ON THE COMPUTATION OF ZEROES OF $J_n(z) - iJ_{n+1}(z) = 0$

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Abstract. The roots of the equation

$$J_n^2(z) + J_{n+1}^2(z) = 0,$$

in which n is a positive integer or zero, are of interest to the specialist in wave reflection from multi-sloped beaches [1]. This note shows how to obtain accurate roots of the equation when n is not large.

1. Introduction. The equation $F_n(z) = 0$, where

$$F_n(z) = J_n^2(z) + J_{n+1}^2(z) \tag{1}$$

and n is a positive integer or zero, has no real or purely imaginary roots. Since $F_n(-z) = F_n(z)$ and $\overline{F_n(z)} = F_n(\bar{z})$, whenever z_k is a root so too are $-z_k$ and $\pm\bar{z}_k$. Hence, when calculating the roots it is sufficient to confine attention to any one quadrant of the z plane. The problem can be simplified by observing that

$$F_n(z) = (J_n(z) + iJ_{n+1}(z))(J_n(z) - iJ_{n+1}(z)).$$

Hence the roots of (1) can be found by solving either one of

$$J_n(z) + iJ_{n+1}(z) = 0, \tag{2}$$

$$J_n(z) - iJ_{n+1}(z) = 0. \tag{3}$$

Since the roots of $J_n(z) = 0$ are all real ([2], §15.25) no root can satisfy both Eqs. (2) and (3). Indeed it has been shown, [1], that the roots of (2) are confined to the upper half of the z plane whereas those of (3) all lie in the lower half plane.

2. Computation of roots. Three methods will be used to evaluate approximations to the roots of

$$J_n(z) - iJ_{n+1}(z) = 0.$$

Received October 6, 1994.

1991 *Mathematics Subject Classification.* Primary 33C10, 41A60.

Method A is analytic and leads to asymptotic formulae for the roots. Methods B and C are numerical and rely on Newton-Raphson iteration: B is strictly valid when $|z| \gg 1$ whereas, due to rounding error difficulties (see §2.3), C is valid for $|z|$ not too large.

2.1. *Method A: The asymptotic formulae.* As in [3], we choose to work with

$$I_n(\zeta) + I_{n+1}(\zeta) = 0, \quad (4)$$

where $z = i\zeta$, rather than (3). Since the roots of (3) lie in the lower half plane, $-\pi < \arg z < 0$. Hence $-\frac{3\pi}{2} < \arg \zeta < -\frac{\pi}{2}$ and ([2], §7.23)

$$I_n(\zeta) \sim \frac{e^\zeta}{\sqrt{2\pi\zeta}} \left[1 + \sum_{m=1}^{\infty} (-1)^m \frac{(n, m)}{(2\zeta)^m} \right] - i(-1)^n \frac{e^{-\zeta}}{\sqrt{2\pi\zeta}} \left[1 + \sum_{m=1}^{\infty} \frac{(n, m)}{(2\zeta)^m} \right], \quad (5)$$

where, for $m = 1, 2, \dots$,

$$(n, m) = \frac{(4n^2 - 1^2)(4n^2 - 3^2) \cdots (4n^2 - [2m - 1]^2)}{2^{2m} m!}.$$

When (5), with each series truncated at its fifth term, is substituted in (4), the logarithm of the resulting expression gives

$$\begin{aligned} 2\zeta = & -\ln(4\zeta) + \ln(2n + 1) + \left(2p + n + \frac{3}{2}\right) \pi i - \frac{1}{4\zeta}(1 - 4n - 4n^2) \\ & + \frac{1}{16\zeta^2}(5 - 4n - 4n^2) - \frac{1}{192\zeta^3}(16n^4 + 32n^3 - 104n^2 - 120n + 73) \\ & + \frac{1}{512\zeta^4}(48n^4 + 96n^3 - 536n^2 - 584n + 419) + O(1/|\zeta|^5), \end{aligned} \quad (6)$$

where p is an integer.

The leading approximation for ζ . For $|\zeta| \gg 1$ and n not large, a first approximation to ζ may be obtained by solving

$$2\zeta = -\ln(4\zeta) + \ln(2n + 1) + \left(2p + n + \frac{3}{2}\right) \pi i - \frac{1}{4\zeta}(1 - 4n - 4n^2) + O(1/|\zeta|^2). \quad (7)$$

Write $\zeta = re^{i\theta}$, where, for the *third quadrant* roots of (4), $-\pi < \theta < -\frac{\pi}{2}$. Then, from (7),

$$\cos \theta = -\frac{1}{2r} \ln \left(\frac{4r}{2n + 1} \right) + O \left(\frac{1}{r^2} \right).$$

Hence, for $r \gg 1$, θ is close to $-\frac{\pi}{2}$. So, when $r \gg 1$, we write $\theta = -\frac{\pi}{2} + \phi$, where $0 < -\phi \ll 1$.

Again, from (7)

$$2r \sin \phi = -\ln \left(\frac{4r}{2n + 1} \right) - \frac{\sin \phi}{4r}(1 - 4n - 4n^2) + O \left(\frac{1}{r^2} \right), \quad (8)$$

$$2r \cos \phi = -2\pi \left(p + 1 + \frac{n}{2} \right) + \phi + \frac{\cos \phi}{4r}(1 - 4n - 4n^2) + O \left(\frac{1}{r^2} \right). \quad (9)$$

We note that (see (8))

$$|\phi| = O \left(\frac{\alpha}{r} \right), \quad \alpha = \frac{1}{2} \ln \left(\frac{4r}{2n + 1} \right).$$

Further, from (9),

$$2r = \left[-2\pi \left(p + 1 + \frac{n}{2} \right) + \phi + \frac{(1 - 4n - 4n^2)}{4r} \left(1 - \frac{\phi^2}{2} + \dots \right) \right] \left(1 + \frac{\phi^2}{2} + \dots \right) + O\left(\frac{1}{r^2}\right).$$

Since $|\phi| = O(\alpha/r)$ and $r = O(-[p + 1 + n/2]\pi)$,

$$2r = -2\pi \left(p + 1 + \frac{n}{2} \right) \left[1 + \frac{\phi^2}{2} \right] + \phi + \frac{1 - 4n - 4n^2}{4r} + O\left(\frac{1}{r^2}\right). \quad (10)$$

From (8),

$$\sin \phi = -\frac{\alpha}{r} \left[1 - \frac{1 - 4n - 4n^2}{8r^2} \right] + O\left(\frac{1}{r^3}\right).$$

Hence, when $n = 0$ or $n = O(1)$,

$$\phi = -\frac{\alpha}{r} + O\left(\frac{\alpha}{r}\right)^3. \quad (11)$$

From Eqs. (10) and (11) it follows that

$$2r = -2\pi \left(p + 1 + \frac{n}{2} \right) \left(1 + \frac{\alpha^2}{2r^2} \right) - \frac{\alpha}{r} + \frac{(1 - 4n - 4n^2)}{4r} + O\left(\frac{1}{r^2}\right), \quad (12)$$

$$\theta = -\frac{\pi}{2} - \frac{\alpha}{r} + O\left(\frac{\alpha}{r}\right)^3. \quad (13)$$

In Eqs. (12) and (13) p must be a *negative* integer, say $p = -k - 1$, $k > n/2$, and α must be positive. Hence the leading term, r_a say, in the asymptotic approximation to r satisfies $r_a = (k - n/2)\pi > (2n + 1)/4$. When $n = 0$ we need $k > 1/(4\pi)$, and the root of smallest modulus corresponds to $k = 1$. This has been confirmed numerically, [4]. By use of Schlöfli's expansion for the product of two Bessel functions, [2], §5.41, Eq. (1) can be expressed as

$$\frac{(z/2)^{2n}}{(n!)^2} \left[1 + (2n + 1)(n!)^2 \sum_{m=1}^{\infty} \frac{(-1)^m (z/2)^{2m} (2m + 2n)!}{[(n + m)!]^2 (2n + m + 1)! m!} \right] = 0.$$

Clearly, for $n > 0$, $z = 0$ is a $2n$ -fold zero of $F_n(z) = 0$. The remaining roots satisfy alternating infinite series which are such that the absolute values of the coefficients of z^{2m} , $m = 1, 2, \dots$, decrease with increase of n .¹ This suggests that, for $n = 0, 1, 2, \dots$, the root of smallest (nonzero) modulus of $F_n(z) = 0$ will increase with increase of n . This conjecture leads to the conclusion that the root of smallest (nonzero) modulus of $F_n(z) = 0$ corresponds to $p = -n - 2$ in (12).

¹Denote $(2n + 1)!(n!)^2 \times m$ th term of the series by $(-1)^m (z/2)^{2m} T(n, m)$. Then

$$\frac{T(n, m)}{T(n + 1, m)} = \frac{8n^4 + (16m + 32)n^3 + (10m^2 + 46[m + 1])n^2 + (2m^3 + 17m^2 + 41m + 28)n + m^3 + 6m^2 + 11m + 6}{8n^4 + (8m + 32)n^3 + (28m + 46)n^2 + (32m + 28)n + 12m + 6}.$$

Iterative solution of Eqs. (12) and (13). The leading approximation to Eqs. (12) and (13) gives purely imaginary roots, with

$$r_a = (j + n/2)\pi$$

being the modulus of the j th root. The second approximation is, for this exercise, as far as we need to go. It gives the approximation

$$r = r_a + \frac{\alpha_a^2 - \alpha_a - (4n^2 + 4n - 1)/4}{2r_a} + O\left(\frac{1}{r_a^2}\right), \quad (14)$$

$$\theta = -\frac{\pi}{2} - \frac{\alpha_a}{r_a} + O\left(\frac{\alpha_a}{r_a}\right)^3, \quad (15)$$

where

$$\alpha_a = \frac{1}{2} \ln \left(\frac{4r_a}{2n+1} \right).$$

The solution correct to $O(\alpha_a\{\alpha_a/r_a\}^3)$. This is obtained from (6) by evaluating the third iterate, r_c , correct to order $1/(r^4)$ and by employing results like

$$\begin{aligned} \alpha_b &= \alpha_a + \frac{1}{16r_a^2} [1 - 4n - 4n^2 + 4\alpha_a^2 - 4\alpha_a] + O\left(\frac{\alpha_a}{r_a}\right)^4, \\ \frac{1}{r_b} &= \frac{1}{r_a} \left[1 - \frac{1}{8r_a^2} (1 - 4n - 4n^2 + 4\alpha_a^2 - 4\alpha_a) + O\left(\frac{\alpha_a}{r_a}\right)^4 \right]. \end{aligned}$$

We obtain, for the j th root of (4),

$$\begin{aligned} r &= r_a + \frac{\alpha_a^2 - \alpha_a - (4n^2 + 4n - 1)/4}{2r_a} \\ &\quad - \frac{1}{384r_a^3} [f(n) + \alpha_a g(n) - \alpha_a^2 h(n) - 256\alpha_a^3 + 48\alpha_a^4] + O\left(\frac{1}{r_a^4}\right), \end{aligned} \quad (16)$$

$$\begin{aligned} \theta &= -\frac{\pi}{2} - \frac{\alpha_a}{r_a} \\ &\quad + \frac{1}{96r_a^3} [3(12n^2 + 12n - 7) - 48\alpha_a(2n^2 + 2n - 1) - 72\alpha_a^2 + 32\alpha_a^3] + O\left(\frac{\alpha_a}{r_a}\right)^5, \end{aligned} \quad (17)$$

where

$$\begin{aligned} f(n) &= 121 - 240n - 128n^2 + 224n^3 + 112n^4, \\ g(n) &= 528n(n+1) - 324, \\ h(n) &= 480n(n+1) - 360, \\ r_a &= (j + n/2)\pi. \end{aligned}$$

When $n = 0$ the $O(1/r_a^3)$ terms independent of α_a and linearly dependent on α_a in (16) do not agree with those given in Eq. (8) of [3]. This is due to an algebraic error in the latter equation. Table (3) demonstrates the improved accuracy of the corrected result given by (16) and (17).

TABLE 1. Results for the s th iteration of the r th root, $x + iy$, of $J_5 - iJ_6 = 0$, when J_5 and J_6 are represented by their asymptotic expansions correct to $O(z^{-13/2})$.

r	s	Approximate root		r	s	Approximate root	
1	0	9.41..	-0.663..	10	0	38.869 95..	-1.334 8..
	1	9.332 41..	-0.695 3..		1	38.869 857 315..	-1.335 404 392 9..
	2	9.332 970 3..	-0.694 453 1..		2	38.869 857 307 368	-1.335 404 390 740
	3	9.332 972 162 7..	-0.694 452 993 2..		3	38.869 857 307 368	-1.335 404 390 740
	4	9.332 972 164 857	-0.694 452 995 816				
	5	9.332 972 164 855	-0.694 452 995 821				

2.2. *Method B: The asymptotic calculation.* By use of the asymptotic expansions for $I_n(\zeta)$ and $I_{n+1}(\zeta)$,² (4) can be replaced by the m th order asymptotic approximation

$$e^\zeta \sum_{k=0}^m a_{n,k} \zeta^{-k} + ie^{-\zeta} \sum_{k=1}^m b_{n,k} \zeta^{-k} = 0, \quad (18)$$

where the $a_{n,k}$ and the $b_{n,k}$ are constants. For $m = 6$ and $n = 0, 1, 2, 3, 5$, this equation has been solved by Newton-Raphson iteration, with initial approximation given by (16) and (17). In all cases studied, convergence correct to at least 12 places after the decimal point was achieved within five iterations. Table (1) shows the rate of convergence for the first and the tenth roots of $J_5(\zeta) - iJ_6(\zeta) = 0$.

2.3. *Method C: The ascending series calculation.* When each of $I_n(\zeta)$ and $I_{n+1}(\zeta)$ is replaced by its ascending series, (4) becomes

$$\sum_{k=0}^{\infty} \frac{(\zeta/2)^{n+2k}}{k!(k+n)!} \left[1 + \frac{(\zeta/2)}{(k+n+1)} \right] = 0. \quad (19)$$

When (19) is truncated at $k = L$ we obtain the $(n + 2L + 1)$ th polynomial equation

$$\sum_{k=0}^{2L+1} c_k (\zeta/2)^{n+k} = 0, \quad (20)$$

where, for $k = 0, 1, \dots, L$

$$c_{2k} = \frac{1}{k!(n+k)!}, \quad c_{2k+1} = \frac{1}{k!(n+1+k)!}.$$

Now the asymptotic formulae show that for $j \gg 1$, $|\zeta_j| \sim (j + n/2)\pi$. Hence, when $j \gg 1$, our polynomial equation has a sum that is zero and certain individual terms that are large in modulus.³ Such ingredients are, of course, a standard recipe for rounding error difficulties. This feature of the calculation places an upper bound on the number of roots— N , say—of (4) that can be obtained by use of the ascending series approach. Clearly $N = N(D)$, where D denotes the number of digits retained in our Maple (computer algebra) calculation.

For given D and suitably small $|\zeta|$, Eq. (20) can be solved by Newton-Raphson iteration, using as initial approximation the asymptotic formulae of §2.1, or, for more rapid

²Or by use of Lommel's polynomials and the asymptotic expansions for $I_0(\zeta)$ and $I_1(\zeta)$.

³When $n = 5$ the modulus of the eighteenth root of $I_5(\zeta) + I_6(\zeta) = 0$ is approximately 64.

convergence, the results of §2.2. For $n = 0, 1, 2, 3, 5$, the leading roots of (4) were evaluated, accuracy being checked by repeating the calculations for different values of L and D . From (19), a necessary condition for accuracy of the j th root, $\zeta_j = r_j e^{i\theta_j}$, is that

$$\frac{(r_j/2)^{2L+n+1}}{(L+n+1)!(L!)} \ll 1.$$

Although L is best determined by numerical experimentation, a useful guide to its value when calculating the j th root can be obtained by use of Stirling's approximation for $L!$. From

$$\frac{(r_j/2)^{2L+n+1}}{(L+n+1)!(L!)} \sim (10)^{-m},$$

where m is some pre-selected positive integer, it follows that

$$[2\pi 10^{-m} L^{L+\frac{1}{2}} (n+L+1)^{n+L+\frac{3}{2}} e^{-(n+2L+1)}]^{1/(n+2L+1)} \sim \frac{r_j}{2} \sim \left(j + \frac{n}{2}\right) \frac{\pi}{2}.$$

For $L \gg 1$, this leads to

$$L \left(1 + \frac{(n+1)}{L}\right)^{\frac{1}{2}(1 + \frac{(n+2)}{2L})} \sim \frac{\pi e}{2} \left(j + \frac{n}{2}\right) 10^{m/(n+2L+1)}.$$

When $n \ll L$ this equation has the approximate solution

$$L \sim \left[\frac{\pi e}{2} \left(j + \frac{n}{2}\right) - n - 1\right] 10^{m/(n+\pi e\{j+n/2\})}.$$

When, for example, $j = 18, n = 5$, and $m = 16$, $L \sim 100$. Table (2) compares the approximate values obtained for the eighteenth root of $J_5(z) - iJ_6(z) = 0$ when L and D are varied.

2.4. *The error in truncating (19) at $k = L$.* In the ascending series calculation we solve

$$\sum_{k=0}^L \frac{(\zeta/2)^{n+2k}}{k!(k+n)!} \left[1 + \frac{(\zeta/2)}{(k+n+1)}\right] = 0$$

rather than

$$\sum_{k=0}^L \frac{(\zeta/2)^{n+2k}}{k!(k+n)!} \left[1 + \frac{(\zeta/2)}{(k+n+1)}\right] = -\varepsilon,$$

where

$$\varepsilon = \sum_{k=L+1}^{\infty} \frac{(\zeta/2)^{n+2k}}{k!(k+n)!} \left[1 + \frac{(\zeta/2)}{(k+n+1)}\right]. \quad (21)$$

Clearly, we need $|\varepsilon| \ll 1$ near the roots of the equation. Now put $\zeta = r e^{i\theta}$, $\varepsilon = \varepsilon_1 + \varepsilon_2$ and consider

$$\varepsilon_1 = \sum_{k=L+1}^{\infty} \frac{(r/2)^{n+2k} e^{(n+2k)i\theta}}{k!(k+n)!}.$$

TABLE 2. Converged results for Newton-Raphson iteration of the ascending series representing $J_5(z) - iJ_6(z) = 0$ when L and D (see text) are varied.

L	D	x_{18}	y_{18}	$\frac{(r/2)^{2L+6}}{L!(L+6)!}$
90	40	64.158 482 488..	-1.579 869 938..	$1.03 \cdot 10^{-8}$
	45	64.158 482 487..	-1.579 869 938..	$1.03 \cdot 10^{-8}$
94	40	64.158 482 444 768..	-1.579 869 867 681..	$1.67 \cdot 10^{-12}$
	45	64.158 482 444 773..	-1.579 869 867 628..	$1.69 \cdot 10^{-12}$
98	40	64.158 482 444 768..	-1.579 869 867 620..	$1.97 \cdot 10^{-16}$
	45	64.158 482 444 768..	-1.579 869 867 616..	$1.98 \cdot 10^{-16}$

Since $L \gg 1$,

$$2\pi\varepsilon_1 \sim \left(\frac{r}{2}e^{(1+i\theta)}\right)^n \sum_{k=L+1}^{\infty} \frac{(\{r/2\}e^{(1+i\theta)})^{2k}}{k^{k+1/2}(n+k)^{n+k+1/2}} = 2\pi\varepsilon_1^a, \text{ say.}$$

Hence

$$\begin{aligned} 2\pi|\varepsilon_1^a| &\leq \left(\frac{re}{2}\right)^n \sum_{k=L+1}^{\infty} \frac{(\{re\}/2)^{2k}}{k^{k+1/2}(n+k)^{n+k+1/2}} \\ &< \left(\frac{re}{2}\right)^n \frac{1}{(L+1)^{1/2}(n+L+1)^{n+1/2}} \sum_{k=L+1}^{\infty} \left(\frac{re}{2(L+1)}\right)^{2k} \\ &= \frac{(\{re\}/2)^n}{(L+1)^{1/2}(n+L+1)^{n+1/2}} \frac{\mu^{(L+1)}}{(1-\mu)}, \end{aligned}$$

where

$$\mu = \left(\frac{re}{2(L+1)}\right)^2.$$

From this result the corresponding approximation for $|\varepsilon_2|$ follows at once and, from $|\varepsilon| \leq |\varepsilon_1| + |\varepsilon_2|$, we then obtain $\varepsilon < \varepsilon_a$, where

$$\varepsilon_a \sim \frac{1}{2\pi} \left(\frac{re}{2}\right)^n \frac{1}{(L+1)^{1/2}} \frac{\mu^{(L+1)}}{(1-\mu)} \left[\frac{1}{(n+L+1)^{n+1/2}} + \frac{(re)}{2(n+L+2)^{n+3/2}} \right].$$

In the case of the eighteenth root of $I_5(\zeta) + I_6(\zeta) = 0$, the approximation $r \sim \pi(j+n/2)$ —which, for $j = 18$ and $n = 5$, is in error by about one third of one percent—and the choice $L = 98$ leads to

$$\varepsilon_a \sim 1.41 \times 10^{-13},$$

which serves as an upper bound for the truncation error in this case.

3. Results. Tables (3), (4), (5), (6), and (7) compare the first thirty approximate roots of $J_n(\zeta) - iJ_{n+1}(\zeta) = 0$ as calculated from the asymptotic formulae and the asymptotic calculation in each of the five cases corresponding to $n = 0, 1, 2, 3$, and 5 , respectively.

Note that the last figure of each root is not rounded. Save for table (3), which lists the first twenty roots, the tables also include the first eighteen roots as calculated by the ascending series approach. Further roots have not been calculated by this approach due to the growth of rounding error as $|\zeta_j|$ increases. The tables show that the accuracy of the asymptotic formulae deteriorates as n increases. This is not surprising since the formulae have been derived on the assumption that $n = O(1)$.

TABLE 3. Unrounded approximate values for the first thirty, fourth-quadrant roots, $x + iy$, of $J_0 - iJ_1 = 0$.

Asymptotic formula		Asymptotic calculation		Ascending series calculation	
2.979	-1.282	2.977 6	-1.279 2	2.980 382 414 790	-1.279 602 540 299
6.175 104	-1.619 1	6.175 108	-1.618 68	6.175 153 070 954	-1.618 717 384 471
9.341 952	-1.818 99	9.341 957	-1.818 869 2	9.341 960 983 461	-1.818 872 787 772
12.498 504	-1.961 50	12.498 506 5	-1.961 458 8	12.498 507 063 958	-1.961 459 538 019
15.650 103 7	-2.072 33	15.650 104 25	-2.072 309 61	15.650 104 385 309	-2.072 309 817 830
18.798 911 4	-2.163 022	18.798 911 644	-2.163 010 912	18.798 911 683 696	-2.163 010 983 274
21.945 979 93	-2.239 779	21.945 979 970	-2.239 772 463	21.945 979 984 381	-2.239 772 492 276
25.091 885 76	-2.306 316 9	25.091 885 758	-2.306 312 793	25.091 885 763 907	-2.306 312 806 675
28.236 973 16	-2.365 038 7	28.236 973 142	-2.365 036 113	28.236 973 145 397	-2.365 036 120 661
31.381 461 01	-2.417 588 7	31.381 460 988 3	-2.417 586 982	31.381 460 989 647	-2.417 586 986 362
34.525 495 55	-2.465 141 2	34.525 495 523 6	-2.465 140 068	34.525 495 524 330	-2.465 140 070 798
37.669 178 40	-2.508 564 5	37.669 178 382 19	-2.508 563 697 6	37.669 178 382 576	-2.508 563 698 916
40.812 582 54	-2.548 518 5	40.812 582 525 02	-2.548 517 903 5	40.812 582 525 250	-2.548 517 904 322
43.955 761 78	-2.585 516 5	43.955 761 763 45	-2.585 516 140 3	43.955 761 763 589	-2.585 516 140 857
47.098 756 71	-2.619 966 0	47.098 756 704 30	-2.619 965 707 1	47.098 756 704 391	-2.619 965 707 516
50.241 598 611	-2.652 195 3	50.241 598 598 51	-2.652 195 131 9	50.241 598 598 574	-2.652 195 132 137
53.384 311 921	-2.682 473 4	53.384 311 910 838	-2.682 473 257 72	53.384 311 910 875	-2.682 473 257 885
56.526 916 091	-2.711 023 00	56.526 916 082 180	-2.711 022 877 648	56.526 916 082 206	-2.711 022 877 758
59.669 426 774	-2.738 030 78	59.669 426 766 956	-2.738 030 685 256	59.669 426 766 973	-2.738 030 685 336
62.811 856 726	-2.763 654 75	62.811 856 719 871	-2.763 654 675 022	62.811 856 719 883	-2.763 654 675 081
65.954 216 449	-2.788 029 79	65.954 216 443 289	-2.788 029 738 475		
69.096 514 672	-2.811 272 00	69.096 514 667 607	-2.811 271 959 593		
72.238 758 717	-2.833 481 98	72.238 758 713 036	-2.833 481 956 161		
75.380 954 769	-2.854 747 53	75.380 954 765 712	-2.854 747 510 576		
78.523 108 094	-2.875 145 68	78.523 108 091 021	-2.875 145 664 058		
81.665 223 203	-2.894 744 411	81.665 223 200 231	-2.894 744 400 454		
84.807 303 984	-2.913 604 019	84.807 303 982 006	-2.913 604 012 522		
87.949 353 809	-2.931 778 224	87.949 353 807 156	-2.931 778 219 939		
91.091 375 614 9	-2.949 315 093 0	91.091 375 612 811	-2.949 315 091 274		
94.233 371 972 4	-2.966 257 809 68	94.233 371 970 601	-2.966 257 809 784		

For $n > 0$, Eqs. (16) and (17) indicate roots for values of $j \leq 0$. When, for example, $n = 5$, roots are indicated for $j = 0$ and $j = -1$. Insertion of these “roots” as initial approximations for the ascending series calculation results in the successive Newton-Raphson iterations converging to zero. This provides numerical verification of the earlier conjecture that the asymptotic approximation to the j th root of $F_n(z) = 0$ given by Eqs. (16) and (17) is valid for $p \leq -n - 1 - j$, where $j \geq 1$ (see Eq. (12)).

TABLE 4. Unrounded approximate values for the first thirty, fourth-quadrant roots, $x + iy$, of $J_1 - iJ_2 = 0$.

Asymptotic formula		Asymptotic calculation		Ascending series calculation	
4.420 4	-0.926	4.420 0	-0.933 1	4.419 943 205 884	-0.932 937 042 896
7.665 98	-1.182 7	7.666 005	-1.183 743	7.666002709895	-1.183 736 098 455
10.854 359	-1.348 7	10.854 375 7	-1.349 025 5	10.854 375 397 749	-1.349 024 657 437
14.023 113	-1.472 86	14.023 121 74	-1.472 942 6	14.023 121 673 932	-1.472 942 493 036
17.182 627	-1.572 14	17.182 631 70	-1.572 177 70	17.182 631 682 732	-1.572 177 640 669
20.337 003	-1.654 951	20.337 006 290	-1.654 963 656	20.337 006 284 216	-1.654 963 633 544
23.488 210 7	-1.725 984	23.488 212 297	-1.725 989 485	23.488 212 295 270	-1.725 989 476 180
26.637 316 6	-1.788 183 3	26.637 317 609 9	-1.788 184 860	26.637 317 608 878	-1.788 184 855 930
29.784 954 0	-1.843 504 1	29.784 954 668 2	-1.843 504 286	29.784 954 667 687	-1.843 504 283 703
32.931 522 5	-1.893 317 8	32.931 523 018 9	-1.893 317 332 7	32.931 523 018 650	-1.893 317 331 424
36.077 287 7	-1.938 621 9	36.077 288 036 5	-1.938 631 137 7	36.077 288 036 420	-1.938 621 136 988
39.222 432 9	-1.980 165 1	39.222 433 177 99	-1.980 164 334 1	39.222 433 177 909	-1.980 164 333 738
42.367 089 3	-2.018 524 1	42.367 089 494 24	-2.018 523 346 2	42.367 089 494 191	-2.018 523 345 918
45.511 353 07	-2.054 152 2	45.511 353 196 05	-2.054 151 471 1	45.511 353 196 026	-2.054 151 470 962
48.655 296 47	-2.087 412 3	48.655 296 572 75	-2.087 411 628 9	48.655 296 572 733	-2.087 411 628 814
51.798 974 95	-2.118 599 5	51.798 975 032 111	-2.118 598 897 11	51.798 975 032 097	-2.118 598 897 033
54.942 431 72	-2.147 957 0	54.942 431 783 315	-2.147 956 431 921	54.942 431 783 306	-2.147 956 431 861
58.085 700 99	-2.175 687 5	58.085 701 037 287	-2.175 686 976 475	58.085 701 037 280	-2.175 686 976 432
61.228 810 20	-2.201 961 8	61.228 810 246 011	-2.201 961 347 271		
64.371 781 67	-2.226 925 2	64.371 781 702 377	-2.226 924 803 728		
67.514 633 68	-2.250 702 2	67.514 633 704 458	-2.250 701 904 556		
70.657 381 39	-2.273 400 6	70.657 381 416 943	-2.273 400 262 795		
73.800 037 501	-2.295 113 7	73.800 037 518 047	-2.295 113 486 295		
76.942 612 677	-2.315 923 78	76.942 612 691 908	-2.315 923 506 869		
80.085 115 996	-2.335 902 69	80.085 116 007 999	-2.335 902 444 565		
83.227 555 206	-2.355 114 33	83.227 555 216 753	-2.355 114 114 137		
86.369 936 973	-2.373 615 45	86.369 936 982 298	-2.373 615 253 076		
89.512 267 059	-2.391 456 71	89.512 267 067 405	-2.391 456 530 756		
92.654 550 475	-2.408 683 55	92.654 550 481 752	-2.408 683 383 912		
95.796 791 596	-2.425 336 86	95.796 791 601 746	-2.425 336 713 113		

TABLE 5. Unrounded approximate values for the first thirty, fourth-quadrant roots, $x + iy$, of $J_2 - iJ_3 = 0$.

Asymptotic formula		Asymptotic calculation		Ascending series calculation	
5.722	-0.803	5.716 100	-0.819 12	5.716 090 565 926	-0.819 169 034 174
9.054 5	-1.017	9.053 802 69	-1.021 671	9.053 802 767 071	-1.021 674 260 831
12.286 8	-1.161 4	12.286 658 79	-1.163 062 3	12.286 658 838 815	-1.163 062 837 072
15.482 20	-1.271 9	15.482 158 48	-1.272 699 98	15.482 158 497 264	-1.272 700 122 514
18.659 65	-1.362 0	18.659 639 189	-1.362 454 98	18.659 639 195 322	-1.362 455 033 267
21.826 958	-1.438 2	21.826 952 051	-1.438 499 621	21.826 952 054 080	-1.438 499 639 187
24.987 934	-1.504 36	24.987 932 188	-1.504 492 147	24.987 932 189 059	-1.504 492 154 786
28.144 689 1	-1.562 70	28.144 688 166 4	-1.562 789 835	28.144 688 166 963	-1.562 789 838 819
31.298 480 2	-1.614 94	31.298 479 874 4	-1.615 004 118	31.298 479 874 715	-1.615 004 120 129
34.450 108 44	-1.662 24	34.450 108 358 40	-1.662 286 601 9	34.450 108 358 544	-1.662 286 603 090
37.600 108 13	-1.705 46	37.600 108 155 67	-1.705 489 405 5	37.600 108 155 755	-1.705 489 406 252
40.748 849 99	-1.745 24	40.748 850 065 72	-1.745 260 990 4	40.748 850 065 769	-1.745 260 990 875
43.896 599 54	-1.782 09	43.896 599 628 65	-1.782 106 400 3	43.896 599 628 686	-1.782 106 400 630
47.043 552 05	-1.816 41	47.043 552 140 84	-1.816 426 715 6	47.043 552 140 861	-1.816 426 715 830
50.189 854 44	-1.848 536	50.189 854 526 17	-1.848 545 790 4	50.189 854 526 191	-1.848 545 790 586
53.335 619 42	-1.878 721	53.335 619 492 862	-1.878 728 902 11	53.335 619 492 870	-1.878 728 902 192
56.480 934 91	-1.907 190	56.480 934 980 481	-1.907 196 085 13	56.480 934 980 487	-1.907 196 085 192
59.625 870 57	-1.934 127	59.625 870 632 976	-1.934 131 872 115	59.625 870 632 980	-1.934 131 872 158
62.770 482 28	-1.959 688	62.770 482 337 485	-1.959 692 544 872		
65.914 815 43	-1.984 008	65.914 815 472 593	-1.984 011 623 167		
69.058 907 23	-2.007 201	69.058 907 275 529	-2.007 204 081 512		
72.202 788 56	-2.029 367	72.202 788 595 564	-2.029 369 632 311		
75.346 485 18	-2.050 593	75.346 485 211 925	-2.050 595 313 009		
78.490 018 81	-2.070 956	78.490 018 837 612	-2.070 957 547 185		
81.633 407 87	-2.090 522	81.633 407 893 270	-2.090 523 802 969		
84.776 668 09	-2.109 352 8	84.776 668 110 411	-2.109 353 939 673		
87.919 812 99	-2.127 500 3	87.919 813 006 398	-2.127 501 310 444		
91.062 854 24	-2.145 012 8	91.062 854 261 979	-2.145 013 672 134		
94.205 802 01	-2.161 933 2	94.205 802 023 938	-2.161 933 941 483		
97.348 665 138	-2.178 300 2	97.348 665 149 693	-2.178 300 827 747		

TABLE 6. Unrounded approximate values for the first thirty, fourth-quadrant roots, $x + iy$, of $J_3 - iJ_4 = 0$.

Asymptotic formula		Asymptotic calculation		Ascending series calculation	
6.976	-0.736	6.953 081	-0.759 105	6.953 093 480 994	-0.759 096 845 464
10.389	-0.922	10.385 566 1	-0.931 074	10.385 556 842 568	-0.931 073 031 515
13.670 3	-1.051	13.669 409 57	-1.055 383 7	13.669 409 657 684	-1.055 383 532 845
16.897 8	-1.152	16.897 476 58	-1.154 028 50	16.897 476 598 033	-1.154 028 436 512
20.097 8	-1.235 0	20.097 723 115	-1.236 113 76	20.097 723 119 991	-1.236 113 783 692
23.281 96	-1.305 8	23.281 902 130	-1.306 507 60	23.281 902 131 961	-1.306 507 593 797
26.455 93	-1.367 7	26.455 901 708 4	-1.368 167 987	26.455 901 708 892	-1.368 167 982 700
29.623 04	-1.422 7	29.623 022 259 7	-1.423 041 774	29.623 022 259 944	-1.423 041 772 018
32.785 276	-1.472 2	32.785 265 994 42	-1.472 483 874	32.785 265 994 499	-1.472 483 873 080
35.943 928	-1.517 3	35.943 921 595 19	-1.517 477 354 1	35.943 921 595 228	-1.517 477 353 390
39.099 861	-1.558 6	39.099 857 660 81	-1.558 759 084 0	39.099 857 660 830	-1.558 759 083 554
42.253 684	-1.596 81	42.253 681 767 980	-1.596 896 084 9	42.253 681 767 986	-1.596 896 084 694
45.405 833	-1.632 27	45.405 832 085 642	-1.632 334 244 7	45.405 832 085 645	-1.632 334 244 602
48.556 634	-1.665 38	48.556 632 804 572	-1.665 430 656 0	48.556 632 804 573	-1.665 430 655 952
51.706 329 9	-1.696 43	51.706 329 072 015	-1.696 475 797 0	51.706 329 072 016	-1.696 475 796 963
54.855 110 4	-1.725 67	54.855 109 781 857	-1.725 709 171 29	54.855 109 781 858	-1.725 709 171 236
58.003 123 3	-1.753 30	58.003 122 887 876	-1.753 330 599 881	58.003 122 887 876	-1.753 330 599 838
61.150 486 3	-1.779 48	61.150 485 960 012	-1.779 508 543 603	61.150 485 960 012	-1.779 508 543 572
64.297 293 8	-1.804 37	64.297 293 626 497	-1.804 386 348 098		
67.443 623 1	-1.828 07	67.443 622 925 577	-1.828 087 005 572		
70.589 537 3	-1.850 70	70.589 537 222 545	-1.850 716 837 538		
73.735 089 2	-1.872 35	73.735 089 122 433	-1.872 368 379 678		
76.880 322 75	-1.893 114	76.880 322 667 006	-1.893 122 667 928		
80.025 275 07	-1.913 043	80.025 275 013 519	-1.913 051 069 144		
83.169 977 78	-1.932 210	83.169 977 732 668	-1.932 216 761 155		
86.314 457 86	-1.950 670	86.314 457 823 023	-1.950 675 939 890		
89.458 738 54	-1.968 474	89.458 738 511 751	-1.968 478 811 867		
92.602 839 91	-1.985 666	92.602 839 892 458	-1.985 670 416 294		
95.746 779 45	-2.002 287	95.746 779 437 588	-2.002 291 310 751		
98.890 572 42	-2.018 374	98.890 572 413 293	-2.018 378 146 745		

TABLE 7. Unrounded approximate values for the first thirty, fourth-quadrant roots, $x + iy$, of $J_5 - iJ_6 = 0$.

Asymptotic formula		Asymptotic calculation		Ascending series calculation	
9.41	-0.66	9.3329	-0.69445	9.333119629036	-0.694442717434
12.964	-0.813	12.9450	-0.829375	12.945103691825	-0.829366713388
16.341	-0.922	16.334615	-0.930806	16.334617793373	-0.930803342054
19.6370	-1.008	19.6342921	-1.0136782	19.634292825814	-1.013677175727
22.8881	-1.081	22.8867958	-1.0841896	22.886796022166	-1.084189240232
26.1116	-1.143	26.11097873	-1.1457220	26.110978807715	-1.145721839349
29.3170	-1.198	29.316667198	-1.20038044	29.316667230893	-1.200380333030
32.5097	-1.2485	32.509549134	-1.24958404	32.509549148968	-1.249583980947
35.6933	-1.2936	35.693168309	-1.29434278	35.693168316421	-1.294342747257
38.86995	-1.3348	38.869857307	-1.33540439	38.869857311224	-1.335404369643
42.04128	-1.3729	42.041218668	-1.373339596	42.041218670385	-1.373339583208
45.20843	-1.4082	45.2083919675	-1.408594768	45.208391968773	-1.408594759819
48.37224	-1.4412	48.3722109023	-1.441526063	48.372210903072	-1.441526057709
51.53332	-1.4722	51.5333001125	-1.472422463	51.533300113022	-1.472422459933
54.69215	-1.5013	54.6921372210	-1.501521825	54.692137221329	-1.501521822656
57.849106	-1.5288	57.8490939107	-1.529022359	57.849093910987	-1.529022357589
61.004473	-1.5549	61.0044638971	-1.555091038	61.004463897257	-1.555091036904
64.158489	-1.5797	64.15848244468	-1.579869868	64.158482444768	-1.579869867616
67.311346	-1.60340	67.31134027888	-1.603480657		
70.463198	-1.62596	70.46319369056	-1.626028685		
73.614175	-1.64754	73.61417199984	-1.647605577		
76.764386	-1.66824	76.76438315325	-1.668291577		
79.913920	-1.68811	79.91391797839	-1.688157357		
83.062855	-1.70722	83.06285345842	-1.707265491		
86.211256	-1.72563	86.21125528061	-1.725671646		
89.359181	-1.74339	89.35917984057	-1.743425564		
92.506676	-1.76054	92.50667583332	-1.760571878		
95.6537864	-1.77712	95.65378552754	-1.777150788		
98.8005465	-1.79317	98.80054579435	-1.793198629		
101.9469896	-1.808730	101.94698894420	-1.808748353		

REFERENCES

- [1] S. Tadeballi and C. E. Synolakis, *Roots of $J_\gamma(z) \pm iJ_{\gamma+1}(z) = 0$ and the evaluation of integrals with cylindrical function kernels*, Quart. Appl. Math. **LII**, 103–111 (1994)
- [2] G. N. Watson, *Theory of Bessel Functions*, Cambridge University Press, 1944
- [3] D. A. MacDonald, *The roots of $J_0(z) - iJ_1(z)$* , Quart. Appl. Math. **XLVII**, 375–378 (1989)
- [4] M. Renardy, *Problems and Solutions*, Ed. M. Klamkin, Siam Review, vol. 31, 1989, pp. 126–127