AN EXACT SOLUTION TO A PROBLEM OF AXISYMMETRIC TORSION OF AN ELASTIC SPACE WITH A SPHERICAL CRACK

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Abstract. It is a well-known fact that the three-dimensional axisymmetric dilatation problem for an elastic space containing a spherical crack has an explicit solution, found and analyzed in [1]–[3]. The corresponding torsion problem, however, also of interest, has not apparently been investigated. To remedy this shortage we present here an exact solution to this problem, which incidentally reveals a remarkable effect concerning the stress intensity.

Denoting by (R, θ, ϕ) a spherical coordinate system, let a crack be situated at $R = a$, $0 \leq \theta \leq \theta_0$, $0 \leq \phi < 2\pi$, in an elastic medium twisted at infinity. Then the displacement vector is $\{0, 0, v_\phi = v(R, \theta)\}$, and the only nonvanishing stress components are

$$\tau_{R\phi} = GR \frac{\partial}{\partial R} \left(\frac{v}{R} \right), \quad \tau_{\theta\phi} = \frac{G}{R} \left(\frac{\partial v}{\partial \theta} - v \cot \theta \right), \quad (1)$$

where G is the shear modulus. We look for a solution in the form

$$v(R, \theta) = u^*(R, \theta) + u(R, \theta), \quad (2)$$

where $u^* = \alpha R^2 \sin \theta \cos \theta$ is the displacement in the crack-free medium, and α is the twist angle per unit length. Setting

$$u = \begin{cases} u_i(R, \theta), & 0 \leq R < a, \\ u_e(R, \theta), & R > a \end{cases} \quad (3)$$

we arrive at the following mixed boundary-value problem:

$$\nabla^2 u - \frac{u}{R^2 \sin^2 \theta} = 0, \quad R \neq a; \quad (4)$$

$$u_e \rightarrow 0 \quad \text{as } R \rightarrow \infty; \quad (5)$$

$$\tau_{R\phi}^{(i)}(a, \theta) = \tau_{R\phi}^{(e)}(a, \theta) = -\alpha a G \sin \theta \cos \theta, \quad 0 \leq \theta < \theta_0; \quad (6)$$

$$\tau_{R\phi}^{(i)}(a, \theta) = \tau_{R\phi}^{(e)}(a, \theta), \quad \theta_0 < \theta \leq \pi; \quad (7)$$

$$u_i(a, \theta) = u_e(a, \theta), \quad \theta_0 < \theta \leq \pi. \quad (8)$$

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A solution that satisfies Eq. (4) and is compatible with the condition (5) at infinity reads

$$u_i = \alpha a^2 \sum_{n=1}^{\infty} A_n \left(\frac{R}{a} \right)^n P_n^1(\cos \theta), \quad 0 \leq R < a; \quad (9)$$

$$u_e = \alpha a^2 \sum_{n=1}^{\infty} B_n \left(\frac{a}{R} \right)^{n+1} P_n^1(\cos \theta), \quad R > a. \quad (10)$$

Here $P_n^1(\cos \theta)$ is the associated Legendre function [4] and A_n, B_n are the unknown dimensionless coefficients. The stress continuity conditions (6)–(7) relate B_n to A_n by

$$B_n = -\frac{n-1}{n+2} A_n, \quad n = 1, 2, \dots, \quad (11)$$

and then (6) and (8) yield the following dual series associated with Legendre functions:

$$\sum_{n=1}^{\infty} (n-1) A_n P_n^1(\cos \theta) = -\sin \theta \cos \theta, \quad 0 \leq \theta < \theta_0; \quad (12)$$

$$\sum_{n=1}^{\infty} \frac{2n+1}{n+2} A_n P_n^1(\cos \theta) = 0, \quad \theta_0 \leq \theta < \pi. \quad (13)$$

This system of dual series differs from the one investigated in [5] and therefore, according to [3], we represent the solution of (12)–(13) in the form

$$A_n = \frac{n+2}{2n(n+1)} \int_0^{\theta_0} \psi(t) \cos \left(n + \frac{1}{2} \right) t dt \quad (14)$$

with $\psi(t) \in C_{[0, \theta_0]}$.

Substituting (14) back in (12)–(13) and resorting to Fourier expansion in Legendre polynomials [4] we find that (13) holds if $\psi(t)$ satisfies the condition

$$\int_0^{\theta_0} \psi(t) \cos \frac{t}{2} dt = 0. \quad (15)$$

After integration of (12) with respect to θ , and taking into account Mehler's integral representation [4] with the necessary discontinuous sum for Legendre polynomials [4], we obtain the Abel integral equation

$$\int_0^{\theta} \left[\psi(x) - \frac{4}{\pi} \int_0^{\theta_0} K(x, t) \psi(t) \right] \frac{dt dx}{\sqrt{2 \cos x - 2 \cos \theta}} = C - \sin^2 \theta, \quad 0 \leq \theta < \theta_0, \quad (16)$$

where

$$\begin{aligned} K(x, t) &= \sum_{n=1}^{\infty} \frac{\cos(n + \frac{1}{2})x \cos(n + \frac{1}{2})t}{n(n+1)} \\ &= 2 \cos \frac{t}{2} \left[(\pi - x) \sin \frac{x}{2} - \cos \frac{x}{2} \right] - 2t \sin \frac{t}{2} \cos \frac{x}{2} \end{aligned} \quad (17)$$

and C is an unknown constant.

Solution of Eq. (16) leads to a Fredholm integral equation of the second kind:

$$\psi(x) - \frac{4}{\pi} \int_0^{\theta_0} K(x, t) \psi(t) dt = \frac{4}{3\pi} \cos \frac{5x}{2} + D \cos \frac{x}{2}, \quad 0 \leq x < \theta, \quad (18)$$

where D is another unknown constant. The kernel (17) of this equation is a degenerate one and thus permits an explicit solution to Eq. (18) that satisfies condition (15) and specifies the value of the unknown constant D , namely

$$\psi(x) = \frac{4}{3\pi} \left[\cos \frac{5x}{2} - \frac{2 \sin 3\theta_0 + 3 \sin 2\theta_0}{6(\theta_0 + \sin \theta_0)} \cos \frac{x}{2} \right]. \quad (19)$$

The shear stress $\tau_{R\phi}$ on the surface $R = a$ follows as

$$\begin{aligned} \tau_{R\phi}(a, \theta) = \alpha a G \left[\sin \theta \cos \theta - \frac{1}{2} \sin \theta \int_0^{\theta_0} \frac{\psi(t) dt}{(2 \cos t - 2 \cos \theta)^{3/2}} \right. \\ \left. + \frac{1}{\sin \theta} \int_0^{\theta_0} \psi(t) (2 \cos t - 2 \cos \theta)^{1/2} dt \right], \quad \theta_0 < \theta \leq \pi, \end{aligned} \quad (20)$$

and the integrals here may be calculated explicitly with the aid of (19).

This formula results in an expression for the stress intensity factor K_{III} in the form

$$\begin{aligned} K_{III} &= \lim_{\theta \rightarrow \theta_0+} \tau_{R\phi} \sqrt{2\pi a(\theta - \theta_0)} \\ &= \frac{2\alpha a^{3/2} G}{3\sqrt{\pi} \sin^{1/2} \theta_0} \left[\frac{2 \sin 3\theta_0 + 3 \sin 2\theta_0}{6(\theta_0 + \sin \theta_0)} \cos \frac{\theta_0}{2} - \cos \frac{5\theta_0}{2} \right]. \end{aligned} \quad (21)$$

Figure 1 (see p. 682) illustrates the dependence of $\bar{K}_{III} = K_{III}(\alpha a^{3/2} G)^{-1}$ on the crack angle θ_0 , which appears to be nonmonotonic. At $\theta_0 \approx 106^\circ$, K_{III} vanishes completely and then changes its sign. Its maximum and minimum values are at $\theta_0 \approx 61^\circ$ and 152° , respectively; the highest value of $|K_{III}|$ is at $\theta_0 = 152^\circ$.

Finally, it is worthwhile to consider the limiting case $\theta_0 \rightarrow 0$, $a \rightarrow \infty$ such that $a\theta_0 \rightarrow b$ (transition to a penny-shaped crack). In that case, we find from (21)

$$K_{III} = \frac{4}{3\sqrt{\pi}} \alpha b^{3/2} G, \quad (22)$$

in agreement with the known result from (6).

FIG. 1. Dependence of the normalized stress intensity factor $\bar{K}_{III}$ on the crack angle

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REFERENCES

- [1] V. A. Ziuzin and V. I. Mossakovskii, *Axisymmetric loading of a space with a spherical cut*, Appl. Math. Mech. **34**, 172–177 (1970)
- [2] N. L. Prokhorova and Iu. I. Solov'ev, *Axisymmetric problem for an elastic space with a spherical cut*, Appl. Math. Mech. **40**, 640–646 (1976)
- [3] M. A. Martynenko and A. F. Ulitko, *Stress state near a vertex of a spherical notch in an unbounded elastic medium*, Soviet Appl. Mech. **14**, 911–918 (1978)
- [4] H. Bateman and A. Erdélyi, *Higher Transcendental Functions*, Vol. I, McGraw-Hill, New York, 1955
- [5] Ia. S. Uflyand, *Method of Dual Equations in Mathematical Physics Problems*, Nauka, Leningrad, 1977 (Russian)
- [6] I. N. Sneddon and M. Lowengrub, *Crack Problem in the Classical Theory of Elasticity*, John Wiley and Sons, New York, 1969