QUARTERLY OF APPLIED MATHEMATICS
VOLUME LIII, NUMBER 2
JUNE 1995, PAGES 273-300

MONOTONIC, COMPLETELY MONOTONIC, AND EXPONENTIAL RELAXATION FUNCTIONS IN LINEAR VISCOELASTICITY

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Summary. A priori restrictions on the relaxation function of linear viscoelasticity are studied under regularity assumptions weaker than those usually made in the literature. The new set of assumptions is sufficient to define, by a limit procedure, the work done in deformation processes in which some parts are subject either to extreme retardations or to extreme accelerations. The use of such processes results in a considerable simplification of the proofs of some classical results. Under the same assumptions, we give a characterization of the monotonicity of the relaxation function in terms of work. We also extend an earlier one-dimensional characterization of complete monotonicity due to Day, and prove that the work done in every closed path in stress-strain space is nonnegative if and only if the relaxation function is of exponential type.

1. Introduction. In a linear viscoelastic material, the stress response to a *deformation process* E is determined by the hereditary law

$$T(t) = \mathbf{G}_0 E(t) + \int_0^{+\infty} \dot{\mathbf{G}}(s) E(t-s)\, ds \tag{1.1}$$

due to Boltzmann and studied extensively by Volterra.[1] Restrictions on the *relaxation function* $\mathbf{G}$ have been deduced from two general requirements on the work: the *postulate of dissipativity* due to König and Meixner [12], which requires that the work done in any finite deformation process starting from the natural state be nonnegative, and *compatibility with thermodynamics* in the sense of Day [4], according to which the work done in any finite cyclic process starting from equilibrium must be nonnegative. The latter requirement has been obtained by Coleman [1, 2] as a consequence of the Clausius-Duhem inequality; the substantial equivalence of the restrictions coming from the two requirements has been proved by Day [4].

These restrictions are not severe; in particular, they do not imply that the relaxation function be monotonic decreasing, a property systematically observed in experiments. An attempt for characterizing the monotonicity of $\mathbf{G}$ by some more

Received September 3, 1992.
1991 *Mathematics Subject Classification.* Primary 73F05.

[1] For reference see, e.g., Leitman and Fisher [13, Secs. 5 and 7].

restrictive property of the work was made by Day in [3]. He proved that, in the one-dimensional case, the work done in retraced paths is increased by delay if and only if the relaxation function is completely monotonic.[2] The main purpose of this paper is to supply some further characterizations of monotonicity in terms of work. There is, however, a preliminary point to be clarified. In the literature, there is no general agreement about the regularity assumed for the functions E and $\mathbf{G}$; moreover, it is usual to introduce supplementary assumptions when proving specific results. For deformation processes, there is a dichotomy in the fact that, on the one hand, physical considerations suggest that E should be continuous and start from a state of equilibrium, since only processes with these properties are accessible to experiments; on the other hand, the mathematical convenience of using step functions and functions extended indefinitely in the past recommends the choice of a more general class of processes.

We assume that $\mathbf{G}$ has a Lebesgue integrable derivative and that E has bounded variation in the past and is continuous from the right. These assumptions are sufficient for the existence of the integral in (1.1) as a Lebesgue integral. They also suffice for establishing a formula of integration by parts involving the Riemann-Stieltjes integral, and for defining the extreme acceleration and the extreme retardation of a part of a given process by means of a limit procedure. The regularity assumptions are discussed in Sec. 2 and the accelerated and retarded processes in Sec. 3. In particular, we are interested in the properties of convergence of the stress and of the work done in accelerated and retarded processes when the accelerations and retardations become extreme. These properties, quite easy to prove when more regular functions are involved, require now a more complicated machinery.

In Sec. 4 the effects of the new assumptions on the work postulates are examined; indeed, it is conceivable that the same postulates imposed on a broader class of processes result in more severe restrictions on the relaxation function. We find that no new restrictions come from dissipativity and from compatibility with thermodynamics; only the property of *strong dissipativity* [7], which must be reformulated in the new context, has stronger implications.

The last three sections are devoted to the characterization in terms of work of monotonicity and related properties. In Sec. 5 we consider processes in which the deformations are proportional to a fixed tensor, and whose magnitude varies monotonically with time. We call such processes *rectilinear monotonic.* We prove that the relaxation function is monotonic if and only if the work done in all rectilinear monotonic processes is decreased by retardation. Section 6 is devoted to completely monotonic relaxation functions; in it, the result of Day [3] mentioned above is extended from one to more dimensions, eliminating at the same time all superfluous regularity assumptions. Relaxation functions of exponential type are studied in the last section. In dimension one, such functions describe the Maxwell rheological model, for which it is easy to prove that the work done in any cycle closed in stress-strain space is nonnegative, so that no work can be extracted from any cycle in stress-strain space, irrespective of the history of the deformation preceding the cycle. In our weaker con-

[2] For precise definitions see Sec. 6 below.

text, we prove that this property of the work is not only necessary, but also sufficient for a relaxation function being of exponential type; it does not hold, for example, if the relaxation function is a linear combination of exponentials.

Let us add some remarks on the notation. We denote by Sym the set of all symmetric linear transformations on the vectors defined on the ordinary Euclidean point space, and by LinSym the set of all linear transformations on Sym. Sym is a finite-dimensional vector space, equipped with the inner product

$$A \cdot B := \mathrm{tr}(AB^{\mathrm{T}}) \tag{1.2}$$

and with the norm

$$|A| := (A \cdot A)^{1/2}. \tag{1.3}$$

Throughout this paper, no use will be made of the fact that the elements of Sym are tensors, except for the definition (1.2) of the inner product. Consequently, the elements of LinSym will be regarded as linear transformations on a finite-dimensional vector space, without reference to the fact that they are indeed fourth-order tensors, and LinSym will be normed by the operator norm

$$\|\mathbf{C}\| := \sup_{A \in \mathrm{Sym} \setminus \{0\}} \frac{|\mathbf{C}A|}{|A|}. \tag{1.4}$$

For any $\mathbf{C}$ in LinSym, we note

$$\mathbf{C}^{\mathrm{S}} := \tfrac{1}{2}(\mathbf{C} + \mathbf{C}^{\mathrm{T}}), \qquad \mathbf{C}^{\mathrm{W}} := \tfrac{1}{2}(\mathbf{C} - \mathbf{C}^{\mathrm{T}}) \tag{1.5}$$

for the symmetric and the skew-symmetric part. The notations

$$\mathbf{C} > \mathbf{D}, \qquad \mathbf{C} \geq \mathbf{D} \tag{1.6}$$

mean that $(\mathbf{C} - \mathbf{D})$ is positive-definite or positive-semidefinite, respectively.

2. Regularity assumptions. According to the constitutive equation (1.1), the response of a linear viscoelastic material is characterized by a tensor $\mathbf{G}_0 \in$ LinSym and by a function $\dot{\mathbf{G}} : (0, +\infty) \to$ LinSym. The tensor $\mathbf{T}(t) \in$ Sym is the *stress* reached at the time t in the *deformation process* $E : \mathbf{R} \to$ Sym. It is clear from (1.1) that only the restriction of E to $(-\infty, t)$ and the actual value $E(t)$ of E contribute to the determination of $\mathbf{T}(t)$. In this section we make, and discuss, the regularity assumptions on the functions E and $\dot{\mathbf{G}}$ which will be used throughout the paper.

The *variation* of E in the closed interval $[p, q]$ is the nonnegative number

$$V_{p,q}(E) := \sup \left\{ \sum_{i=1}^{n} |E(t_i) - E(t_{i-1})| \right\}, \tag{2.1}$$

with the supremum taken over all finite collections $\{t_i, i = 0, 1, \ldots, n; n \in \mathbf{N}\}$ of points in $[p, q]$ such that $t_{i-1} < t_i$, and the variation of E in the unbounded interval $(-\infty, q]$ is

$$V_{-\infty,q}(E) := \lim_{p \to -\infty} V_{p,q}(E). \tag{2.2}$$

The restriction of E to some (bounded or unbounded) interval is a *function of bounded variation* if the variation of E in that interval is finite. A function of bounded variation is bounded. Moreover, it is known that a function of bounded variation has at most countably many discontinuity points and admits a left limit and a right limit at all interior points of its domain. It is also known that the finiteness of the limit (2.2) implies that the limit

$$E(-\infty) := \lim_{p \to -\infty} E(p) \tag{2.3}$$

exists in Sym.

We make the following regularity assumptions on E and $\dot{\mathbf{G}}$:

(A1) for all $t \in \mathbf{R}$ the restriction of E to $(-\infty, t]$ is a function of bounded variation;
(A2) E is continuous from the right;
(A3) $\dot{\mathbf{G}}$ is Lebesgue integrable in $[0, +\infty)$.

If we denote by $E(t^-)$ and $E(t^+)$ the left and right limits of E at t, with the second assumption we set

$$E(t) = E(t^+). \tag{2.4}$$

The assumption (A3) implies that $\dot{\mathbf{G}}$ has a primitive, i.e., that there is an absolutely continuous function $\mathbf{G} : [0, +\infty) \to \text{LinSym}$ such that $\dot{\mathbf{G}}$ coincides almost everywhere with the derivative of $\mathbf{G}$. This function has the form

$$\mathbf{G}(s) = \mathbf{G}(0) + \int_0^s \dot{\mathbf{G}}(r)\, dr \tag{2.5}$$

and is determined to within the initial value $\mathbf{G}(0)$. It is convenient to choose $\mathbf{G}(0)$ equal to the tensor $\mathbf{G}_0$ appearing in (1.1), so that the stress is completely determined by E and $\mathbf{G}$. $\mathbf{G}$ is called the *relaxation function* of the viscoelastic material. Another consequence of (A3) is that the limit

$$\mathbf{G}_\infty := \lim_{s \to +\infty} \mathbf{G}(s) \tag{2.6}$$

exists in LinSym [5, p. 111]. The existence of $\mathbf{G}_\infty$ and the continuity of $\mathbf{G}$ also imply the boundedness of $\mathbf{G}$.

Let H and $\mathbf{C}$ be maps from $\mathbf{R}$ into Sym and LinSym, respectively. The *Riemann-Stieltjes integral* of $\mathbf{C}$ with respect to H from p to q is the second-order tensor

$$\int_p^q \mathbf{C}(r)\, dH(r) := \lim_{n \to +\infty} \sum_{i=1}^n \mathbf{C}(r_i^n)(H(t_i^n) - H(t_{i-1}^n)), \tag{2.7}$$

each $\{t_i^n, i = 0, 1, \ldots, n; n \in \mathbf{N}\}$ being a finite collection of points with $t_0^n = p$, $t_n^n = q$, $t_{i-1}^n < t_i^n$ and with the maximum of the distances $(t_i^n - t_{i-1}^n)$ approaching zero when $n \to \infty$, and each r_i^n being an arbitrary point in $[t_{i-1}^n, t_i^n]$. Here we list some properties of this integral that will be used in this paper, and for which we refer to [14] and [10].

2.1. PROPOSITION. The Riemann-Stieltjes integral has the following properties:

(i) if the integral (2.7) exists, then

$$\left| \int_p^q \mathbf{C}(r)\, dH(r) \right| \le \sup_{r \in [p,q]} \|\mathbf{C}(r)\|\; V_{p,q}(H). \tag{2.8}$$

(ii) A sufficient condition for the existence of (2.7) is that $\mathbf{C}$ be continuous and H be of bounded variation in $[p, q]$.

(iii) If Condition (ii) is verified, the function

$$F(t) := \int_p^t \mathbf{C}(r)\, dH(r), \quad t \in [p, q] \tag{2.9}$$

is of bounded variation. Its right and left limits at t are given by

$$\begin{aligned} F(t^+) - F(t) &= \mathbf{C}(t)[H(t^+) - H(t)], \\ F(t) - F(t^-) &= \mathbf{C}(t)[H(t) - H(t^-)]. \end{aligned} \tag{2.10}$$

(iv) If (ii) holds and if $\mathbf{C}$ has a Lebesgue integrable derivative $\dot{\mathbf{C}}$, then the formula of integration by parts applies:

$$\int_p^q \mathbf{C}(r)\, dH(r) = -\int_p^q \dot{\mathbf{C}}(r) H(r)\, dr + \mathbf{C}(q)H(q) - \mathbf{C}(p)H(p). \tag{2.11}$$

(v) If H has bounded variation in $(-\infty, q]$ and if $\mathbf{C}$ is continuous and has a Lebesgue integrable derivative in $(-\infty, q]$, then the formula (2.11) holds with p replaced by $-\infty$. □

Under our assumptions (A1)–(A3), the conditions for the validity of the formula (2.11) are satisfied by $\mathbf{C}(r) = \mathbf{G}(q - r)$, $r \in (-\infty, q]$. In particular, taking $p = -\infty$, $q = t$ and substituting into (1.1) we get the following alternative form of the constitutive equation:

$$T(t) = \int_{-\infty}^t \mathbf{G}(t - r)\, dE(r) + \mathbf{G}_\infty E(-\infty). \tag{2.12}$$

By this way, the *stress process* T associated with E, defined by (1.1), is represented by an integral of the type (2.9). Consequently, T has bounded variation in $(-\infty, t]$ for all $t \in \mathbf{R}$ and its right and left limits are given by (2.10) with $\mathbf{C}(r) = \mathbf{G}(t - r)$. Recalling that $E(t) = E(t^+)$ by (2.4), we have

$$\begin{aligned} T(t^+) - T(t) &= 0, \\ T(t) - T(t^-) &= \mathbf{G}_0[E(t) - E(t^-)]. \end{aligned} \tag{2.13}$$

The first equation shows that T is continuous from the right and leads to the following conclusion.

2.2. Proposition. Let E and $\mathbf{G}$ satisfy the assumptions (A1)–(A3). Then the stress associated with E satisfies (A1)–(A2). □

The second equation shows that T and E have the same jump points and that the jump of T at a point t is given by $\mathbf{G}_0$ applied to the jump of E at t.

The constitutive equation in the form (2.12) also shows that the stress corresponding to the constant deformation process $E(t) = A$, $t \in \mathbf{R}$, $A \in \mathrm{Sym}$, is

$$T(t) = \mathbf{G}_\infty A \qquad \forall t \in \mathbf{R}. \tag{2.14}$$

We say that $\mathbf{G}_\infty A$ is the *equilibrium stress* associated with A, and that the deformation-stress pair $(A, \mathbf{G}_\infty A)$ is an *equilibrium state* for the material. A particular equilibrium state is the *natural state* $(0, 0)$. Notice that, by (2.12), $T(-\infty) = \mathbf{G}_\infty E(-\infty)$, i.e., that *every pair* (T, E) *of processes related by the constitutive equation* (1.1) *starts from an equilibrium state.*

The *work* done in the process E in the time interval $[p, q]$ is

$$w(E; p, q) := \int_p^q T(t) \cdot dE(t), \tag{2.15}$$

with the Riemann-Stieltjes integral on the right defined as in (2.7), with the tensors $\mathbf{C}(r_i^n)H(t_j^n)$ replaced by the scalars $T(r_i^n) \cdot E(t_j^n)$. By (ii) of Prop. 2.1, the existence of this integral is ensured whenever T, and therefore E, is continuous in $[p, q]$. A counterexample [10, Note 8.1] shows that, if T is discontinuous even at a single point, the value of the integral may depend upon the choice of the points r_i^n. The definition of the work done in a discontinuous process is one of the purposes of the next section.

3. Accelerations and retardations. Let $[a, b]$ be a fixed interval. For any real positive α consider the *time rescaling* f_α:

$$f_\alpha(t) := \begin{cases} t + (1-\alpha)(b-a) & \text{for } t < a, \\ \alpha t + (1-\alpha)b & \text{for } a \le t < b, \\ t & \text{for } t \ge b, \end{cases} \tag{3.1}$$

which maps $[a, b]$ into the interval $[\alpha a + (1-\alpha)b, b]$, with a uniform contraction if $\alpha < 1$ and a uniform dilatation if $\alpha > 1$. If E is a deformation process, the process E_α:

$$E_\alpha(f_\alpha(t)) := E(t), \quad t \in \mathbf{R}, \tag{3.2}$$

is also a deformation process, called the *α-acceleration of E in* $[a, b]$ if $\alpha < 1$, and the *α-retardation of E in* $[a, b]$ if $\alpha > 1$. In particular, for $t \ge b$,

$$E_\alpha(t) = E(t) \quad \forall \alpha > 0, \tag{3.3}$$

and, for $t < b$,

$$E_\alpha(t) := \begin{cases} E(t - (1-\alpha)(b-a)) & \text{for } t < f_\alpha(a), \\ E(b - \alpha^{-1}(b-t)) & \text{for } t \ge f_\alpha(a). \end{cases} \tag{3.4}$$

Recalling that $f_\alpha(a) = b - \alpha(b-a)$ and that E is continuous from the right by (A2), for all $t < b$ we have

$$\lim_{\alpha \to 0} E_\alpha(t) = E(t - b + a), \qquad \lim_{\alpha \to +\infty} E_\alpha(t) = E(b^-). \tag{3.5}$$

From (3.3) and (3.4) it follows that the one-parameter family $\alpha \mapsto E_\alpha$ converges pointwise to the deformation process E_0:

$$E_0(t) := \begin{cases} E(t) & \text{for } t \ge b, \\ E(t - b + a) & \text{for } t < b, \end{cases} \tag{3.6}$$

when $\alpha \to 0$, and to the deformation process E_∞:

$$E_\infty(t) := \begin{cases} E(t) & \text{for } t \ge b, \\ E(b^-) & \text{for } t < b, \end{cases} \tag{3.7}$$

when $\alpha \to +\infty$. Denote by T_α the stress process associated with E_α:

$$T_\alpha(t) = \mathbf{G}_0 E_\alpha(t) + \int_0^{+\infty} \dot{\mathbf{G}}(s) E_\alpha(t-s)\, ds, \tag{3.8}$$

and by T_0, T_∞ the stress processes associated with E_0, E_∞, respectively. The next proposition shows that the family $\alpha \mapsto T_\alpha$ converges pointwise to T_0 when $\alpha \to 0$ and to T_∞ when $\alpha \to +\infty$.

3.1. Proposition. Let E and $\mathbf{G}$ satisfy (A1)–(A3). Then, for all $t \in \mathbf{R}$,

$$\lim_{\alpha\to 0} T_\alpha(t) = T_0(t), \qquad \lim_{\alpha\to +\infty} T_\alpha(t) = T_\infty(t). \tag{3.9}$$

Proof. For the integrand function in (3.8) we have

$$|\dot{\mathbf{G}}(s) E_\alpha(t-s)| \le \|\dot{\mathbf{G}}(s)\| \sup_{r\in(-\infty,t]} |E_\alpha(r)|. \tag{3.10}$$

If $t \ge b$, then E_α and E take the same values in $(-\infty, t]$, so that the suprema in $(-\infty, t]$ of E_α and E coincide. If $t < b$, then

$$\sup_{r\in(-\infty,t]} |E_\alpha(r)| \le \sup_{r\in(-\infty,b]} |E_\alpha(r)| = \sup_{r\in(-\infty,b]} |E(r)|. \tag{3.11}$$

In both cases, the supremum of E_α in $(-\infty, t]$ is bounded by a constant independent of α. Since $\dot{\mathbf{G}}$ is Lebesgue integrable in $[0, +\infty)$ by (A3), we conclude that for all $\alpha > 0$ and for any fixed $t \in \mathbf{R}$ the integrand function in (3.8) is bounded by a Lebesgue integrable function. Thus, by the Lebesgue dominated convergence theorem, the limit can be taken inside the integral, and Eqs. (3.9) follow from the pointwise convergence of $\alpha \mapsto E_\alpha$ to E_0 and E_∞, respectively. □

We say that a process E is subject to an *extreme acceleration* in $[a, b]$ when $\alpha \to 0$ and to an *extreme retardation* when $\alpha \to +\infty$. We have shown that a process E_α converges pointwise to E_0 in an extreme acceleration and to E_∞ in an extreme retardation, and that the stress converges pointwise to the stresses T_0, T_∞ associated with E_0, E_∞ by the constitutive equation. To evaluate the corresponding limits of the work, we start from a preparatory result, which shows the behaviour of the stress response at the interior points of the interval $[f_\alpha(a), f_\alpha(b)]$ when $\alpha \to 0$ and $\alpha \to +\infty$.

3.2. Proposition. Let E and $\mathbf{G}$ satisfy (A1)–(A3). Then, for all $t \in [a, b)$,

$$\lim_{\alpha\to 0} T_\alpha(f_\alpha(t)) = T(a) + \mathbf{G}_0(E(t) - E(a)) \tag{3.12}$$

and

$$\lim_{\alpha\to +\infty} T_\alpha(f_\alpha(t)) = \mathbf{G}_\infty E(t^-) + \mathbf{G}_0(E(t) - E(t^-)). \tag{3.13}$$

Proof. By the constitutive equation (1.1),

$$T_\alpha(f_\alpha(t)) = \mathbf{G}_0 E_\alpha(f_\alpha(t)) + \int_0^{+\infty} \dot{\mathbf{G}}(s) E_\alpha(f_\alpha(t) - s)\, ds, \tag{3.14}$$

and, by (3.4),

$$E_\alpha(f_\alpha(t) - s) = \begin{cases} E(f_\alpha(t) - s - (1-\alpha)(b-a)) & \text{for } f_\alpha(t) - s < f_\alpha(a), \\ E(b - \alpha^{-1}(b - f_\alpha(t) + s)) & \text{for } f_\alpha(t) - s \ge f_\alpha(a). \end{cases} \tag{3.15}$$

For $t \in [a, b)$, from (3.1) we get

$$E_\alpha(f_\alpha(t) - s) = \begin{cases} E(\alpha t - s + (1-\alpha)a) & \text{for } s > \alpha(t-a), \\ E(t - \alpha^{-1} s) & \text{for } s \leq \alpha(t-a), \end{cases} \tag{3.16}$$

and (3.14) is transformed into

$$T_\alpha(f_\alpha(t)) = \mathbf{G}_0 E(t) + \int_0^{\alpha(t-a)} \dot{\mathbf{G}}(s) E(t - \alpha^{-1} s)\, ds + \int_{\alpha(t-a)}^{+\infty} \dot{\mathbf{G}}(s) E(\alpha t - s + (1-\alpha)a)\, ds. \tag{3.17}$$

Using inequalities of the type (3.10), (3.11) it can be proved that the integrand functions are uniformly bounded by a Lebesgue integrable function. Therefore, by the dominated convergence theorem, we can take the limits inside the integrals. For $\alpha \to 0$, the first integral converges to zero and the second one converges to

$$\int_0^{+\infty} \dot{\mathbf{G}}(s) E(a - s)\, ds = T(a) - \mathbf{G}_0 E(a), \tag{3.18}$$

and for $\alpha \to +\infty$ the first integral converges to

$$\int_0^{+\infty} \dot{\mathbf{G}}(s) E(t^-)\, ds = (\mathbf{G}_\infty - \mathbf{G}_0) E(t^-) \tag{3.19}$$

and the second integral converges to zero. □

Note that Eq. (3.12) associates with $E(t)$ the same stress which, according to Eq. $(2.13)_2$, would follow from a jump of E at a, of amount $E(t) - E(a)$. Moreover, Eq. (3.13) tells us that, if E is continuous at t, an extreme retardation associates with $E(t)$ the equilibrium stress $\mathbf{G}_\infty E(t)$. Thus, Eqs. (3.12) and (3.13) can be interpreted by saying that *an extreme acceleration has the same effect as a jump*, and that *in an extreme retardation of a continuous process the material undergoes a sequence of equilibrium states.*

For the work done in a retardation or acceleration E_α we introduce the notation

$$w_\alpha(E; p, q) := w(E_\alpha; f_\alpha(p), f_\alpha(q)), \tag{3.20}$$

where $[p, q]$ is any interval of the real line, not related with the interval $[a, b]$ in which E has been accelerated or retarded. For the moment, this definition applies only to processes that are continuous in $[p, q]$, because only for such processes is the integral (2.15) well defined. The work done in $[p, q]$ in an extreme acceleration and in an extreme retardation of $[a, b]$ is defined as the limit of $w_\alpha(E; p, q)$ when $\alpha \to 0$ and $\alpha \to +\infty$, respectively. For the particular case $[p, q] = [a, b]$, these limits are evaluated in the next proposition and in the following corollary, which generalize results established by Gurtin and Herrera [7] and by Day [4, 5] under stronger regularity assumptions.

3.3. Proposition. Let E and $\mathbf{G}$ satisfy (A1)–(A3) and let E be continuous in $(a, b]$. Then,

$$\lim_{\alpha \to 0} w_\alpha(E; a, b) = (T(a) - \mathbf{G}_0 E(a)) \cdot (E(b) - E(a)) + \int_a^b \mathbf{G}_0 E(t) \cdot dE(t) \tag{3.21}$$

and

$$\lim_{\alpha\to+\infty} w_\alpha(E;a,b) = \int_a^b \mathbf{G}_\infty E(t)\cdot dE(t). \tag{3.22}$$

Proof. By (2.15) and by the fact that $E_\alpha(f_\alpha(t)) = E(t)$,

$$w_\alpha(E;a,b) = \int_{f_\alpha(a)}^{f_\alpha(b)} T_\alpha(f_\alpha(t))\cdot dE_\alpha(f_\alpha(t)) = \int_a^b T_\alpha(f_\alpha(t))\cdot dE(t). \tag{3.23}$$

If E is continuous in $(a,b]$, so are $E_\alpha(f_\alpha(\cdot))$ and $T_\alpha(f_\alpha(\cdot))$, and the above integral is well defined. We recall that $T_\alpha(f_\alpha(\cdot))$ continuous also implies that the Riemann-Stieltjes integral coincides with the Lebesgue-Stieltjes integral, i.e., with the Lebesgue integral with respect to the Stieltjes measure corresponding to E [11, Sec. 36]. By the preceding proposition, the family $\alpha\mapsto T_\alpha(f_\alpha(\cdot))$ converges to the limits (3.12), (3.13) at all points of $[a,b)$. Since E is continuous at b by assumption, this convergence property extends to the whole interval $[a,b]$. Moreover, it follows from (3.14) that $\alpha\mapsto T_\alpha(f_\alpha(\cdot))$ is uniformly bounded in $[a,b]$ by

$$|T_\alpha(f_\alpha(t))| \le \sup_{r\in(-\infty,b]} |E(r)| \left(\|\mathbf{G}_0\| + \int_0^{+\infty} \|\dot{\mathbf{G}}(s)\|\,ds\right). \tag{3.24}$$

Thus, the Lebesgue dominated convergence theorem applies, and we can take the limits inside the integral (3.23). The desired results follow at once from (3.12) and (3.13). □

3.4. Corollary. Let E and $\mathbf{G}$ be as in the preceding proposition and let $\mathbf{G}_0$ and $\mathbf{G}_\infty$ be symmetric. Then

$$\lim_{\alpha\to0} w_\alpha(E;a,b) = T(a)\cdot(E(b)-E(a)) + \tfrac12\mathbf{G}_0(E(b)-E(a))\cdot(E(b)-E(a)) \tag{3.25}$$

and

$$\lim_{\alpha\to+\infty} w_\alpha(E;a,b) = \tfrac12\mathbf{G}_\infty E(b)\cdot E(b) - \tfrac12\mathbf{G}_\infty E(a)\cdot E(a). \tag{3.26}$$

Proof. The formula of integration by parts for the Stieltjes integral [10, Sec. 8.1] yields

$$\int_a^b \mathbf{G}_0 E(t)\cdot dE(t) = -\int_a^b \mathbf{G}_0^{\mathrm{T}} E(t)\cdot dE(t) + [\mathbf{G}_0 E(t)\cdot E(t)]_a^b. \tag{3.27}$$

Thus, if $\mathbf{G}_0$ is symmetric,

$$\int_a^b \mathbf{G}_0 E(t)\cdot dE(t) = \tfrac12[\mathbf{G}_0 E(t)\cdot E(t)]_a^b, \tag{3.28}$$

and (3.25) follows from (3.21). The proof of (3.26) is similar. □

For intervals $[p,q]$ other than $[a,b]$, the evaluation of the limits of $w_\alpha(E;p,q)$ can be done without difficulty. For $q\le a$, it is immediate to check that $w_\alpha(E;p,q) = w(E;p,q)$ for all $\alpha>0$, so that

$$\lim_{\alpha\to0} w_\alpha(E;p,q) = \lim_{\alpha\to+\infty} w_\alpha(E;p,q) = w(E;p,q). \tag{3.29}$$

It is also not difficult to prove that, for all $p\ge b$,

$$\lim_{\alpha\to0} w_\alpha(E;p,q) = w(E_0;p,q), \qquad \lim_{\alpha\to+\infty} w_\alpha(E;p,q) = w(E_\infty;p,q). \tag{3.30}$$

Any other interval can be decomposed into the sum of subintervals of the type considered above, and the work can be evaluated as the sum of the works done in each subinterval. For example, for an interval $[p, q]$ with $p < a$ and $q > b$,

$$\lim_{\alpha\to+\infty} w_\alpha(E; p, q) = w(E; p, a) + \lim_{\alpha\to+\infty} w_\alpha(E; a, b) + w(E_\infty; b, q). \quad (3.31)$$

The same procedure can be used to evaluate the work done in processes with more than one interval subject to an extreme acceleration or retardation.

The fact, expressed by (3.12), that an extreme acceleration has the same effect on the material response as a jump, suggests that discontinuous processes could be identified with the extreme accelerations of suitable continuous processes. For example, a process E_0 discontinuous at b could be identified with the extreme acceleration in $[a, b]$ of any continuous process E such that, for some $a < b$,

$$E(t) = E_0(t + b - a) \quad \text{for } t < a, \qquad E(t) = E_0(t) \quad \text{for } t \geq b. \quad (3.32)$$

The fundamental advantage of this identification would be the possibility of defining the work done at the jump point by

$$w(E_0; b) := \lim_{\alpha\to 0} w_\alpha(E; a, b). \quad (3.33)$$

Unfortunately, the expression (3.21) of this limit shows that it depends upon the values taken by E in (a, b), which are arbitrary. However, as shown by (3.25), if $\mathbf{G}_0$ is symmetric the limit becomes independent of these values and the definition (3.33) becomes meaningful. In the following, *we identify the extreme acceleration of E in $[a, b]$ with the process E_0 defined by* (3.6), *and we take* (3.33) *as the definition of the work done in a jump*, bearing in mind that this takes for granted the symmetry of $\mathbf{G}_0$.

Let us prove a useful alternative expression for the work (3.33).

3.5. PROPOSITION. Let E_0 be a deformation process discontinuous at b, and let T_0 be the corresponding stress. Let E_0 and $\mathbf{G}$ obey (A1)–(A3) and let $\mathbf{G}_0$ be symmetric. Then

$$w(E_0; b) = \tfrac{1}{2}(T_0(b) + T_0(b^-)) \cdot (E_0(b) - E_0(b^-)). \quad (3.34)$$

Proof. After identifying E_0 with the extreme acceleration in $[a, b]$ of a continuous process E obeying (3.32), from (3.33) and (3.25) we have

$$w(E_0; b) = T(a) \cdot (E(b) - E(a)) + \tfrac{1}{2}\mathbf{G}_0(E(b) - E(a)) \cdot (E(b) - E(a)). \quad (3.35)$$

By (3.32),

$$E(b) = E_0(b), \qquad E(a) = E_0(b^-), \quad (3.36)$$

and by the fact that E, and therefore T, is continuous at a and the restriction of E to $(-\infty, a)$ differs from that of E_0 to $(-\infty, b)$ only by a shift of the time scale,

$$T(a) = T(a^-) = T_0(b^-). \quad (3.37)$$

Moreover, by the formula $(2.13)_2$ relating the jumps of T and E,

$$T_0(b) = T_0(b^-) + \mathbf{G}_0(E_0(b) - E_0(b^-)) = T(a) + \mathbf{G}_0(E(b) - E(a)). \quad (3.38)$$

Substitution into (3.35) yields the desired result. □

The identification of the extreme retardation of E in $[a, b]$ with the process E_∞ defined by (3.7) is impossible. The reason is that the work done by E_∞ is different, in general, from the limit (3.31), which is the work done in the extreme retardation of E. The impossibility of this identification reflects the fact that E_∞ bears no trace of the work done in the interval $(-\infty, f_\alpha(a))$ preceding the retarded interval $[f_\alpha(a), f_\alpha(b)]$. Rather than with a process, an extreme retardation can be identified with a path in stress-strain space; indeed, consider a process E continuous in $[a, b]$ and take its retardations E_α in $[a, b]$. For all $t \in [a, b]$ we have $E_\alpha(f_\alpha(t)) = E(t)$ from (3.2); moreover, by (3.13), for $\alpha \to +\infty$ the stress $T_\alpha(f_\alpha(t))$ converges to $\mathbf{G}_\infty E(t)$. In the limit for $\alpha \to +\infty$, with any $t \in [a, b]$ we can associate the pair $(E(t), \mathbf{G}_\infty E(t))$, which represents an equilibrium state for the material. Thus, in the extreme retardation the material traverses the *equilibrium path*

$$t \mapsto (E(t), \mathbf{G}_\infty E(t)), \quad t \in [a, b]. \tag{3.39}$$

Notice that the parameter t does not coincide any more with the time. Indeed, the time required to traverse the whole path is infinite.

To account for extreme retardations, rather than of the work done in a process, it is convenient to speak of the work done in an *admissible path in stress-strain space*, where by *admissible* we mean either a path in which E is a process obeying (A1)–(A2) and T is the process associated with E by the constitutive equation (1.1), or an equilibrium path associated with an extreme retardation. In both cases the work done in the path has the expression (2.15), but only in the first case does the parameter t coincide with the time.

4. Work postulates. As already remarked, a priori restrictions on the relaxation function have been deduced from the study of two special classes of functions: the dissipative functions and the functions compatible with thermodynamics. Let us introduce the following terminology: we say that a deformation process E is *finite* if there is an interval $[a, b]$ such that $E(t) = E(a)$ for all $t < a$ and $E(t) = E(b)$ for all $t > b$. We say that E *starts from the natural state* if $E(-\infty) = 0$ and that E is *cyclic* if the limit $E(+\infty)$ exists and coincides with $E(-\infty)$.

4.1. Definition (Gurtin and Herrera [7]). A relaxation function is *dissipative* if the work done in any finite process starting from the natural state is nonnegative. □

4.2. Definition (Day [4]). A relaxation function is *compatible with thermodynamics* if the work done in any finite cyclic process is nonnegative. □

The relationship between these two classes of functions and the restrictions to which they are subject are summarized in the following statement.

4.3. Proposition (Day [4]). A relaxation function $\mathbf{G}$ is dissipative if and only if it is compatible with thermodynamics and $\mathbf{G}_\infty$ is positive-semidefinite; $\mathbf{G}$ is compatible with thermodynamics if and only if $\mathbf{G}_\infty$ is symmetric and the function $\overline{\mathbf{G}}$:

$$\overline{\mathbf{G}}(s) := \mathbf{G}(s) - \mathbf{G}_\infty \tag{4.1}$$

is dissipative. If $\mathbf{G}$ is dissipative, then

$$\mathbf{G}_0 \text{ and } \mathbf{G}_\infty \text{ are symmetric and positive-semidefinite} \tag{4.2}$$

and

$$\mathbf{G}_0 \pm \mathbf{G}(s) \geq \mathbf{0}, \quad \forall s \in [0, +\infty). \quad \square \tag{4.3}$$

A requirement that can be removed from Definitions 4.1, 4.2 is that the processes be finite; for example, an *extension of the dissipation principle* to processes that are not finite, but are continuous and have a Lebesgue integrable derivative, can be found in [6]. In fact, we prove below that all restrictions imposed to the work done in finite processes by dissipativity and compatibility with thermodynamics are automatically satisfied by the work done in any admissible path. In the following statement, a *cyclic admissible path* is an admissible path whose final deformation coincides with the initial one.

4.4. Proposition. Assume that (A1)–(A3) hold. Then $\mathbf{G}$ is dissipative if and only if the work done in any admissible path starting from the natural state is nonnegative, and $\mathbf{G}$ is compatible with thermodynamics if and only if the work done in any cyclic admissible path is nonnegative.

Proof. Only the *only if* parts need to be proved. Moreover, we can restrict ourselves to those admissible paths that are associated with deformation processes obeying (A1)–(A2). Indeed, if we are able to prove that $w(E; -\infty, t) \geq 0$ for all $t \in \mathbf{R}$ and for all E that either start from the natural state or are cyclic, then the same holds for their retardations E_α. By definition, the work done in the extreme retardation of E is the limit of $w_\alpha(E; -\infty, b)$ when $\alpha \to +\infty$, and therefore is nonnegative as well.

The proof is based on the fact that, if the variation of E in $(-\infty, b]$ is bounded, then for any $\varepsilon > 0$ there is a $t_\varepsilon < b$ such that

$$V_{-\infty, t_\varepsilon}(E) < \varepsilon. \tag{4.4}$$

Fix a deformation process E, two instants a, b with $a < b$, an $\varepsilon > 0$, and take $t_\varepsilon < a$ satisfying (4.4). Consider the finite process E^ε:

$$E^\varepsilon(r) := \begin{cases} E(-\infty) & \text{for } r < t_\varepsilon \text{ and for } r \geq b, \\ E(r) & \text{for } t_\varepsilon \leq r \leq b, \end{cases} \tag{4.5}$$

and denote by T^ε the stress process associated with E^ε. Using the fact that $E^\varepsilon(r) = E(r)$ for all $r \in (t_\varepsilon, b)$, for any $t \in (t_\varepsilon, b)$ we get

$$w(E; -\infty, t) - w(E^\varepsilon; -\infty, t) = \int_{-\infty}^{t_\varepsilon} T(r) \cdot dE(r) - \int_{-\infty}^{t_\varepsilon} T^\varepsilon(r) \cdot dE^\varepsilon(r) + \int_{t_\varepsilon}^{t} (T(r) - T^\varepsilon(r)) \cdot dE(r). \tag{4.6}$$

For the first integral, by (4.4),

$$\left| \int_{-\infty}^{t_\varepsilon} T(r) \cdot dE(r) \right| \leq \sup_{r \in (-\infty, t_\varepsilon]} |T(r)| \, V_{-\infty, t_\varepsilon}(E) \leq \varepsilon \sup_{r \in (-\infty, a]} |T(r)|. \tag{4.7}$$

Recalling that E^ε takes constant values in $(-\infty, t_\varepsilon)$, the second integral reduces to the work done at the jump point t_ε. Since by Prop. 4.3 both dissipativity and compatibility with thermodynamics imply the symmetry of $\mathbf{G}_0$, the work at the jump can be evaluated by the formula (3.35), with $E(b)$, $E(a)$, and $T(a)$ replaced by $E^\varepsilon(t_\varepsilon) = E(t_\varepsilon)$, $E^\varepsilon(t_\varepsilon^-) = E(-\infty)$, and $T^\varepsilon(t_\varepsilon^-) = \mathbf{G}_\infty E(-\infty)$, respectively. Keeping in mind that, by its very definition (2.1), the variation of E in some interval is greater than the norm of the difference of the values taken by E at any two points of the same interval, from (4.4) we have

$$|E(r) - E(-\infty)| \leq V_{-\infty, t_\varepsilon}(E) < \varepsilon \quad \forall r \in (-\infty, t_\varepsilon], \tag{4.8}$$

and, therefore,

$$\begin{aligned}\left|\int_{-\infty}^{t_\varepsilon} T^\varepsilon(r) \cdot dE^\varepsilon(r)\right| &\leq |(\mathbf{G}_\infty E(-\infty) + \tfrac{1}{2}\mathbf{G}_0(E(t_\varepsilon) - E(-\infty))) \cdot (E(t_\varepsilon) - E(-\infty))| \\ &\leq \|\mathbf{G}_\infty\|\, |E(-\infty)|\varepsilon + \tfrac{1}{2}\|\mathbf{G}_0\|\varepsilon^2.\end{aligned} \tag{4.9}$$

Finally, from (1.1) and (4.8), for all $r \in [t_\varepsilon, t]$,

$$\begin{aligned}|T(r) - T^\varepsilon(r)| &= \left|\int_0^{+\infty} \dot{\mathbf{G}}(s)(E(r-s) - E^\varepsilon(r-s))\, ds\right| \\ &= \left|\int_{r-t_\varepsilon}^{+\infty} \dot{\mathbf{G}}(s)(E(r-s) - E(-\infty))\, ds\right| \\ &\leq \sup_{r\in(-\infty, t_\varepsilon]} |E(r) - E(-\infty)| \int_{r-t_\varepsilon}^{+\infty} \|\dot{\mathbf{G}}(s)\|\, ds \\ &\leq \varepsilon \int_0^{+\infty} \|\dot{\mathbf{G}}(s)\|\, ds.\end{aligned} \tag{4.10}$$

Thus, all terms on the right-hand side of (4.6) are bounded by ε multiplied by a constant independent of ε, t, and b. Therefore,

$$w(E; -\infty, t) \geq w(E^\varepsilon; -\infty, t) - M\varepsilon \quad \forall \varepsilon > 0,\ \forall t \in (a, b), \tag{4.11}$$

with M independent of ε, t, and b.

Let $\mathbf{G}$ be dissipative and let E start from the natural state. Then E^ε is a finite process starting from the natural state, and this implies $w(E^\varepsilon; -\infty, t) \geq 0$. Since ε is arbitrarily small, it follows from (4.11) that $w(E; -\infty, t) \geq 0$ for all t in (a, b), and, since (a, b) has been chosen arbitrarily, the work $w(E; -\infty, t)$ is nonnegative for all $t \in \mathbf{R}$.

Let now $\mathbf{G}$ be compatible with thermodynamics and let E be cyclic. Then E^ε is a finite cyclic process, and this implies $w(E^\varepsilon; -\infty, +\infty) \geq 0$. Using the fact that M does not depend upon b, we can take the limit of (4.11) for $b \to +\infty$ to get

$$w(E; -\infty, +\infty) \geq -M\varepsilon, \tag{4.12}$$

and the nonnegativeness of the work done in the cyclic process E follows by letting $\varepsilon \to 0$. □

We are also interested in *strongly dissipative* relaxation functions, whose definition, due to Gurtin and Herrera [7], is the following.

4.5. DEFINITION. A relaxation function is *strongly dissipative* if it is dissipative and if the only finite deformation process starting from the natural state that satisfies $w(E\,;\,-\infty\,,\,t)=0$ for all $t\in\mathbf{R}$ is the null process $E(t)=0\,,\ \forall t\in\mathbf{R}$. □

When considering admissible paths instead of finite deformation processes, it seems reasonable to replace *finite deformation process* by *admissible path* in the above definition. We recall that, in this case, the parameter t need not be identified with the time. It is also convenient to consider the following alternative definition of strong dissipativity.

4.6. DEFINITION. A relaxation function is *strongly dissipative* if it is dissipative and if, for any deformation process E starting from the natural state,

$$E(t)\neq 0\Rightarrow w(E\,;\,-\infty\,,\,t)>0. \quad \square \tag{4.13}$$

Since any process can be frozen at any time t by setting $E(r)=E(t)\,,\ \forall r>t$, the new definition states that a relaxation function is strongly dissipative if a positive work is required to deform the material starting from the natural state. The equivalence of the two definitions and a characterization of both are given by the following statement.

4.7. PROPOSITION. Let (A1)–(A3) hold. Then the assertions

(i) the relaxation function is strongly dissipative according to Definition 4.6,
(ii) the relaxation function is strongly dissipative according to Definition 4.5, with *finite deformation process* replaced by *admissible path*,
(iii) the relaxation function is dissipative and $\mathbf{G}_\infty$ is positive-definite

are equivalent.

Proof. Assume that the relaxation function satisfies Definition 4.6 and assume that there is an admissible path E starting from the natural state in which $w(E\,;\,-\infty\,,\,t)=0\,,\ \forall t\in\mathbf{R}$. Then (4.13) implies $E(t)=0$ for all $t\in\mathbf{R}$. This proves that (i) $\Rightarrow$ (ii).

Assume now that (ii) is true, and that $\mathbf{G}_\infty$ is symmetric and positive-semidefinite, as required by (4.2), without being positive-definite. Then there is an $A\in\mathrm{Sym}\setminus\{0\}$, such that $\mathbf{G}_\infty A=0$. Consider the process $E(t)=\lambda(t)A$, with λ a continuous scalar-valued function with $\lambda(t)=0$ for all $t<a$ and $\lambda(t)=1$ for all $t\geq b$. Consider the retardations E_α of E in $[a\,,\,b]$. A simple computation based on (3.31) and (3.26) shows that the work done in the extreme retardation is

$$\lim_{\alpha\to+\infty} w_\alpha(E\,;\,-\infty\,,\,t)=\tfrac{1}{2}\lambda^2(t)\mathbf{G}_\infty A\cdot A=0\quad \forall t\in\mathbf{R}. \tag{4.14}$$

Thus, we have a non-null admissible path starting from the natural state, for which the work is zero for all t, in contradiction with (ii). This shows that the assumption that $\mathbf{G}_\infty$ is only positive-semidefinite is false, and proves that (ii) $\Rightarrow$ (iii).

Assume, finally, that (iii) holds. Fix an interval $[a\,,\,b]$, and take a process E starting from the natural state, continuous in $[a\,,\,b]$, and with $E(t)=0$ for all $t\geq b$. If we consider the retardations of E in $[a\,,\,b]$, from (3.26) and (3.31) we have

$$\lim_{\alpha\to+\infty} w_\alpha(E\,;\,-\infty\,,\,b)=w(E\,;\,-\infty\,,\,a)-\tfrac{1}{2}\mathbf{G}_\infty E(a)\cdot E(a). \tag{4.15}$$

For a dissipative relaxation function, $w_\alpha(E; -\infty, b)$ is nonnegative for all α, and therefore the limit is nonnegative. Thus,

$$w(E; -\infty, a) \geq \tfrac{1}{2}\mathbf{G}_\infty E(a) \cdot E(a), \tag{4.16}$$

and (4.13) follows from the positive-definiteness of $\mathbf{G}_\infty$. Since the restriction of E to $(-\infty, a]$ is arbitrary, we have proved that (iii) $\Rightarrow$ (i). □

4.8. REMARK. It follows from the first assertion in Prop. 4.3 that, if $\mathbf{G}_\infty$ is positive-definite, then $\mathbf{G}$ is dissipative if and only if $\mathbf{G}$ is compatible with thermodynamics and $\mathbf{G}_\infty > \mathbf{0}$. Thus, the result just proved can be restated as follows: under the assumptions (A1)–(A3), *a relaxation function* $\mathbf{G}$ *is strongly dissipative if and only if it is compatible with thermodynamics and* $\mathbf{G}_\infty$ *is positive-definite.* □

4.9. REMARK. In [7], Gurtin and Herrera proved that, if $\mathbf{G}$ is twice continuously differentiable and if all deformation processes are continuous and piecewise smooth, then $\mathbf{G}$ is strongly dissipative in the sense of Def. 4.5 if and only if $\mathbf{G}$ is dissipative and $\mathbf{G}_0$ is positive-definite. Here we have proved that the modified definition of strong dissipativity is equivalent to $\mathbf{G}$ dissipative and $\mathbf{G}_\infty$ positive-definite. Since (4.3) implies $\mathbf{G}_0 \geq \mathbf{G}_\infty$, the modified definition of strong dissipativity given here is more restrictive than the original one. □

5. Monotonicity of the relaxation function. A relaxation function $\mathbf{G}$ is *monotonic* (*nonincreasing*) if

$$r > s \Rightarrow \mathbf{G}(r) \leq \mathbf{G}(s). \tag{5.1}$$

$\mathbf{G}$ is nonincreasing if and only if for each $A \in \mathrm{Sym}$ the scalar function g_A:

$$g_A(s) := \mathbf{G}(s)A \cdot A, \quad s \in [0, +\infty), \tag{5.2}$$

is nonincreasing. Therefore, monotonicity is indeed a condition on the symmetric part of $\mathbf{G}$. Although there is some experimental evidence about the monotonicity of the relaxation function, this property has not yet been related with general postulates, such as those discussed in the preceding section. An explicit example of a function that is dissipative but not monotonic, due to Gurtin and Herrera [7], excludes that monotonicity be a consequence of dissipativity or compatibility with thermodynamics. The most successful attempt to characterize monotonicity in terms of work is due to Day [3], who was able to show that, in dimension one, the complete monotonicity of the relaxation function is equivalent to the property of the work being increased by delay in retraced paths. We shall discuss and extend Day's result later.

We say that a deformation process E is *rectilinear* if there is a real-valued function λ such that

$$E(t) = \lambda(t)A \quad \forall t \in \mathbf{R}, \tag{5.3}$$

for some $A \in \mathrm{Sym}$, and that a rectilinear process is *monotonic* if the function λ is monotonic (nonincreasing or nondecreasing). Our characterization of the monotonicity of the relaxation function is that a relaxation function is nonincreasing if and only if the work done in any monotonic rectilinear process is decreased by retardation.

5.1. THEOREM. Let (A1)–(A3) hold and let $\mathbf{G}_0$ be symmetric. Then $\mathbf{G}$ is nonincreasing if and only if, for all monotonic rectilinear processes E and for all finite intervals $[a, b]$, the work done in all retardations E_α of E in $[a, b]$ satisfies

$$w_\alpha(E; a, b) \leq w(E; a, b). \tag{5.4}$$

Proof. By (3.23) and (2.12),

$$\begin{aligned} w_\alpha(E; a, b) - w(E; a, b) &= \int_a^b (T_\alpha(f_\alpha(t)) - T(t)) \cdot dE(t) \\ &= \int_a^b \left(\int_{-\infty}^{f_\alpha(t)} \mathbf{G}(f_\alpha(t) - r)\, dE_\alpha(r) - \int_{-\infty}^t \mathbf{G}(t - r)\, dE(r) \right) \cdot dE(t). \end{aligned} \tag{5.5}$$

We claim that

$$\int_{-\infty}^{f_\alpha(t)} \mathbf{G}(f_\alpha(t) - r)\, dE_\alpha(r) = \int_{-\infty}^t \mathbf{G}(f_\alpha(t) - f_\alpha(r))\, dE(r). \tag{5.6}$$

Indeed, consider a finite interval $[p, t]$, and take points t_i^n, r_i^n as in (2.7). In the interval $[f_\alpha(p), f_\alpha(t)]$, take the points $f_\alpha(t_i^n)$, $f_\alpha(r_i^n)$. Using the definitions (2.7) and (3.2) we get

$$\begin{aligned} \int_{f_\alpha(p)}^{f_\alpha(t)} \mathbf{G}(f_\alpha(t) - r)\, dE_\alpha(r) &= \lim_{n\to+\infty} \sum_{i=1}^n \mathbf{G}(f_\alpha(t) - f_\alpha(r_i^n))(E_\alpha(f_\alpha(t_i^n)) - E_\alpha(f_\alpha(t_{i-1}^n))) \\ &= \lim_{n\to+\infty} \sum_{i=1}^n \mathbf{G}(f_\alpha(t) - f_\alpha(r_i^n))(E(t_i^n) - E(t_{i-1}^n)) \\ &= \int_p^t \mathbf{G}(f_\alpha(t) - f_\alpha(r))\, dE(r), \end{aligned} \tag{5.7}$$

and (5.6) follows for $p \to -\infty$. Substitution into (5.5) yields

$$w_\alpha(E; a, b) - w(E; a, b) = \int_a^b \left(\int_{-\infty}^t (\mathbf{G}(f_\alpha(t) - f_\alpha(r)) - \mathbf{G}(t - r))\, dE(r) \right) \cdot dE(t). \tag{5.8}$$

For a rectilinear process, we have $dE(t) = A\, d\lambda(t)$, and therefore

$$w_\alpha(E; a, b) - w(E; a, b) = \int_a^b \left(\int_{-\infty}^t (g_A(f_\alpha(t) - f_\alpha(r)) - g_A(t - r))\, d\lambda(r) \right) d\lambda(t). \tag{5.9}$$

Let $\mathbf{G}$, and therefore g_A, be nonincreasing. Then the fact that $f_\alpha(t) - f_\alpha(r) \geq t - r$ for all retardations implies

$$g_A(f_\alpha(t) - f_\alpha(r)) \leq g_A(t - r), \tag{5.10}$$

so that the integrand function in (5.9) is nonpositive. If λ is nondecreasing (nonincreasing), then (5.10) and the formula (2.7) imply that the integral

$$\int_{-\infty}^t (g_A(f_\alpha(t) - f_\alpha(r)) - g_A(t - r))\, d\lambda(r) \tag{5.11}$$

is nonpositive (nonnegative). For the same reason, the integral (5.9) is nonpositive, both for nondecreasing and for nonincreasing λ.

Assume now that (5.4) holds. Take a monotonic rectilinear process $E(t) = \lambda(t)A$ with

$$\lambda(t) := \begin{cases} 0 & \text{for } t < a, \\ \lambda_1 & \text{for } a \le t < b, \\ \lambda_1 + \lambda_2 & \text{for } t \ge b, \end{cases} \tag{5.12}$$

and with $\lambda_1\lambda_2 > 0$. Since E is piecewise constant, the work is concentrated at the jump points a, b, and, since $\mathbf{G}_0$ is symmetric, this work can be evaluated using the formula (3.34). From the constitutive equation we easily get

$$T(b^-) = \lambda_1\mathbf{G}(b-a)A, \qquad T(b) = \lambda_1\mathbf{G}(b-a)A + \lambda_2\mathbf{G}_0A, \tag{5.13}$$

so that, by (3.34),

$$w(E; a, b) = (\lambda_1 g_A(b-a) + \tfrac{1}{2}\lambda_2 g_A(0))\lambda_2. \tag{5.14}$$

For the retardations of E in $[a, b]$ we have the same result with $(b-a)$ replaced by $\alpha(b-a)$. Therefore,

$$w_\alpha(E; a, b) - w(E; a, b) = (g_A(\alpha(b-a)) - g_A(b-a))\lambda_1\lambda_2. \tag{5.15}$$

Since the left-hand side is nonpositive and $\lambda_1\lambda_2$ is positive by assumption, we have proved that

$$g_A(\alpha(b-a)) \le g_A(b-a) \quad \forall\alpha > 1. \tag{5.16}$$

The monotonicity of $\mathbf{G}$ follows from the arbitrariness of $(b-a)$ and A. □

6. Complete monotonicity. Consider a function $\mathbf{G} : [0, +\infty) \to \mathrm{LinSym}$. For a given $h > 0$, the *finite difference* of $\mathbf{G}$ at s is

$$\Delta_h\mathbf{G}(s) := \mathbf{G}(s+h) - \mathbf{G}(s), \tag{6.1}$$

and the *finite difference of order* n *at* s is

$$\Delta_h^n\mathbf{G}(s) := \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathbf{G}(s+kh), \quad n \in \mathbf{N}. \tag{6.2}$$

Note that $\Delta_h^0\mathbf{G}(s) = \mathbf{G}(s)$ and $\Delta_h^1\mathbf{G}(s) = \Delta_h\mathbf{G}(s)$. $\mathbf{G}$ is said to be *completely monotonic* if

$$(-1)^n\Delta_h^n\mathbf{G}(s) \ge \mathbf{0} \quad \forall s \in [0, +\infty) \tag{6.3}$$

for all natural integers n and for all $h > 0$. These definitions are obvious extensions of the standard definition for real-valued functions; see, e.g., [14, Chapt. IV]. In particular, for any $A \in \mathrm{Sym}$ we have

$$\Delta_h^n\mathbf{G}(s)A \cdot A = \Delta_h^n g_A(s), \tag{6.4}$$

with g_A the real-valued function defined in (5.2). Thus, $\mathbf{G}$ is completely monotonic if and only if for all $A \in \mathrm{Sym}$ the function g_A is completely monotonic. Since only the symmetric part $\mathbf{G}^{\mathrm{S}}$ of $\mathbf{G}$ enters in the definition of g_A, we have that, just as monotonicity, complete monotonicity is a condition on the symmetric part of $\mathbf{G}$.

6.1. REMARK. It is known that a real-valued function g is completely monotonic if and only if it has all derivatives $D^n g$ and

$$(-1)^n D^n g(s) \geq 0 \tag{6.5}$$

for all $n \in \mathbf{N}$ and for all s in the domain of g [14, Sec. 4.7]. By applying this result to the functions g_A and using the fact that $D^n \mathbf{G}(s)A \cdot A = D^n g_A(s)$, we have that $\mathbf{G}$ is completely monotonic if and only if the function $s \mapsto \mathbf{G}(s)A \cdot A$ has all derivatives and satisfies

$$(-1)^n D^n \mathbf{G}(s)A \cdot A \geq 0. \quad \square \tag{6.6}$$

In dimension one, completely monotonic functions are characterized by Bernstein's representation formula.

6.2. THEOREM (Bernstein). A real-valued function g is completely monotonic if and only if there is a bounded nondecreasing function $\mathbf{k}$ such that

$$g(s) = \int_0^{+\infty} e^{-\omega s}\, d\mathbf{k}(\omega). \quad \square \tag{6.7}$$

For a proof of this theorem see, e.g., [14, Sec. 4.12]. This result can be extended to more dimensions in the following way.

6.3. COROLLARY. A function $\mathbf{G} : [0, +\infty) \to \mathrm{LinSym}$ is completely monotonic if and only if there is a bounded, nondecreasing function $\mathbf{K} : [0, +\infty) \to \mathrm{LinSym}$ such that

$$\mathbf{G}^{\mathrm{S}}(s) = \int_0^{+\infty} e^{-\omega s}\, d\mathbf{K}(\omega). \tag{6.8}$$

Proof. Let $\mathbf{G}^{\mathrm{s}}$ be as in (6.8) and set $k_A(\omega) := \mathbf{K}(\omega)A \cdot A$. Then,

$$g_A(s) = \int_0^{+\infty} e^{-\omega s}\, dk_A(\omega). \tag{6.9}$$

Clearly, $\mathbf{K}$ bounded and nondecreasing implies k_A bounded and nondecreasing. Thus, g_A is completely monotonic by Bernstein's Theorem. Since this holds for all $A \in \mathrm{Sym}$, the function $\mathbf{G}$ is completely monotonic.

Assume now that $\mathbf{G}$ is completely monotonic. Then g_A is completely monotonic for each $A \in \mathrm{Sym}$ and, by Bernstein's Theorem, there is a bounded, nondecreasing function $k_A : [0, +\infty) \to \mathbf{R}$ such that g_A has the representation (6.9). Take an orthonormal basis $\{A^i\}$ of Sym. In view of the identity

$$\begin{aligned} 2\mathbf{G}^{\mathrm{S}}(s)A^i \cdot A^j &= \mathbf{G}^{\mathrm{S}}(s)(A^i + A^j) \cdot (A^i + A^j) - \mathbf{G}^{\mathrm{S}}(s)A^i \cdot A^i - \mathbf{G}^{\mathrm{S}}(s)A^j \cdot A^j \\ &= g_{A^i+A^j}(s) - g_{A^i}(s) - g_{A^j}(s), \end{aligned} \tag{6.10}$$

from (6.9) we get

$$2\mathbf{G}^{\mathrm{S}}(s)A^i \cdot A^j = \int_0^{+\infty} e^{-\omega s}[dk_{A^i+A^j}(\omega) - dk_{A^i}(\omega) - dk_{A^j}(\omega)]. \tag{6.11}$$

Define $\mathbf{K} \in \mathrm{LinSym}$ by

$$\mathbf{K}(\omega) := \sum_{i,j=1}^{m} K_{ij}(\omega) A^i \otimes A^j, \tag{6.12}$$

with m the dimension of Sym and

$$K_{ij}(\omega) := \tfrac{1}{2}[k_{A^i+A^j}(\omega) - k_{A^i}(\omega) - k_{A^j}(\omega)]. \tag{6.13}$$

Then (6.11) implies

$$2\mathbf{G}^{\mathrm{S}}(s)A^i \cdot A^j = 2\left(\int_0^{+\infty} e^{-\omega s}\, d\mathbf{K}(\omega)\right) A^i \cdot A^j, \tag{6.14}$$

and, therefore, (6.8). It remains to prove that $\mathbf{K}$ is bounded, symmetric, and nondecreasing. The first two properties are direct consequences of (6.13). To prove that $\mathbf{K}$ is nondecreasing, it is sufficient to prove that

$$\mathbf{K}(\omega)A \cdot A = k_A(\omega) \quad \forall A \in \mathrm{Sym}, \ \forall \omega \in [0, +\infty). \tag{6.15}$$

Indeed, k_A is nondecreasing for all A. For any fixed A, take the basis $\{A^i\}$ such that $A^1 = A/|A|$. By the fact that, for any positive λ, $k_{\lambda A}(\omega) = \lambda^2 k_A(\omega)$ as a consequence of (5.2) and (6.9), it is sufficient to prove that $\mathbf{K}(\omega)A^1 \cdot A^1 = k_{A^1}(\omega)$. This is done by observing that, by (6.12) and (6.13),

$$\mathbf{K}(\omega)A^1 \cdot A^1 = K_{11}(\omega) = \tfrac{1}{2}k_{2A^1}(\omega) - k_{A^1}(\omega). \quad \square \tag{6.16}$$

We say that a relaxation function is of *positive type* if

$$\int_0^{+\infty}\int_0^{+\infty} \mathbf{G}(r+s)\, dH(r) \cdot dH(s) \geq 0 \tag{6.17}$$

for all functions $H : [0, +\infty) \to \mathrm{Sym}$ with bounded variation. Just as complete monotonicity, this is in fact a condition on the symmetric part of $\mathbf{G}$. Indeed, r and s can be interchanged in (6.17), so that $\mathbf{G}$ can be replaced by $\mathbf{G}^{\mathrm{T}}$ or by $\mathbf{G}^{\mathrm{S}}$.

In [3] Day shows that, in dimension one and considering only continuous functions H, $\mathbf{G}$ is of positive type if and only if it is completely monotonic. Here we give a proof in finite dimension, removing at the same time the requirement that the functions H be continuous.

6.4. Proposition. Let $\mathbf{G}$ satisfy (A3). Then $\mathbf{G}$ is of positive type if and only if it is completely monotonic.

Proof. Let $\mathbf{G}$ be completely monotonic. Recalling that, in (6.17), $\mathbf{G}$ can be replaced by $\mathbf{G}^{\mathrm{S}}$, the representation formula (6.8) yields

$$\int_0^{+\infty}\int_0^{+\infty} \mathbf{G}(r+s)\, dH(r) \cdot dH(s) = \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} e^{-\omega(s+r)}\, d\mathbf{K}(\omega)\, dH(r) \cdot dH(s), \tag{6.18}$$

with $\mathbf{K} : [0, +\infty) \to \mathrm{LinSym}$ symmetric-valued, bounded, and nondecreasing. The function $(\omega, s, r) \mapsto |e^{-\omega(s+r)}|$ being integrable in $(0, +\infty)^3$, we can invoke Fubini's Theorem to interchange the order of integration:

$$\int_0^{+\infty}\int_0^{+\infty} \mathbf{G}(r+s)\, dH(r) \cdot dH(s) = \int_0^{+\infty} d\mathbf{K}(\omega)\left(\int_0^{+\infty} e^{-\omega r}\, dH(r)\right) \cdot \left(\int_0^{+\infty} e^{-\omega s}\, dH(s)\right). \tag{6.19}$$

Denote by $F(\omega)$ the integral in brackets. By the definition (2.7) of the Riemann-Stieltjes integral we get

$$\int_0^{+\infty} d\mathbf{K}(\omega)F(\omega)\cdot F(\omega) = \lim_{n\to\infty}\sum_{j=1}^{n}(\mathbf{K}(t_j^n) - \mathbf{K}(t_{j-1}^n))F(r_j^n)\cdot F(r_j^n), \tag{6.20}$$

where each term in the sum is nonnegative because $\mathbf{K}$ is nondecreasing. Thus, we have proved that $\mathbf{G}$ completely monotonic implies $\mathbf{G}$ of positive type.

Assume now that $\mathbf{G}$ is of positive type. Take a finite subdivision $\{s_i\,,\ i = 0, 1, \dots, n;\ n \in \mathbf{N}\}$ of $[0, +\infty)$, with $s_{i+1} > s_i$, and take $n+1$ tensors $A_0, A_1, \dots, A_n$ in Sym. Consider the piecewise constant function H:

$$H(s) := \begin{cases} 0 & \text{for } s < s_0\,, \\ \sum_{i=0}^{k} A_i & \text{for } s_k \le s < s_{k+1}\,,\quad k = 0, 1, \dots, n-1\,, \\ \sum_{i=0}^{n} A_i & \text{for } s_n \le s. \end{cases} \tag{6.21}$$

Using the formula (2.7) twice, we get

$$\int_0^{+\infty}\int_0^{+\infty} \mathbf{G}(r+s)\,dH(r)\cdot dH(s) = \sum_{i=0}^{n}\int_0^{+\infty}\mathbf{G}(s_i+s)A_i\cdot dH(s) = \sum_{i,j=0}^{n}\mathbf{G}(s_i+s_j)A_i\cdot A_j\,. \tag{6.22}$$

Consider the following particular choice of s_i and A_i. For fixed positive reals a, h and for any fixed $A \in \text{Sym}$, take

$$s_i = \tfrac{1}{2}a + ih\,, \qquad A_i = (-1)^{n-i}\binom{n}{i}A, \quad i = 0, 1, \dots, n. \tag{6.23}$$

The right-hand side of (6.22) takes the form

$$\sum_{i,j=0}^{n}(-1)^{2n-i-j}\binom{n}{i}\binom{n}{j}\mathbf{G}(a+(i+j)h)A\cdot A. \tag{6.24}$$

Setting $k := i + j$, from the identity

$$\sum_{i=0}^{k}\binom{n}{i}\binom{n}{k-i} = \binom{2n}{k} \tag{6.25}$$

and from the definition (6.2) we get

$$\sum_{i,j=0}^{n}\mathbf{G}(s_i+s_j)A_i\cdot A_j = \sum_{k=0}^{2n}(-1)^{2n-k}\binom{2n}{k}\mathbf{G}(a+kh)A\cdot A = \Delta_h^{2n}\mathbf{G}(a)A\cdot A. \tag{6.26}$$

If $\mathbf{G}$ is of positive type, comparison with (6.22) shows that

$$\Delta_h^{2n}\mathbf{G}(a) \ge \mathbf{0} \tag{6.27}$$

for all natural integers n and for all positive reals a and h. It remains to prove that, for all n, a, and h,

$$\Delta_h^{2n+1}\mathbf{G}(a) \le \mathbf{0}. \tag{6.28}$$

Let us first observe that, if $\mathbf{G}(a)$ has a limit when $a \to +\infty$, as implied by (A3), then, by (6.2),

$$\lim_{a\to+\infty} \Delta_h^n \mathbf{G}(a) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \mathbf{G}_\infty = \mathbf{0} \tag{6.29}$$

for all n, a, and h. Consider the identity

$$\Delta_h^{2n}\mathbf{G}(a+2h) - 2\Delta_h^{2n}\mathbf{G}(a+h) + \Delta_h^{2n}\mathbf{G}(a) = \Delta_h^{2n+2}\mathbf{G}(a), \tag{6.30}$$

coming again from (6.2), and note that the right side is positive-semidefinite by (6.27). Writing the same identity at all points $a + kh$, $k = 0, 1, \dots, \ell$ and summing in k we get

$$\Delta_h^{2n}\mathbf{G}(a+(\ell+2)h) - \Delta_h^{2n}\mathbf{G}(a+(\ell+1)h) - \Delta_h^{2n}\mathbf{G}(a+h) + \Delta_h^{2n}\mathbf{G}(a) \geq \mathbf{0}. \tag{6.31}$$

For $\ell \to +\infty$ we get the inequality

$$-\Delta_h^{2n}\mathbf{G}(a+h) + \Delta_h^{2n}\mathbf{G}(a) \geq \mathbf{0} \tag{6.32}$$

which implies (6.28). □

Adopting a terminology introduced in [3], we call *retraced path* a deformation process such that, for some $a \in \mathbf{R}$,

$$E(a-s) = E(a+s), \quad \forall s \geq 0. \tag{6.33}$$

A *retraced path delayed by time* h with respect to E is the retraced path E_h defined by

$$E_h(a+s) := \begin{cases} E(a) & \text{for } 0 \leq s < h, \\ E(a+s-h) & \text{for } s \geq h. \end{cases} \tag{6.34}$$

For simplicity, we denote by $w(E)$ the work $w(E; -\infty, +\infty)$ done in the whole process E. We say that the work is *increased by delay in retraced paths* if $w(E_h) \geq w(E)$ for all retraced paths E and for all $h > 0$. The following theorem generalizes the one-dimensional characterization of complete monotonicity given in [3].

6.5. THEOREM. Let (A1)–(A3) hold. Then the work is increased by delay in retraced paths if and only if the function $\overline{\mathbf{G}}$ defined in (4.1) is completely monotonic.

Proof. By (6.34), the work $w(E_h; -\infty, a-h)$ is equal to $w(E; -\infty, a)$, and $w(E_h; a-h, a+h)$ is equal to zero. Therefore, denoting by T_h the stress process associated with E_h,

$$w(E_h) - w(E) = \int_{a+h}^{+\infty} T_h(t) \cdot dE_h(t) - \int_a^{+\infty} T(t) \cdot dE(t), \tag{6.35}$$

and, after a change of variable in the first integral, using the fact that $E_h(t) = E(t-h)$ in $(a+h, +\infty)$,

$$w(E_h) - w(E) = \int_a^{+\infty} (T_h(t+h) - T(t)) \cdot dE(t). \tag{6.36}$$

From the constitutive equation (2.12), using again (6.34) we get

$$T_h(t+h) - T(t) = \int_{-\infty}^{a} (\mathbf{G}(t+2h-r) - \mathbf{G}(t-r))\, dE(r), \tag{6.37}$$

so that

$$w(E_h) - w(E) = \int_a^{+\infty} \int_{-\infty}^a (\mathbf{G}(t+2h-r) - \mathbf{G}(t-r))\, dE(r) \cdot dE(t). \tag{6.38}$$

After the changes of variable $t = a+p$ and $r = a-q$ and after use of (6.33) we get

$$w(E_h) - w(E) = \int_0^{+\infty} \int_0^{+\infty} (-\mathbf{G}(p+q+h) + \mathbf{G}(p+q))\, dE(a+q) \cdot dE(a+p). \tag{6.39}$$

Since the restriction of E to $(a, +\infty)$ is an arbitrary function of bounded variation, we have proved that the work is increased in retraced paths if and only if the finite difference $-\Delta_h \mathbf{G}$ is of positive type for all $h > 0$. By Proposition 6.4, $-\Delta_h \mathbf{G}$ is of positive type if and only if it is completely monotonic. It remains to prove that $-\Delta_h \mathbf{G}$ is completely monotonic for all $h > 0$ if and only if $\overline{\mathbf{G}}$ is completely monotonic. To do this, it is sufficient to note that, by the definition (6.2),

$$(-1)^n \Delta_h^n \overline{\mathbf{G}}(s) = (-1)^{n-1} \Delta_h^{n-1}(-\Delta_h \mathbf{G}(s)), \quad \forall n \in \mathbf{N}\backslash\{0\}. \tag{6.40}$$

Therefore, $\overline{\mathbf{G}}$ completely monotonic implies $-\Delta_h \mathbf{G}$ completely monotonic for all $h > 0$. Conversely, if $-\Delta_h \mathbf{G}$ is completely monotonic for all $h > 0$, then $\overline{\mathbf{G}}$ satisfies (6.3) for all $n \in \mathbf{N}\backslash\{0\}$. The missing inequality $\overline{\mathbf{G}} \geq \mathbf{0}$ comes from the identity

$$-\mathbf{G}(s+h) + \mathbf{G}(s) = -\Delta_h \mathbf{G}(s) \geq \mathbf{0}, \quad \forall h > 0, \tag{6.41}$$

in the limit for $h \to +\infty$. □

7. Relaxation functions of exponential type. In this section we consider relaxation functions of the form

$$\mathbf{G}(s) = \mathbf{A} + \mathbf{B} e^{s\mathbf{H}}, \tag{7.1}$$

with $\mathbf{A}, \mathbf{B}, \mathbf{H} \in \text{LinSym}$ and with

$$e^{s\mathbf{H}} := \sum_{k=0}^{\infty} \frac{s^k}{k!} \mathbf{H}^k. \tag{7.2}$$

For this particular class of functions we wish to prove that our assumption (A3) is satisfied if and only if all eigenvalues of $\mathbf{H}$ whose eigenspaces are not contained in the null space of $\mathbf{B}$ have a negative real part. The proof is based on the $\mathbf{S}+\mathbf{N}$ decomposition of linear operators on a real finite-dimensional vector space introduced by Hirsch and Smale [9, Sec. 6.2].

7.1. THEOREM (Hirsch and Smale). Any $\mathbf{H} \in \text{LinSym}$ admits a unique decomposition into the sum of two commuting operators $\mathbf{S}$ and $\mathbf{N}$:

$$\mathbf{H} = \mathbf{S} + \mathbf{N}, \qquad \mathbf{S}, \mathbf{N} \in \text{LinSym}, \qquad \mathbf{SN} = \mathbf{NS}, \tag{7.3}$$

with $\mathbf{S}$ semisimple and $\mathbf{N}$ nilpotent. Moreover, $\mathbf{H}$ and $\mathbf{S}$ have the same eigenvalues. □

We recall that $\mathbf{N}$ is *nilpotent* if there is a positive integer q such that $\mathbf{N}^q = \mathbf{0}$, and that $\mathbf{S}$ is *semisimple* if its complexification is diagonalizable. This means that with each complex eigenvalue λ_h, $h = 1, 2, \ldots, \ell$ of $\mathbf{S}$ it is possible to associate a (complex) projection $\mathbf{E}^h$ of the complexification of Sym into the eigenspace corresponding to λ_h such that

$$\mathbf{S}\mathbf{E}^h = \lambda_h \mathbf{E}^h, \qquad \mathbf{E}^h\mathbf{E}^k = \delta_{hk}\mathbf{E}^h, \qquad \sum_{h=1}^{\ell} \mathbf{E}^h = \mathbf{I}, \tag{7.4}$$

and, therefore,

$$\mathbf{S} = \sum_{h=1}^{\ell} \lambda_h \mathbf{E}^h. \tag{7.5}$$

This spectral representation of $\mathbf{S}$ is discussed in [8, Sec. 80] in the particular case of a normal operator, in which the projections $\mathbf{E}^h$ are all orthogonal.

Let us examine the consequences of our assumption (A3) on relaxation functions of the type (7.1).

7.2. PROPOSITION. Let $\mathbf{G}$ be as in (7.1). Then the assertions

(i) $\mathbf{G}$ satisfies (A3),
(ii) $\mathbf{G}_\infty = \mathbf{A}$,
(iii) the eigenvalues of $\mathbf{H}$ whose eigenspaces are not contained in the null space of $\mathbf{B}$ have a negative real part

are equivalent.

Proof. Let $\lambda_h = \alpha_h + i\beta_h$ be an eigenvalue of $\mathbf{H}$ and let L_h be a corresponding (complex) eigenvector:

$$\mathbf{H}L_h = \lambda_h L_h, \quad h = 1, 2, \ldots, \ell. \tag{7.6}$$

Then, by (7.2),

$$e^{s\mathbf{H}}L_h = e^{s\lambda_h}L_h, \qquad \mathbf{H}e^{s\mathbf{H}}L_h = \lambda_h e^{s\lambda_h}L_h \tag{7.7}$$

and, recalling the definition (1.4) of the operator norm and the identities $|e^{s\lambda_h}| = e^{s\alpha_h}$ and $|\lambda_h e^{s\lambda_h}| = |\lambda_h| e^{s\alpha_h}$,

$$\|\mathbf{B}e^{s\mathbf{H}}\| \geq e^{s\alpha_h}\frac{|\mathbf{B}L_h|}{|L_h|}, \qquad \|\mathbf{B}\mathbf{H}e^{s\mathbf{H}}\| \geq |\lambda_h| e^{s\alpha_h}\frac{|\mathbf{B}L_h|}{|L_h|}, \tag{7.8}$$

for all $h = 1, 2, \ldots, \ell$. On the other hand, using the $\mathbf{S}+\mathbf{N}$ decomposition of $\mathbf{H}$, from the fact that $\mathbf{S}$ and $\mathbf{N}$ commute we have

$$\mathbf{B}e^{s\mathbf{H}} = \mathbf{B}e^{s\mathbf{S}}e^{s\mathbf{N}}. \tag{7.9}$$

Since $\mathbf{S}$ and $\mathbf{H}$ have the same eigenvalues, the spectral representation (7.5) of $\mathbf{S}$ leads to the expansion

$$\mathbf{B}e^{s\mathbf{S}} = \sum_{h'} e^{\lambda_{h'}}\mathbf{B}\mathbf{E}^{h'}, \tag{7.10}$$

the sum being made over all indexes h' such that $\mathbf{B}\mathbf{E}^{h'} \neq \mathbf{0}$. Denoting by $\alpha_{h''}$ the maximum of the $\alpha_{h'}$, we get

$$\|\mathbf{B}e^{s\mathbf{S}}\| \leq e^{s\alpha_{h''}}\sum_{h'}\|\mathbf{B}\mathbf{E}^{h'}\|. \tag{7.11}$$

For the nilpotent operator $\mathbf{N}$, the existence of a q such that $\mathbf{N}^q = \mathbf{0}$ implies

$$\|e^{s\mathbf{N}}\| = \left\| \sum_{k=0}^{q-1} \frac{s^k}{k!}\mathbf{N}^k \right\| \leq \varphi(s) \left\| \sum_{k=0}^{q-1} \frac{\mathbf{N}^k}{k!} \right\| = \varphi(s)\|e^{\mathbf{N}}\|, \tag{7.12}$$

with $\varphi(s) = 1$ for $0 \leq s < 1$ and $\varphi(s) = s^{q-1}$ for $s \geq 1$. In conclusion, we have proved that there is a constant M such that

$$\|\mathbf{B}e^{s\mathbf{H}}\| \leq M\varphi(s)e^{s\alpha_{h''}}. \tag{7.13}$$

This inequality, together with $(7.8)_2$ and with the fact that $\dot{\mathbf{G}}(s) = \mathbf{BH}e^{s\mathbf{H}}$, shows that

$$|\lambda_{h''}|\frac{|\mathbf{B}L_{h''}|}{|L_{h''}|}\int_0^{+\infty} e^{s\alpha_{h''}}\,ds \leq \int_0^{+\infty} \|\dot{\mathbf{G}}(s)\|\,ds \leq M\|\mathbf{H}\|\int_0^{+\infty} \varphi(s)e^{s\alpha_{h''}}\,ds, \tag{7.14}$$

and proves that (i) $\Leftrightarrow$ (iii), since (iii) is the same as assuming $\alpha_{h''} < 0$. Moreover, (7.13) together with $(7.8)_1$ implies

$$e^{s\alpha_{h''}}\frac{|\mathbf{B}L_{h''}|}{|L_{h''}|} \leq \|\mathbf{B}e^{s\mathbf{H}}\| \leq Ms^{q-1}e^{s\alpha_{h''}} \tag{7.15}$$

for all $s \geq 1$, and taking the limit for $s \to +\infty$ shows that (ii) $\Leftrightarrow$ (iii). □

From this proposition it follows that a relaxation function of the type (7.1) obeying the assumption (A3) takes the form

$$\overline{\mathbf{G}}(s) = \overline{\mathbf{G}}_0 e^{s\mathbf{H}} \tag{7.16}$$

with $\overline{\mathbf{G}}(s) := \mathbf{G}(s) - \mathbf{G}_\infty$ as in (4.1). In particular, the tensors $\mathbf{A}$ and $\mathbf{B}$ appearing in (7.1) are identified with $\mathbf{G}_\infty$ and $\overline{\mathbf{G}}_0$, respectively.

7.3. Definition. A relaxation function of the type (7.16) is called a *relaxation function of exponential type.* □

In what follows, relaxation functions of exponential type are related with the sign of the work done in closed paths in stress-strain space. If E is a deformation process and T is the corresponding stress process, we say that the restriction of E to the interval $[a, b]$ is a *closed path in stress-strain space* if $E(a) = E(b)$ and $T(a) = T(b)$. It seems reasonable to suppose that the work done in any such path be nonnegative. It turns out that this assumption is very restrictive: it is verified if and only if the relaxation function is of exponential type and compatible with thermodynamics. The proof of this assertion is preceded by the following preliminary results.

7.4. Lemma. Let $\mathbf{G}$ be a relaxation function compatible with thermodynamics, and let $\mathbf{P}$ be the orthogonal projection of LinSym onto the range of $\overline{\mathbf{G}}_0$. Then

$$\overline{\mathbf{G}}_0 = \mathbf{P}\overline{\mathbf{G}}_0 = \overline{\mathbf{G}}_0\mathbf{P} = \mathbf{P}\overline{\mathbf{G}}_0\mathbf{P}, \tag{7.17}$$

and

$$\overline{\mathbf{G}}(s) = \mathbf{P}\overline{\mathbf{G}}(s) \quad \forall s \in [0, +\infty). \tag{7.18}$$

Moreover, if $\mathbf{G}$ is of exponential type,

$$\dot{\mathbf{G}}(0) = \mathbf{P}\dot{\mathbf{G}}(0) = \dot{\mathbf{G}}(0)\mathbf{P} = \mathbf{P}\dot{\mathbf{G}}(0)\mathbf{P}. \tag{7.19}$$

Proof. Equation $(7.17)_1$ is a consequence of the fact that the restriction of $\mathbf{P}$ to the range of $\overline{\mathbf{G}}_0$ coincides with the identity mapping in the range of $\overline{\mathbf{G}}_0$. If $\mathbf{G}$ is compatible with thermodynamics, then $\overline{\mathbf{G}}_0$, and therefore $\mathbf{P}\overline{\mathbf{G}}_0$, is symmetric by Prop. 4.3. Thus, Eq. $(7.17)_2$ follows from the symmetry of $\mathbf{P}$. The last equation follows from $(7.17)_1$.

To prove (7.18), take $A, B \in \mathrm{Sym}$, $a, b \in \mathbf{R}$, with $b > a$ and $p > 0$. Consider the piecewise-constant process

$$E(t) = \begin{cases} 0 & \text{for } t < a - p \text{ and for } t \geq b, \\ A & \text{for } a - p \leq t < a, \\ A + B & \text{for } a \leq t < b. \end{cases} \tag{7.20}$$

The work done in the whole process is the sum of the works done at the discontinuity points, and can be evaluated using the formula (3.35):

$$\begin{aligned} w(E; -\infty, +\infty) = {} & \tfrac{1}{2}\mathbf{G}_0 A \cdot A + \tfrac{1}{2}(2\mathbf{G}(p)A + \mathbf{G}_0 B) \cdot B \\ & + \tfrac{1}{2}(2\mathbf{G}(p + b - a)A + 2\mathbf{G}(b - a)B - \mathbf{G}_0(A + B)) \cdot (-A - B). \end{aligned} \tag{7.21}$$

Since E is a finite cyclic process and $\mathbf{G}$ is compatible with thermodynamics, this work is nonnegative. In particular, for $b - a \to +\infty$ we get

$$\overline{\mathbf{G}}_0(A + B) \cdot (A + B) + (\overline{\mathbf{G}}(p) - \overline{\mathbf{G}}_0)A \cdot B \geq 0, \tag{7.22}$$

and, if B belongs to the null space of $\overline{\mathbf{G}}_0$,

$$\overline{\mathbf{G}}_0 A \cdot A + \overline{\mathbf{G}}(p)A \cdot B \geq 0 \quad \forall A \in \mathrm{Sym}. \tag{7.23}$$

This implies $\overline{\mathbf{G}}(p)A \cdot B = 0$. After introducing the orthogonal projection onto the null space of $\overline{\mathbf{G}}_0$:

$$\mathbf{P}^{\perp} := \mathbf{I} - \mathbf{P}, \tag{7.24}$$

we obtain

$$\overline{\mathbf{G}}(p)A \cdot \mathbf{P}^{\perp} B = 0 \qquad \forall\, A, B \in \mathrm{Sym}. \tag{7.25}$$

This implies $\mathbf{P}^{\perp}\overline{\mathbf{G}}(p) = \mathbf{0}$, and therefore (7.18). Finally, if $\mathbf{G}$ is of exponential type, the differentiation of (2.16) yields

$$\dot{\mathbf{G}}(0) = \overline{\mathbf{G}}_0 \mathbf{H} \tag{7.26}$$

and $(7.19)_1$ follows from $(7.17)_1$. The restriction (4.3) imposed by compatibility with thermodynamics requires that $\dot{\mathbf{G}}(0)$ be negative-semidefinite. Thus,

$$\mathbf{P}\dot{\mathbf{G}}(0)A \cdot A \leq 0 \quad \forall A \in \mathrm{Sym}. \tag{7.27}$$

After introducing the projection $\mathbf{P}^{\perp}$, this condition takes the form

$$\dot{\mathbf{G}}(0)(\mathbf{P}A + \mathbf{P}^{\perp}A) \cdot \mathbf{P}A \leq 0 \quad \forall A \in \mathrm{Sym}, \tag{7.28}$$

and, since $\mathbf{P}A$ and $\mathbf{P}^{\perp}A$ may vary independently, this implies $\dot{\mathbf{G}}(0)\mathbf{P}^{\perp} = \mathbf{0}$, i.e., $\dot{\mathbf{G}}(0) = \dot{\mathbf{G}}(0)\mathbf{P}$. □

7.5. THEOREM. Let the assumptions (A1)–(A3) hold. Then the work done in all closed paths in stress-strain space is nonnegative if and only if $\mathbf{G}$ is of exponential type and compatible with thermodynamics.

Proof. It follows from (7.17) that there is a unique $\mathbf{L}$ in LinSym such that

$$\mathbf{L} = \mathbf{PLP} \quad \text{and} \quad \mathbf{L}\overline{\mathbf{G}}_0 = \overline{\mathbf{G}}_0\mathbf{L} = \mathbf{P}. \tag{7.29}$$

Consequently, for a relaxation function of the form (7.16) and compatible with thermodynamics,

$$\dot{\mathbf{G}}(s) = \dot{\mathbf{G}}(0)e^{s\mathbf{H}} = \dot{\mathbf{G}}(0)\mathbf{P}e^{s\mathbf{H}} = \dot{\mathbf{G}}(0)\mathbf{L}\overline{\mathbf{G}}_0 e^{s\mathbf{H}} = \dot{\mathbf{G}}(0)\mathbf{L}\overline{\mathbf{G}}(s), \tag{7.30}$$

and the constitutive equation (1.1) takes the form

$$T(t) - \mathbf{G}_0 E(t) = \int_{-\infty}^{t} \dot{\mathbf{G}}(t-r)E(r)\,dr = \dot{\mathbf{G}}(0)\mathbf{L}\int_{-\infty}^{t} \overline{\mathbf{G}}(t-r)E(r)\,dr. \tag{7.31}$$

By differentiation, we get

$$\begin{aligned}(T(t) - \mathbf{G}_0 E(t))^{\cdot} &= \dot{\mathbf{G}}(0)\mathbf{L}(\overline{\mathbf{G}}_0 E(t) + T(t) - \mathbf{G}_0 E(t)) \\ &= \dot{\mathbf{G}}(0)\mathbf{L}(T(t) - \mathbf{G}_\infty E(t)).\end{aligned} \tag{7.32}$$

The multiplication of the left-hand side by $\mathbf{L}(T(t) - \mathbf{G}_\infty E(t)) = \mathbf{L}(T(t) - \mathbf{G}_0 E(t) + \overline{\mathbf{G}}_0 E(t))$, followed by integration over $[a, b]$, yields

$$\int_a^b \mathbf{L}(T(t) - \mathbf{G}_0 E(t)) \cdot (T(t) - \mathbf{G}_0 E(t))^{\cdot}\,dt + \int_a^b \mathbf{L}\overline{\mathbf{G}}_0 E(t) \cdot (T(t) - \mathbf{G}_0 E(t))^{\cdot}\,dt. \tag{7.33}$$

In (7.29), the symmetry of $\mathbf{P}$ and $\overline{\mathbf{G}}_0$ implies the symmetry of $\mathbf{L}$. The use of the symmetry of $\mathbf{L}$ in the first integral and integration by parts of the second integral yield

$$\begin{aligned}&[\tfrac{1}{2}\mathbf{L}(T(t) - \mathbf{G}_0 E(t)) \cdot (T(t) - \mathbf{G}_0 E(t))]_a^b - \int_a^b (T(t) - \mathbf{G}_0 E(t)) \cdot \mathbf{L}\overline{\mathbf{G}}_0\,dE(t) \\ &\quad + [\mathbf{L}\overline{\mathbf{G}}_0 E(t) \cdot (T(t) - \mathbf{G}_0 E(t))]_a^b.\end{aligned} \tag{7.34}$$

If the restriction of E to $[a, b]$ is a closed path in stress-strain space, the terms in brackets vanish. Recalling that $\mathbf{L}\overline{\mathbf{G}}_0 = \mathbf{P}$ by (7.29), we get from (7.32)

$$-\int_a^b (T(t) - \mathbf{G}_0 E(t)) \cdot \mathbf{P}\,dE(t) = \int_a^b \dot{\mathbf{G}}(0)\mathbf{L}(T(t) - \mathbf{G}_\infty E(t)) \cdot \mathbf{L}(T(t) - \mathbf{G}_\infty E(t))\,dt. \tag{7.35}$$

Moreover, recalling that $\dot{\mathbf{G}}(0) = \mathbf{P}\dot{\mathbf{G}}(0)$ by (7.19), it follows from (7.31) that $(T(t) - \mathbf{G}_0 E(t)) = \mathbf{P}(T(t) - \mathbf{G}_0 E(t))$. Thus, the operator $\mathbf{P}$ can be omitted from the first integral, which by this way reduces to

$$-\int_a^b (T(t) - \mathbf{G}_0 E(t)) \cdot dE(t) = [\tfrac{1}{2}\mathbf{G}_0 E(t) \cdot E(t)]_a^b - w(E; a, b). \tag{7.36}$$

The term in the bracket vanishes because we are considering a closed path in stress-strain space. Thus, we find that the work done in $[a, b]$ is the opposite of the integral on the right-hand side of (7.35). But this integral is nonpositive by the negative-semidefiniteness of $\dot{\mathbf{G}}(0)$. Thus, we have proved that for a relaxation function of exponential type and compatible with thermodynamics the work done in any path closed in stress-strain space is nonnegative.

Assume now that the work is nonnegative in any closed paths in stress-strain space. Since finite cyclic processes are particular closed paths in stress-strain space, it follows from Def. 4.2 that the relaxation function is compatible with thermodynamics. To prove that $\mathbf{G}$ is of exponential type, take $A, B, C \in \mathrm{Sym}$, $a, b \in \mathbf{R}$ with $b > a$, and $p, q > 0$. Consider the deformation process

$$E(t) = \begin{cases} 0 & \text{for } t < a - p - q, \\ A & \text{for } a - p - q \le t < a - p, \\ A + B & \text{for } a - p \le t < a, \\ A + B + C & \text{for } a \le t < b, \\ A + B & \text{for } b \le t. \end{cases} \tag{7.37}$$

For any fixed $\varepsilon \in (0, p)$, take the restriction of E to $[a - \varepsilon, b]$. Since $E(a-\varepsilon) = E(b) = A + B$, this is a closed path in stress-strain space if

$$\begin{aligned} 0 &= T(b) - T(a - \varepsilon) \\ &= \mathbf{G}(p + q + b - a)A + \mathbf{G}(p + b - a)B + \mathbf{G}(b - a)C - \mathbf{G}_0 C \\ &\quad - \mathbf{G}(p + q - \varepsilon)A - \mathbf{G}(p - \varepsilon)B. \end{aligned} \tag{7.38}$$

The corresponding work is

$$\begin{aligned} w(E; a - \varepsilon, b) = {} & \tfrac{1}{2}(2\mathbf{G}(p+q)A + 2\mathbf{G}(p)B + \mathbf{G}_0 C) \cdot C \\ & + \tfrac{1}{2}(2\mathbf{G}(p + q + b - a)A + 2\mathbf{G}(p + b - a)B \\ & \qquad + 2\mathbf{G}(b - a)C - \mathbf{G}_0 C) \cdot (-C), \end{aligned} \tag{7.39}$$

as given by the formula (3.35) for the work done at discontinuity points. By assumption, this work is nonnegative if A, B, C satisfy (7.38). For $b - a \to +\infty$, the work converges to

$$(\overline{\mathbf{G}}(p + q)A + \overline{\mathbf{G}}(p)B + \overline{\mathbf{G}}_0 C) \cdot C \tag{7.40}$$

and (7.38) reduces to

$$\overline{\mathbf{G}}(p + q - \varepsilon)A + \overline{\mathbf{G}}(p - \varepsilon)B + \overline{\mathbf{G}}_0 C = 0. \tag{7.41}$$

By the property (7.18) of $\overline{\mathbf{G}}$, this is an equation in the range of $\overline{\mathbf{G}}_0$. Thus, there is no loss of generality in taking $C = \mathbf{P}C$ and in pre-multiplying by $\mathbf{L}$. From (7.29) we get

$$C = -\mathbf{L}(\overline{\mathbf{G}}(p + q - \varepsilon)A + \overline{\mathbf{G}}(p - \varepsilon)B), \tag{7.42}$$

and the nonnegativeness of the work is expressed by the inequality

$$\mathbf{L}[\overline{\mathbf{G}}(p{+}q)A{+}\overline{\mathbf{G}}(p)B{-}\overline{\mathbf{G}}(p{+}q{-}\varepsilon)A{-}\overline{\mathbf{G}}(p{-}\varepsilon)B]{\cdot}[\overline{\mathbf{G}}(p{+}q{-}\varepsilon)A{+}\overline{\mathbf{G}}(p{-}\varepsilon)B] \le 0 \tag{7.43}$$

for all $A, B \in \mathrm{Sym}$, for all positive reals p, q, and for all $\varepsilon \in (0, p)$. This implies that the function f:

$$f(p) := |\mathbf{L}^{1/2}(\overline{\mathbf{G}}(p + q)A + \overline{\mathbf{G}}(p)B)| \tag{7.44}$$

is nonincreasing. In particular, the choice

$$B := -\mathbf{L}\overline{\mathbf{G}}(q)A \tag{7.45}$$

yields $f(0) = 0$ and therefore $f(p) = 0$ for all p, because f cannot take negative values. By the arbitrariness of A, this implies

$$\mathbf{L}^{1/2}(\overline{\mathbf{G}}(p+q) - \overline{\mathbf{G}}(p)\mathbf{L}\overline{\mathbf{G}}(q)) = \mathbf{0}, \tag{7.46}$$

and, since the restriction of $\overline{\mathbf{G}}_0$, and therefore of $\mathbf{L}$, to the range of $\overline{\mathbf{G}}_0$ is positive-definite, we conclude that

$$\overline{\mathbf{G}}(p+q) = \overline{\mathbf{G}}(p)\mathbf{L}\overline{\mathbf{G}}(q) \tag{7.47}$$

for all positive p and q. Differentiation with respect to p at $p = 0$ yields

$$\dot{\mathbf{G}}(q) = \dot{\mathbf{G}}(0)\mathbf{L}\overline{\mathbf{G}}(q). \tag{7.48}$$

The solution of this differential equation under the initial condition $\overline{\mathbf{G}}(0) = \overline{\mathbf{G}}_0$ is (7.16), with $\mathbf{PH} = \mathbf{PHP} = \mathbf{L}\dot{\mathbf{G}}(0)$ and $\mathbf{P}^{\perp}\mathbf{H}$ arbitrary. Thus, $\mathbf{G}$ is of exponential type. □

Acknowledgment. This research has been supported by the Italian Ministry for University and Scientific and Technological Research (MURST).

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