EQUILIBRIUM OF MULTIVALUED PERMANENT SYSTEMS WITH NONCONVEX RIGHT-HAND SIDES

By

LI YONG, WANG HUAIZHONG, AND LU XIANRUI

Department of Mathematics, Jilin University, P. R. China

Abstract. The existence of periodic solutions for differential inclusions with nonconvex right-hand sides is proved. As an application, an affirmative answer to the existence of an equilibrium for permanent multivalued systems with nonconvex right-hand sides is given.

1. Introduction. Deterministic modelling in the biological sciences often leads to ordinary differential equations defined on a certain state space,

$$x' = f(x), \quad (1)$$

for example ecological differential equations with the state space $\mathbf{R}_+^n$ (the positive cone):

$$x'_i = x_i f_i(x), \quad i = 1, \dots, n, \quad x = (x_1, \dots, x_n).$$

For such equations, an important and interesting problem is whether permanence implies the existence of an equilibrium state or not.

Here (1) is said to be permanent if there exist $m, M \in (0, \infty)$ such that, given any $x \in \text{int } \mathbf{R}_+^n$ (the interior of $\mathbf{R}_+^n$), there is a T_x such that every solution $x(t)$ starting at x satisfies

$$m \leq x_i(t) \leq M, \quad i = 1, \dots, n, \quad t \geq T_x. \quad (2)$$

Biologically, permanence ensures the long-term survival of all species. There has been a large amount of work devoted to this topic (see, for example, [3, 5, 7, 12–14, 16–19]). As argued in [17], it is reasonable to expect a positive answer to the existence question above. This has indeed been proved by Hofbauer and Sigmund [14], Hutson and Moran [18], and Hutson [17]. In particular, in [17] Hutson gives a simple and efficient discussion of the problem by using the asymptotic Schauder fixed point theorem. There he also proposed the following open problem:

For the multivalued system

$$x' \in F(x), \quad (3)$$

does permanence imply existence of an equilibrium state?

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When $F(x)$ is a compact convex-valued upper semicontinuous map, this problem has been positively solved in our paper [21], recently, by utilizing the asymptotic Schauder fixed point theorem and the approximation theorem of multivalued maps due to Zaremba (see [1] and [28]). However, for the case in which $F(x)$ is not necessarily convex valued, the problem remains open, and we would like to discuss it in the present paper.

As a general observation, we consider the existence problem of periodic solutions for differential inclusions of the form

$$x' \in f(t, x) \quad (4)$$

where f is a multivalued map from $\mathbf{R} \times X$ into X with a real separable Banach space. The problem has been investigated by many authors; see, for example, [8–10, 20, 23, 25].

Here we are concerned with the following problem:

If solutions of (4) are uniformly bounded and uniformly-ultimately bounded, does (4) have periodic solutions?

For ordinary or functional differential equations satisfying the uniqueness condition with respect to initial values, such results have been established by Yoshizawa [27], Hale and Lopes [11], Burton and Zhang [4]. However, solutions of (4) with respect to initial values are not necessarily unique, or convex valued. Hence, direct applications of asymptotic fixed point theorems, due to Browder [2], Nussbaum [24], and Horn [15], to the Poincaré map of (4) seem to be impossible. To overcome these difficulties, we exploit a continuous selection theorem due to Colombo, Fryszkowski, Rzeżuchowski, and Staicu [6]. Of course, in our discussion, some additional conditions on $F(x)$ and $f(t, x)$ are required.

The plan of the paper is as follows. In Sec. 2, using the continuous selection theorem and Horn's asymptotic fixed point theorem (see [15]), we prove that under certain conditions, the uniform boundedness and uniform-ultimate boundedness of solutions for (4) imply the existence of periodic solutions; then applying this result, we give an affirmative answer to the equilibrium problem of (3) in Sec. 3.

2. Periodic solutions. The aim of this section is to present a general existence theorem on periodic solutions of (4).

First, let us recall some basic concepts, definitions and facts (see [26]).

Let X be a real separable space with norm $|\cdot|$, and let $\text{Comp}(X)$ denote the metric space of nonempty compact subsets of X with the Hausdorff metric h defined by

$$h(A, B) = \max\{\sup_{x \in B} \text{dist}(x, A), \sup_{x \in A} \text{dist}(x, B)\}.$$

Let $\omega > 0$ and denote by $\mathcal{L}$ the σ -algebra of all Lebesgue measurable subsets of $[0, \omega]$ and by $\mathcal{B}(X)$ the family of all Borel subsets. By $C([0, \omega], X)$ (resp. $L^1([0, \omega], X)$) we mean the Banach space of all continuous (resp. Bochner integrable) maps $x: [0, \omega] \rightarrow X$ endowed with norm $\|x\| = \sup\{|x(t)|: t \in [0, \omega]\}$ (resp. $\|x\|_1 = \int_0^\omega |x(t)| dt$). $\text{AC}([0, \omega], X)$ stands for the Banach space of all absolutely continuous maps $x: [0, \omega] \rightarrow X$ with norm $\|x\|_{\text{AC}} = |x(0)| + \|x'\|_1$.

Let S be a separable metric space and let $\mathcal{A}$ be a σ -algebra of subsets of S . A map $G: S \rightarrow \text{Comp}(X)$ is said to be $\mathcal{A}$ -measurable if for each closed subset C of X , the set $\{s \in S: G(s) \cap C \neq \emptyset\} \in \mathcal{A}$.

Let $f: [0, \omega] \times X \rightarrow \text{Comp}(X)$ satisfy the following conditions.

(H₁) f is $\mathcal{L} \otimes \mathcal{B}(X)$ -measurable.

(H₂) There exists $k \in L^1([0, \omega], R)$ such that $h(f(t, x), f(t, y)) \leq k(t)|x - y|$, for all $x, y \in X$, a.e. in $[0, \omega]$.

(H₃) There is $\beta \in L^1([0, \omega], R)$ such that $\text{dist}(0, f(t, 0)) \leq \beta(t)$, $t \in [0, \omega]$ a.e.

Consider the Cauchy problem

$$x' \in f(t, x), \quad x(0) = x_0. \quad (5)$$

A map $x: [0, \omega] \rightarrow X$ is said to be a solution of (5), if $x \in \text{AC}([0, \omega], X)$ with $x(0) = x_0$ such that

$$x'(t) \in f(t, x(t)) \text{ a.e.}$$

The following theorem is vital to our discussion.

CONTINUOUS SELECTION THEOREM ([6, Theorem 3.1]). Suppose f satisfies (H₁)–(H₃). Then there exists a map $x: [0, \omega] \times X \rightarrow X$ such that

(I) for each x_0 , $x(t, x_0)$ is a solution of (5) defined on $[0, \omega]$.

(II) The map $x_0 \rightarrow x(\cdot, x_0)$ is continuous from X into $\text{AC}([0, \omega], X)$.

(III) $|x(t, x_0) - x_0| \leq \exp(\int_0^t k(s) ds) + |x_0|(\exp(\int_0^t k(s) ds) - 1) + \int_0^t \beta(\tau) \exp(\int_\tau^t k(s) ds) d\tau$, $t \in [0, \omega]$.

In the following, we also assume

(H₄) for each $x \in X$, $f(t + \omega, x) = f(t, x)$, for all $t \in R$.

Thus a solution $x(t)$ of (4) defined on R is said to be an ω -periodic solution of (4) if $x(t + \omega) = x(t)$, for all $t \in R$.

DEFINITION 1. Let $D \subset X$ be nonempty and bounded. Solutions of (4) are said to be uniformly compact for D , if there is an $S \in \text{Comp}(X)$ such that for each $x_0 \in D$, each solution $x(t)$ of (4) with initial data $x(0) = x_0$ satisfies

$$x(t) \in S, \text{ for } t > 0;$$

if given a bounded $D \subset X$, there is an $H > 0$, such that for each $x_0 \in X$ with $x_0 \in D$, each solution $x(t)$ of (4) with $x(0) = x_0$ satisfies

$$|x(t)| \leq H \text{ for } t \geq 0,$$

then solutions of (4) are said to be uniformly bounded for D .

DEFINITION 2. Let $D_1, D_2 \subset X$ be nonempty and bounded. Solutions of (4) are said to be uniformly D_1 - D_2 bounded if there exists a $T > 0$ such that for each $x_0 \in D_1$, each solution $x(t)$ of (4) with $x(0) = x_0$ satisfies

$$x(t) \in D_2 \text{ for } t \geq T.$$

Let $B > 0$. If given any $M > 0$, there is a $T = T(M) > 0$ such that for each $x_0 \in X$ with $|x_0| \leq M$, each solution $x(t)$ of (4) with $x(0) = x_0$ satisfies

$$|x(t)| \leq B \text{ for } t \geq T(M),$$

then solutions of (4) are said to be *uniformly ultimately bounded*.

We are now in a position to state our main result.

THEOREM 1. Let $D_1, D_2 \subset X$ be nonempty, bounded, and convex such that $D_2 \subset D_1$, $\text{dist}(\partial D_1, D_2) > 0$, and D_1 is open and D_2 is closed. If solutions of (4) are uniformly compact for D_1 and uniformly D_1 - D_2 bounded, then (4) has an ω -periodic solution lying in D_2 .

To prove this theorem, we need the following asymptotic fixed point theorem.

HORN'S FIXED POINT THEOREM ([15]). Let $S_0 \subset S_1 \subset S_2$ be convex subsets of the Banach space X such that S_0 and S_2 are closed and S_1 is open relative to S_2 . Let $P: S_2 \rightarrow X$ be continuous, compact, and such that for some integer $N > 0$,

$$P^j(S_1) \subset S_2, \quad 0 \leq j \leq N-1,$$

and

$$P^j(S_1) \subset S_0, \quad N \leq j \leq 2N-1.$$

Then P has a fixed point in S_0 .

Proof of Theorem 1. By the uniform compactness of solutions for D_1 , there is an $S \in \text{Comp}(X)$ such that for each $x_0 \in D_1$, each solution $x(t)$ of (4) with $x(0) = x_0$ satisfies

$$x(t) \in S \quad \text{for } t > 0. \quad (6)$$

From the uniform D_1 - D_2 boundedness of solutions it is known that there is a positive integer N such that for each $x_0 \in D_1$, each solution $x(t)$ of (4) with $x(0) = x_0$ satisfies

$$x(t) \in D_2 \quad \text{for } t \in [N\omega, (2N-1)\omega]. \quad (7)$$

According to the continuous selection theorem, there exists a map $x: [0, 2N\omega] \times X \rightarrow X$ such that

(a) for each $x_0 \in X$, $x(t, x_0)$ is a solution of (4) defined on $[0, 2N\omega]$;

(b) the map $x_0 \rightarrow x(\cdot, x_0)$ is continuous from X into $\text{AC}([0, 2N\omega], X)$. Define the Poincaré map $P: X \rightarrow X$ by

$$P(x_0) = x(\omega, x_0),$$

and set

$$S_2 = \overline{\text{CO}} \left(\bigcup_{j=0}^N \{x(j\omega, x_0) : x_0 \in D_1\} \right).$$

$$S_1 = D_1, \quad S_0 = D_2.$$

By (6) and the boundedness of D_1 , S_2 is bounded, convex closed, and P is continuous and compact. Obviously, $S_0 \subset S_1 \subset S_2$. Thus, using (6) and (7) we get

$$P^j(S_1) \subset S_2, \quad 0 \leq j \leq N-1,$$

$$P^j(S_1) \subset S_0, \quad N \leq j \leq 2N-1.$$

By Horn's fixed point theorem, P has a fixed point $x^* \in S_0$, which implies that

$$x(\omega, x^*) = x(0, x^*).$$

Define

$$\tilde{x}(t) = \begin{cases} x(t, x^*), & t \in [0, \omega], \\ x(t - j\omega, x^*), & t \in [j\omega, (j+1)\omega], \quad j = \pm 1, \pm 2, \dots \end{cases}$$

Then $\tilde{x}(t)$ is an ω -periodic solution of (4), which completes the proof.

3. Equilibrium states. Now we answer the existence problem of equilibria for (3) by the following theorem.

THEOREM 2. Let the following conditions be fulfilled.

- (I) $F: \mathbf{R}^n \rightarrow \text{Comp}(\mathbf{R}^n)$ is a Lipschitz map, that is, for each $x \in \mathbf{R}^n$ there exist $k, r > 0$ such that $h(F(y), F(z)) \leq k(x)|y - z|$ for $y, z \in S(x, r)$, where $S(x, r) = \{y \in \mathbf{R}^n: \text{dist}(y, x) < r\}$.
- (II) Given any $D \in \text{Comp}(\mathbf{R}_+^n)$, solutions of (3) are uniformly bounded for D .
- (III) There are $M > m > 0$ such that given any $D \in \text{Comp}(\mathbf{R}_+^n)$, there is a $T > 0$ such that for each $x_0 \in D$, each solution $x(t)$ of (3) satisfies

$$m \leq x_i(t) \leq M, \quad t \geq T.$$

Then (3) has an equilibrium in $\text{int}(\mathbf{R}_+^n)$, i.e., there is an $x^* \in \mathbf{R}_+^n$ such that

$$0 \in F(x^*).$$

Proof. Set

$$D_1 = \{y \in \mathbf{R}^n: 0 \leq y_i \leq 2m, \quad i = 1, \dots, n\},$$

$$D_2 = \left\{y \in \mathbf{R}^n: \frac{m}{2} \leq y_i \leq \frac{3M}{2}, \quad i = 1, \dots, n\right\}.$$

By Theorem 1, given any $\omega > 0$, (3) has an ω -periodic solution $x(t, x_0(\omega))$ satisfying

$$\frac{m}{2} \leq x_i(t, x_0(\omega)) \leq \frac{3M}{2}, \quad t \in \mathbf{R}, \quad i = 1, \dots, n.$$

Applying [17, Lemma 2], we know that there is $x^* \in \mathbf{R}^n$ with $\frac{m}{2} \leq x_i^* \leq \frac{3M}{2}$, $i = 1, \dots, n$ such that

$$x(t, x^*) \equiv x^*, \quad t \in \mathbf{R}.$$

By the continuous selection theorem (I), $x(t, x^*)$ is a solution of (3). Hence

$$0 \in F(x^*),$$

which proves the theorem.

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