Lemma 1(c) and $E_n(t_0) \le 1$ for all positive integers n, the first term on the right-hand side is bounded by $I^2(\Psi + Lg)(0)$. From Lemma 1(a) and Condition (B), the second term is bounded by

$$k_1 k_6 \sum_{n=1}^{\infty} \left[\int_0^{t_0} E_n(t_0 - \tau)\, d\tau \right]^2 \leq k_1 k_6 \sum_{n=1}^{\infty} \lambda_n^{-2}.$$

By using the Schwarz inequality on the third term, we obtain

$$I^2(u_t)(t_0) \leq I^2(\Psi + Lg)(0) + k_1 k_6 \sum_{n=1}^{\infty} \lambda_n^{-2}$$
$$+ 2[I^2(\Psi + Lg)(0)]^{1/2} \left[k_1 k_6 \sum_{n=1}^{\infty} \lambda_n^{-2} \right]^{1/2}.$$

We note that the right-hand side is independent of t_0. Hence, $I^2(u_t)(t)$ is bounded on $(0, \Gamma]$. As for $I^2(u_t)(0)$, it follows from Lemma 1(e) that for x in $(0, 1)$,

$$u_t(x, 0) = \sum_{n=1}^{\infty} I_n(\Psi + Lg)(0)\phi_n(x)$$
$$= \Psi(x, 0) + Lg(x),$$

from which,

$$I^2(u_t)(0) = I^2(\Psi + Lg)(0).$$

Thus, $I^2(u_t)(t) \leq k_8$ on $[0, \Gamma]$ for some constant k_8.

(c) Since $\ell u = u_t - \Psi$, it follows from the Schwarz inequality that

$$I^2(\ell u)(t) = I^2(u_t - \Psi)(t)$$
$$= I^2(u_t)(t) + I^2(\Psi)(t) - 2 \int_0^1 x^b u_t(x, t)\Psi(x, t)\, dx$$
$$\leq I^2(u_t)(t) + I^2(\Psi)(t) + 2[I^2(u_t)(t) I^2(\Psi)(t)]^{1/2}.$$

Then from Theorem 5(b) and Condition (A), $I^2(\ell u)(t) < \infty$ on $[0, \Gamma]$.

3. Nonhomogeneous boundary conditions. In this section, we assume $|b| < 1$; we also assume as in Sec. 2 that $g(x)$, $Lg(x)$, and $\Psi(x, t)$ satisfy the hypotheses of Theorem 3, except that $g(0) = 0 = g(1)$. Let us consider the linear problem, (1.1), subject to

$$u(x, 0) = g(x) \quad \text{for } 0 \leq x \leq 1,$$
$$u(0, t) = r_1(t) \quad \text{and} \quad u(1, t) = r_2(t) \quad \text{for } 0 < t \leq \Gamma < \infty, \tag{3.1}$$

where $r_1(t)$ and $r_2(t)$ are in $C^2[0, \infty)$ such that $r_1(0) = g(0)$ and $r_2(0) = g(1)$.

THEOREM 6. The problem (1.1) and (3.1) has a unique solution.

Proof. Let us consider the problem,

$$Lw(x, t) = -[\Psi(x, t) + Ls(x, t)] \quad \text{in } Q_\Gamma,$$
$$w(x, 0) = g(x) - s(x, 0) \text{ for } 0 \leq x \leq 1, \qquad w(0, t) = 0 = w(1, t) \text{ for } 0 < t \leq \Gamma,$$

where

$$s(x, t) = (1 - x^{2\nu})r_1(t) + x^{2\nu} r_2(t).$$

It follows from the assumptions on $\Psi(x, t)$, $g(x)$, $r_1(t)$, and $r_2(t)$ that $\Psi(x, t) + Ls(x, t)$ and $g(x) - s(x, 0)$ satisfy the conditions for $\Psi(x, t)$ and $g(x)$, respectively, in Theorem 3. Hence, $w(x, t)$ exists and is unique. It follows that u given by $u = w + s$ is the unique solution of the problem (1.1) and (3.1).

We note that the solution u in Theorem 6 has the properties stated in Theorem 5.

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