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THE VIBRATION OF AN ELASTIC DIELECTRIC WITH PIEZOELECTROMAGNETISM

By

J. S. YANG (*Dept. of Civil Engineering and Operations Research, Princeton University, Princeton, NJ*)

AND

X. Y. WU (*Dept. of Civil and Environmental Engineering, Carleton University, Ottawa, Ontario, Canada*)

Abstract. This paper presents a few results on the free vibration of a finite elastic dielectric with linear piezoelectromagnetism. Following the proof of selfadjointness, the orthogonality of modes corresponding to different frequencies is proved. A variational principle is given in Rayleigh quotient form for the natural frequency. The variational principle is mixed in the sense that all field variables can be varied independently, and it can be used to generate other variational principles.

1. Introduction. The theory of piezoelectromagnetism (dynamic piezoelectricity) has fully dynamic electromagnetic fields. Reciprocity, uniqueness, and minimum principles have been proved in [1]. The vibration of piezoelectromagnetic plates has been studied [2–4] to consider the effect of electromagnetic radiation. The prediction of electromagnetic radiation from vibrating piezoelectric bodies is important in the resonator industry. Knowledge of the amount of energy radiated is needed in computing the quality factor of the resonator. The exact treatment of radiation phenomena requires a fully dynamic theory. The quasi-static theory of piezoelectricity can at most give an approximation of the radiation. A variational principle for piezoelectromagnetism is given in [5], which can be used to derive field equations for piezoelectromagnetism. A mixed variational principle for the field equations of piezoelectromagnetism is given in [6].

In this paper, a few basic properties of the eigenvalue problem for the free vibration of a finite elastic dielectric with linear piezoelectromagnetism are established. The selfadjointness of the eigenvalue problem is proved first, which then leads to the orthogonality of modes corresponding to different frequencies. A mixed variational formulation for the natural frequency is derived in Rayleigh quotient form with all field variables as independent variables. The variational principle can be used to derive other variational principles. This will be shown by an example. The variational principles given here generalize the results in [8] from the quasistatic case to the dynamic case.

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2. Governing equations. Let the finite spatial region occupied by the piezoelectromagnetic elastic dielectric be V, the boundary surface of V be S, the unit outward normal of S be n_i, and S be partitioned in the following ways:

$$
\begin{aligned}
&S_u \cup S_T = S_\phi \cup S_D = S_A \cup S_H = S, \\
&S_u \cap S_T = S_\phi \cap S_D = S_A \cap S_H = \varnothing.
\end{aligned} \tag{1}
$$

The governing equations for the free motion of a finite piezoelectromagnetic body in V are [5]

$$
\begin{gathered}
T_{ji,j} = \rho \dot{v}_i, \qquad -\rho v_i = -\rho \dot{u}_i \quad \text{in } V, \\
S_{ij} - \tfrac{1}{2}(u_{j,i} + u_{i,j}) = 0 \quad \text{in } V, \\
-E_i - \phi_{,i} = \dot{A}_i, \qquad B_i - \varepsilon_{ijk} A_{k,j} = 0 \quad \text{in } V, \\
D_{i,i} = 0, \qquad -\varepsilon_{ijk} H_{k,j} = -\dot{D}_i \quad \text{in } V, \\
T_{ij} - (c_{ijkl} S_{kl} - e_{kij} E_k) = 0 \quad \text{in } V, \\
-D_i - (-e_{ikl} S_{kl} - \varepsilon_{ik} E_k) = 0 \quad \text{in } V, \\
H_i - \frac{1}{\mu_0} B_i = 0 \quad \text{in } V,
\end{gathered} \tag{2}
$$

with homogeneous boundary conditions

$$
\begin{aligned}
u_i &= 0 \quad \text{on } S_u, & -T_{ji} n_j &= 0 \quad \text{on } S_T, \\
\phi &= 0 \quad \text{on } S_\phi, & -D_i n_i &= 0 \quad \text{on } S_D, \\
\varepsilon_{ijk} n_j A_k &= 0 \quad \text{on } S_A, & \varepsilon_{ijk} n_j H_k &= 0 \quad \text{on } S_H,
\end{aligned} \tag{3}
$$

where ρ is mass density, T_{ij} stress, S_{ij} strain, u_i displacement, v_i velocity, E_i electric field, D_i electric displacement, B_i magnetic induction, H_i magnetic field, ϕ and A_i the scalar and vector potentials of the electromagnetic fields in the dielectric, and μ_0 is the magnetic permeability of free space. c_{ijkl}, e_{kij}, and ε_{ij} are all material constants. ε_{ijk} is the permutation tensor. Because of the potential representation $(2)_3$ of the electromagnetic fields, only two $((2)_4)$ of the four Maxwell's equations are left and the other two are identically satisfied.

We note that the stress equation of motion $(2)_1$ has been written in terms of the velocity v_i so that only the first-order time derivative appears. This is for consistency in form with the first-order time derivative in Maxwell's equations and the potential representation of the electromagnetic fields.

The homogeneous electromagnetic boundary conditions $(3)_{2,3}$ include the two common electromagnetic boundary conditions [7] of short circuit boundaries (electric wall, on which tangential $\mathbf{E}$ and normal $\mathbf{B}$ vanish) and open circuit boundary (magnetic wall, on which tangential $\mathbf{H}$ and normal $\mathbf{D}$ vanish) as special cases.

For time harmonic motions, let

$$
\begin{aligned}
&u_i(\mathbf{x}, t) = u_i(\mathbf{x})\cos\omega t, && v_i(\mathbf{x}, t) = v_i(\mathbf{x})\sin\omega t,\\
&T_{ij}(\mathbf{x}, t) = T_{ij}(\mathbf{x})\cos\omega t, && S_{ij}(\mathbf{x}, t) = S_{ij}(\mathbf{x})\cos\omega t,\\
&E_i(\mathbf{x}, t) = E_i(\mathbf{x})\cos\omega t, && D_i(\mathbf{x}, t) = D_i(\mathbf{x})\cos\omega t,\\
&\phi(\mathbf{x}, t) = \phi(\mathbf{x})\cos\omega t, && \\
&H_i(\mathbf{x}, t) = H_i(\mathbf{x})\sin\omega t, && B_i(\mathbf{x}, t) = B_i(\mathbf{x})\sin\omega t,\\
&A_i(\mathbf{x}, t) = A_i(\mathbf{x})\sin\omega t. &&
\end{aligned}
\tag{4}
$$

Then (2) and (3) become

$$
\begin{aligned}
&T_{ji,j} = \omega\rho v_i, \qquad -\rho v_i = \omega\rho u_i \quad \text{in } V,\\
&S_{ij} - \tfrac{1}{2}(u_{j,i} + u_{i,j}) = 0 \quad \text{in } V,\\
&-E_i - \phi_{,i} = \omega A_i, \qquad B_i - \varepsilon_{ijk}A_{k,j} = 0 \quad \text{in } V,\\
&D_{i,i} = 0, \qquad -\varepsilon_{ijk}H_{k,j} = \omega D_i \quad \text{in } V,\\
&T_{ij} - (c_{ijkl}S_{kl} - e_{kij}E_k) = 0 \quad \text{in } V,\\
&-D_i - (-e_{ikl}S_{kl} - \varepsilon_{ik}E_k) = 0 \quad \text{in } V,\\
&H_i - \frac{1}{\mu_0}B_i = 0 \quad \text{in } V,
\end{aligned}
\tag{5}
$$

and

$$
\begin{aligned}
u_i &= 0 \quad \text{on } S_u, & -T_{ji}n_j &= 0 \quad \text{on } S_T,\\
\phi &= 0 \quad \text{on } S_\phi, & -D_in_i &= 0 \quad \text{on } S_D,\\
\varepsilon_{ijk}n_jA_k &= 0 \quad \text{on } S_A, & \varepsilon_{ijk}n_jH_k &= 0 \quad \text{on } S_H.
\end{aligned}
\tag{6}
$$

Values of ω are sought corresponding to which nontrivial solutions of u_i, v_i, S_{ij}, T_{ij}, ϕ, E_i, D_i, A_i, H_i, and B_i exist. Hence (5) and (6) constitute an eigenvalue problem. For (5) and (6), it is convenient to introduce the electric enthalpy density function

$$
\begin{aligned}
H(\mathbf{S}, \mathbf{E}, \mathbf{B}) = {}& \tfrac{1}{2}c_{ijkl}S_{ij}S_{kl} - e_{ijk}E_iS_{jk}\\
&- \tfrac{1}{2}\varepsilon_{ij}E_iE_j + \tfrac{1}{2}\mu_0^{-1}B_iB_i \quad \text{in } V.
\end{aligned}
\tag{7}
$$

Then (5) and (6) can be written as

$$
\begin{aligned}
&T_{ji,j} = \omega\rho v_i, \qquad -\rho v_i = \omega\rho u_i \quad \text{in } V,\\
&S_{ij} - \tfrac{1}{2}(u_{j,i} + u_{i,j}) = 0 \quad \text{in } V,\\
&-E_i - \phi_{,i} = \omega A_i, \qquad B_i - \varepsilon_{ijk}A_{k,j} = 0 \quad \text{in } V,\\
&D_{i,i} = 0, \qquad -\varepsilon_{ijk}H_{k,j} = \omega D_i \quad \text{in } V,\\
&T_{ij} - \frac{\partial H}{\partial S_{ij}} = 0, \qquad -D_i - \frac{\partial H}{\partial E_i} = 0, \qquad H_i - \frac{\partial H}{\partial B_i} = 0 \quad \text{in } V,
\end{aligned}
\tag{8}
$$

and

$$
\begin{aligned}
u_i &= 0 \quad \text{on } S_u, & -T_{ji}n_j &= 0 \quad \text{on } S_T,\\
\phi &= 0 \quad \text{on } S_\phi, & -D_in_i &= 0 \quad \text{on } S_D,\\
\varepsilon_{ijk}n_jA_k &= 0 \quad \text{on } S_A, & \varepsilon_{ijk}n_jH_k &= 0 \quad \text{on } S_H.
\end{aligned}
\tag{9}
$$

3. Selfadjointness and orthogonality. We introduce the following abstract notations of vector $\mathbf{U}$, operators $\mathbf{L}$ and $\mathbf{M}$, and inner product $\langle\ ,\ \rangle$:

$$\mathbf{U} = \{u_i, v_i, T_{ij}, D_i, H_i, \phi, A_i, S_{ij}, E_i, B_i\}, \tag{10}$$

$$\mathbf{LU} = \left\{ T_{ji,j}, -\rho v_i, S_{ij} - \tfrac{1}{2}(u_{j,i} + u_{i,j}), D_{i,i}, -E_i - \phi_{,i}, \right. \\ \left. B_i - \varepsilon_{ijk}A_{k,j}, -\varepsilon_{ijk}H_{k,j}, T_{ij} - \frac{\partial H}{\partial S_{ij}}, -D_i - \frac{\partial H}{\partial E_i}, H_i - \frac{\partial H}{\partial B_i} \right\}, \tag{11}$$

$$\mathbf{MU} = \{\rho v_i, \rho u_i, 0, A_i, 0, 0, D_i, 0, 0, 0\}, \tag{12}$$

$$\langle \mathbf{U}, \mathbf{U}^* \rangle = \int_V [u_i u_i^* + v_i v_i^* + T_{ij}T_{ij}^* + D_i D_i^* + H_i H_i^* \\ + \phi\phi^* + A_i A_i^* + S_{ij}S_{ij}^* + E_i E_i^* + B_i B_i^*]\, dV, \tag{13}$$

where $\mathbf{U}^*$ is another arbitrary abstract vector. It is clear that the above inner product is symmetric, that is, $\langle \mathbf{U}, \mathbf{U}^* \rangle = \langle \mathbf{U}^*, \mathbf{U} \rangle$. With the above definitions, Eq. (8) in V can be written as

$$\mathbf{LU} = \omega \mathbf{MU}. \tag{14}$$

It can be verified with integration by parts that for any two abstract vectors $\mathbf{U}$ and $\mathbf{U}^*$ satisfying homogeneous boundary conditions (9) the following is true:

$$\begin{aligned}
\langle \mathbf{LU}, \mathbf{U}^* \rangle &= \int_V \left\{ T_{ji,j}u_i^* - \rho v_i v_i^* + [S_{ij} - \tfrac{1}{2}(u_{j,i} + u_{i,j})]T_{ij}^* \right. \\
&\quad - (E_i + \phi_{,i})D_i^* + (B_i - \varepsilon_{ijk}A_{k,j})H_i^* + D_{i,i}\phi^* - \varepsilon_{ijk}H_{k,j}A_i^* \\
&\quad \left. + \left(T_{ij} - \frac{\partial H}{\partial S_{ij}}\right) S_{ij}^* - \left(D_i + \frac{\partial H}{\partial E_i}\right) E_i^* + \left(H_i - \frac{\partial H}{\partial B_i}\right) B_i^* \right\} dV \\
&= \int_V \left\{ u_i T_{ji,j}^* - v_i \rho v_i^* + T_{ij}[S_{ij}^* - \tfrac{1}{2}(u_{j,i}^* + u_{i,j}^*)] \right. \\
&\quad - D_i(E_i^* + \phi_{,i}^*) + H_i(B_i^* - \varepsilon_{ijk}A_{k,j}^*) + \phi D_{i,i}^* - A_i \varepsilon_{ijk}H_{k,j}^* \\
&\quad \left. + S_{ij}\left(T_{ij}^* - \frac{\partial H^*}{\partial S_{ij}^*}\right) - E_i\left(D_i^* + \frac{\partial H^*}{\partial E_i^*}\right) + B_i\left(H_i^* - \frac{\partial H^*}{\partial B_i^*}\right) \right\} dV \\
&= \langle \mathbf{U}, \mathbf{LU}^* \rangle, \\
\langle \mathbf{MU}, \mathbf{U}^* \rangle &= \int_V (\rho v_i u_i^* + \rho u_i v_i^* + A_i D_i^* + D_i A_i^*)\, dV \\
&= \int_V (u_i \rho v_i^* + v_i \rho u_i^* + D_i A_i^* + A_i D_i^*)\, dV \\
&= \langle \mathbf{U}, \mathbf{MU}^* \rangle.
\end{aligned} \tag{15}$$

Hence the operators $\mathbf{L}$ and $\mathbf{M}$ are selfadjoint for abstract vectors satisfying homogeneous boundary conditions (9). With the selfadjointness, we can now proceed to prove the orthogonality of eigenvectors corresponding to different eigenvalues. Let

ω and ω^* be two different eigenvalues and let their corresponding eigenvectors be $\mathbf{U}$ and $\mathbf{U}^*$; we have

$$\begin{aligned}\mathbf{LU} &= \omega\mathbf{MU},\\ \mathbf{LU}^* &= \omega^*\mathbf{MU}^*.\end{aligned} \tag{16}$$

Taking the inner product of both sides of $(16)_1$ by $\mathbf{U}^*$ and both sides of $(16)_2$ by $\mathbf{U}$, and then subtracting the resulting equations, we obtain

$$0 = (\omega - \omega^*)\langle \mathbf{U}^*, \mathbf{MU}\rangle. \tag{17}$$

Since $\omega \neq \omega^*$, (17) implies the orthogonality condition

$$\langle \mathbf{U}^*, \mathbf{MU}\rangle = \int_V (\rho v_i u_i^* + \rho u_i v_i^* + A_i D_i^* + D_i A_i^*)\, dV = 0. \tag{18}$$

We note that (18) further implies $\langle \mathbf{U}^*, \mathbf{LU}\rangle = 0$, which is another form of the orthogonality condition.

3. A variational principle. In this section, we will give a variational formulation for the eigenvalue problem (8) and (9). Different from the variational formulations for the quasi-static case [8] which are for ω^2, the following variational principle is for ω. This is consistent with the corresponding variational principle for pure electromagnetic fields of a finite body [9].

Generally, for a fractional functional

$$\Pi = \frac{\Lambda}{\Gamma} \tag{19}$$

we have

$$\delta\Pi = \frac{1}{\Gamma^2}(\Gamma\delta\Lambda - \Lambda\delta\Gamma) = \frac{1}{\Gamma}(\delta\Lambda - \Pi\delta\Gamma). \tag{20}$$

Therefore, $\delta\Pi = 0$ implies

$$\delta\Lambda - \Pi\delta\Gamma = 0. \tag{21}$$

Now we consider the following functional of those $\mathbf{U}$ that satisfy (9):

$$\Pi_0(\mathbf{U}) = \frac{\frac{1}{2}\langle \mathbf{LU}, \mathbf{U}\rangle}{\frac{1}{2}\langle \mathbf{MU}, \mathbf{U}\rangle}. \tag{22}$$

With the selfadjointness of $\mathbf{L}$ and $\mathbf{M}$, it can be verified that the stationary condition of Π_0 is

$$\langle \mathbf{LU} - \Pi_0\mathbf{MU}, \delta\mathbf{U}\rangle = 0. \tag{23}$$

Because of the arbitrariness of $\delta\mathbf{U}$, the stationary condition of Π_0 gives (14), with the stationary value of Π_0 as ω. Here the boundary conditions (9) are constraints that must be satisfied by all the admissible vectors $\mathbf{U}$ for Π_0. To include boundary conditions (9) as stationary conditions of variations, we can use Lagrange multipliers to release (9). This leads to the following functional Π_1 which has no constraints and gives (14) or (8), and (9) as stationary conditions. To be specific, we define

$$\begin{aligned}
&\Lambda_1(\mathbf{u}, \mathbf{v}, \mathbf{T}, \mathbf{D}, \mathbf{H}, \phi, \mathbf{A}, \mathbf{S}, \mathbf{E}, \mathbf{B}) \\
&\quad = \int_V \{-\tfrac{1}{2}\rho v_i v_i + [S_{ij} - \tfrac{1}{2}(u_{j,i} + u_{i,j})]T_{ij} - (E_i + \phi_{,i})D_i \\
&\qquad\qquad + (B_i - \varepsilon_{ijk}A_{k,j})H_i - H(\mathbf{S}, \mathbf{E}, \mathbf{B})\}\, dV \\
&\qquad + \int_{S_u} T_{ji}n_j u_i\, dS + \int_{S_\phi} D_i n_i \phi\, dS + \int_{S_A} \varepsilon_{ijk} n_j A_k H_i\, dS, \qquad (24) \\
&\Gamma_1(\mathbf{u}, \mathbf{v}, \mathbf{T}, \mathbf{D}, \mathbf{H}, \phi, \mathbf{A}, \mathbf{S}, \mathbf{E}, \mathbf{B}) = \int_V (\rho u_i v_i + A_i D_i)\, dV, \\
&\Pi_1(\mathbf{u}, \mathbf{v}, \mathbf{T}, \mathbf{D}, \mathbf{H}, \phi, \mathbf{A}, \mathbf{S}, \mathbf{E}, \mathbf{B}) = \frac{\Lambda_1}{\Gamma_1}.
\end{aligned}$$

Then we have, after integration by parts,

$$\begin{aligned}
\delta\Lambda_1 = &\int_V \Big\{ -\rho v_i \delta v_i + T_{ji,j}\delta u_i + D_{i,i}\delta\phi - \varepsilon_{ijk}H_{k,j}\delta A_i \\
&\quad + [S_{ij} - \tfrac{1}{2}(u_{j,i} + u_{i,j})]\delta T_{ij} - (E_i + \phi_{,i})\delta D_i + (B_i - \varepsilon_{ijk}A_{k,j})\delta H_i \\
&\qquad + \left(T_{ij} - \frac{\partial H}{\partial S_{ij}}\right)\delta S_{ij} - \left(D_i + \frac{\partial H}{\partial E_i}\right)\delta E_i + \left(H_i - \frac{\partial H}{\partial B_i}\right)\delta B_i \Big\}\, dV \\
&+ \int_{S_u} u_i \delta T_{ji} n_j\, dS - \int_{S_T} T_{ji} n_j \delta u_i\, dS \qquad (25) \\
&+ \int_{S_\phi} \phi \delta D_i n_i\, dS - \int_{S_D} D_i n_i \delta\phi\, dS \\
&+ \int_{S_A} \varepsilon_{ijk} n_j A_k \delta H_i\, dS + \int_{S_H} \varepsilon_{ijk} n_j H_k \delta A_i\, dS, \\
\delta\Gamma_1 = &\int_V (\rho u_i \delta v_i + \rho v_i \delta u_i + A_i \delta D_i + D_i \delta A_i)\, dV.
\end{aligned}$$

Therefore, $\delta\Pi_1 = 0$ implies

$$\begin{gathered}
T_{ji,j} = \Pi_1 \rho v_i, \qquad -\rho v_i = \Pi_1 p u_i \quad \text{in } V, \\
S_{ij} - \tfrac{1}{2}(u_{j,i} + u_{i,j}) = 0 \quad \text{in } V, \\
-E_i - \phi_{,i} = \Pi_1 A_i, \qquad B_i - \varepsilon_{ijk}A_{k,j} = 0 \quad \text{in } V, \\
D_{i,i} = 0, \qquad -\varepsilon_{ijk}H_{k,j} = \Pi_1 D_i \quad \text{in } V, \\
T_{ij} - \frac{\partial H}{\partial S_{ij}} = 0, \qquad -D_i - \frac{\partial H}{\partial E_i} = 0, \qquad H_i - \frac{\partial H}{\partial B_i} = 0 \quad \text{in } V, \qquad (26) \\
u_i = 0 \quad \text{on } S_u, \qquad -T_{ji}n_j = 0 \quad \text{on } S_T, \\
\phi = 0 \quad \text{on } S_\phi, \qquad -D_i n_i = 0 \quad \text{on } S_D, \\
\varepsilon_{ijk} n_j A_k = 0 \quad \text{on } S_A, \qquad \varepsilon_{ijk} n_j H_k = 0 \quad \text{on } S_H.
\end{gathered}$$

Comparing (26) with (8) and (9), we conclude that the stationary conditions of Π_1 in (24) give the eigenvalue problem (8) and (9) with the stationary value of Π_1 as ω.

This variational formulation is of mixed type in the sense that various mechanical and electromagnetic fields can vary independently and there are no constraints.

5. Other variational principles. The variational principles for the vibration of quasi-static piezoelectricity were summarized and systematically developed in [8]. It was shown that for each Legendre transform of the electric enthalpy H there exists a variational principle. The situation is similar for piezoelectromagnetism. Since the electric enthalpy function for piezoelectromagnetism has more variables, there can be more versions of variational principles. We will just show one as an example and not try to exhaust them.

First we introduce a function M from H through Legendre transform as follows:

$$M(\mathbf{T}, \mathbf{D}, \mathbf{H}) = H(\mathbf{S}, \mathbf{E}, \mathbf{B}) - T_{ij}S_{ij} + D_iE_i - H_iB_i, \tag{27}$$

which generates the constitutive relations in the following form:

$$S_{ij} = -\frac{\partial M}{\partial T_{ij}}, \qquad E_i = \frac{\partial M}{\partial D_i}, \qquad B_i = -\frac{\partial M}{\partial H_i}. \tag{28}$$

Then we define

$$\begin{aligned}
&\Lambda_2(\mathbf{u}, \mathbf{v}, \mathbf{T}, \mathbf{D}, \mathbf{H}, \phi, \mathbf{A}) \\
&\quad = \int_V [-\tfrac{1}{2}\rho v_i v_i - \tfrac{1}{2}(u_{j,i} + u_{i,j})T_{ij} - \phi_{,i}D_i \\
&\quad - \varepsilon_{ijk}A_{k,j}H_i - M(\mathbf{T}, \mathbf{D}, \mathbf{H})]\,dV \\
&\quad + \int_{S_u} T_{ji}n_j u_i\,dS + \int_{S_\phi} D_i n_i \phi\,dS + \int_{S_A} \varepsilon_{ijk}n_j A_k H_i\,dS, \qquad (29) \\
&\Gamma_2(\mathbf{u}, \mathbf{v}, \mathbf{T}, \mathbf{D}, \mathbf{H}, \phi, \mathbf{A}) = \int_V (\rho u_i v_i + A_i D_i)\,dV, \\
&\Pi_2(\mathbf{u}, \mathbf{v}, \mathbf{T}, \mathbf{D}, \mathbf{H}, \phi, \mathbf{A}) = \frac{\Lambda_2}{\Gamma_2}.
\end{aligned}$$

Then we have, after integration by parts,

$$\begin{aligned}
\delta\Lambda_2 = &\int_V \Bigg\{ -\rho v_i \delta v_i + T_{ji,j}\delta u_i + D_{i,i}\delta\phi - \varepsilon_{ijk}H_{k,j}\delta A_i \\
&- \left[\frac{\partial M}{\partial T_{ij}} + \tfrac{1}{2}(u_{j,i} + u_{i,j})\right]\delta T_{ij} - \left(\frac{\partial M}{\partial E_i} + \phi_{,i}\right)\delta D_i - \left(\frac{\partial M}{\partial H_i} + \varepsilon_{ijk}A_{k,j}\right)\delta H_i \Bigg\} \\
&+ \int_{S_u} u_i \delta T_{ji} n_j\,dS - \int_{S_T} T_{ji}n_j\,\delta u_i\,dS \\
&+ \int_{S_\phi} \phi\delta D_i n_i\,dS - \int_{S_D} D_i n_i \delta\phi\,dS \\
&+ \int_{S_A} \varepsilon_{ijk}n_j A_k \delta H_i\,dS + \int_{S_H} \varepsilon_{ijk}n_j H_k \delta A_i\,dS, \\
\delta\Gamma_2 = &\int_V (\rho u_i \delta v_i + \rho v_i \delta u_i + A_i \delta D_i + D_i \delta A_i)\,dV.
\end{aligned}$$

Therefore, $\delta\Pi_2 = 0$ implies

$$\begin{aligned}
&T_{ji,j} = \Pi_2 \rho v_i, \qquad -\rho v_i = \Pi_2 \rho u_i \quad \text{in } V, \\
&-\frac{\partial M}{\partial T_{ij}} - \tfrac{1}{2}(u_{j,i} + u_{i,j}) = 0 \quad \text{in } V, \\
&-\frac{\partial M}{\partial D_i} - \phi_{,i} = \Pi_2 A_i, \qquad -\frac{\partial M}{\partial H_i} - \varepsilon_{ijk} A_{k,j} = 0 \quad \text{in } V, \\
&D_{i,i} = 0, \qquad -\varepsilon_{ijk} H_{k,j} = \Pi_2 D_i \quad \text{in } V, \\
&\qquad u_i = 0 \quad \text{on } S_u, \qquad -T_{ji} n_j = 0 \quad \text{on } S_T, \\
&\qquad \phi = 0 \quad \text{on } S_\phi, \qquad -D_i n_i = 0 \quad \text{on } S_D, \\
&\varepsilon_{ijk} n_j A_k = 0 \quad \text{on } S_A, \qquad \varepsilon_{ijk} n_j H_k = 0 \quad \text{on } S_H.
\end{aligned} \tag{31}$$

Hence, the stationary conditions of Π_2 give the eigenvalue problem (31), with the stationary value of Π_2 as the eigenvalue. It can be seen that the elimination of $\mathbf{S}$, $\mathbf{E}$, and $\mathbf{B}$ in (8) through the constitutive relation $(8)_5$ results in (31). Hence (31) is equivalent to the original eigenvalue problem (8) and (9). The variational principles for Π_1 and Π_2 here generalize indirectly those in [8] to the dynamic case.

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