ON THE SUMMATION OF A PARTICULAR NON-UNIFORMLY-CONVERGENT SERIES

BY

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Abstract. In a previous article the force between a dielectric cylinder and a conducting half-space for the cylinder approaching the half-space was obtained as the limit of a non-uniformly-convergent series. A special asymptotic analysis was performed to get a closed-form solution for this limit. The present article focuses on a delicate and crucial step in this analysis and gives a justification by a mathematically rigorous proof.

1. Introduction. In [1] the force between a dielectric cylinder and a conducting half-space was obtained. The limit case for the cylinder approaching the half-space was expressed as the limit of a non-uniformly-convergent series. It was demonstrated that the use of asymptotic techniques led to an explicit expression for this limit. This was first shown for a conducting cylinder, for which the resulting expressions are somewhat simpler. Finally, the same technique was applied to the dielectric cylinder case.

In the present article the series obtained for the conducting cylinder will be focused on. First, a rigorous proof will be given for the non-uniform convergence. Then, a crucial step in the asymptotic technique will be scrutinized. It will be demonstrated that this step is not trivial by looking at the necessary and sufficient conditions for its validity in a simpler case. Finally, a rigorous proof will justify its application to the present series.

2. Basic problem. In [1] the force K_x^* between a conducting cylinder and a conducting half-space was shown to satisfy (taking the same notation as in [1]):

$$\begin{aligned} \frac{K_x^*}{2\pi} u_0 &= u_0 \sinh u_0 \sum_{n=1}^{\infty} n \frac{[n \sinh u_0 \cosh nu_0 - \cosh u_0 \sinh nu_0]}{\sinh^2 nu_0 \sinh(n+1)u_0} \\ &\doteq -u_0 \sinh u_0 \sum_{n=1}^{\infty} \frac{N_n}{D_n} \doteq I^*. \end{aligned} \tag{1}$$

To obtain the force when the cylinder approaches the half-space the limit for $u_0 \rightarrow 0$ has to be taken. By performing numerical calculations of Eq. (1) for small values of u_0 one

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soon discovers that switching the limit with the summation sign is not allowed, i.e.,

$$\lim_{u_0 \rightarrow 0} u_0 \sinh u_0 \sum_{n=1}^{\infty} \frac{N_n}{D_n} \neq \sum_{n=1}^{\infty} \lim_{u_0 \rightarrow 0} u_0 \sinh u_0 \frac{N_n}{D_n}. \quad (2)$$

Indeed, numerical calculations point to a limit value of $1/2$, whereas it is not difficult to show (by expansion in Taylor series or application of the rule of de l'Hôpital) that each term on the right of Eq. (2) tends to zero. This implies that the limit function is not continuous in 0 and consequently that the series is not uniformly convergent in u_0 over the interval $[0, \alpha]$ where α is some positive real number. Intuitively this can be better understood if one realizes that no matter how small u_0 is chosen, it is always possible to find a value of n big enough to yield a significant contribution of nu_0 .

In [1] the problem of taking the limit was tackled by formally letting $u_0 \rightarrow 0$ without making any assumption about the size of nu_0 . This led to

$$N_n \sim n(\sinh nu_0 - nu_0 \cosh nu_0), \quad u_0 \rightarrow 0 \quad (3)$$

and

$$\begin{aligned} D_n &\sim \sinh^2 nu_0 [e^{nu_0}(1 + u_0) - e^{-nu_0}(1 - u_0)]/2, \quad u_0 \rightarrow 0, \\ &\sim \sinh^3 nu_0, \quad u_0 \rightarrow 0. \end{aligned} \quad (4)$$

Since Eqs. (3) and (4) are crucial in the theoretical derivation in [1], it is worthwhile to analyse these steps critically. For instance, although it is clear that

$$\sinh(n+1)u_0 \not\sim \sinh nu_0, \quad u_0 \rightarrow 0, \quad n \in \mathbb{N}^+, \quad n \text{ fixed}, \quad (5)$$

one could be led to believe the contrary by comparing Eqs. (1) and (4). An important point is that in the deduction of Eq. (4) n is not fixed, since at fixed n the expression $u_0 \rightarrow 0$ implies $nu_0 \rightarrow 0$, in contradiction to the no-size-assumption of nu_0 . In the following sections these intuitive thoughts will be justified by rigorous mathematical proofs.

3. Proof of the non-uniform convergence. In what follows the terms of the series in Eq. (1) will be denoted by $a_n(u_0)$, i.e.,

$$a_n(u_0) \doteq nu_0 \sinh u_0 \frac{n \sinh u_0 \cosh nu_0 - \cosh u_0 \sinh nu_0}{\sinh^2 nu_0 \sinh(n+1)u_0}. \quad (6)$$

The present proof basically consists in proving that

$$\lim_{u_0 \rightarrow 0} \sum_{n=1}^{\infty} a_n(u_0) > 0.1 \quad (7)$$

whereas

$$\sum_{n=1}^{\infty} \lim_{u_0 \rightarrow 0} a_n(u_0) = 0. \quad (8)$$

First, two preliminary theorems will be given.

THEOREM 1. Let

$$f(x) \doteq x \coth xu_0 - \coth u_0, \quad x > 1, \quad u_0 > 0. \quad (9)$$

Then

$$\begin{aligned} f(x) &> \frac{e^{-2u_0}}{6u_0} (8u_0^3 + 12u_0^2 + 12u_0 + 6) \\ &\quad - \frac{e^{-2xu_0}}{6u_0} (8x^3u_0^3 + 12x^2u_0^2 + 12xu_0 + 6) > 0. \end{aligned} \quad (10)$$

Proof.

$$\frac{df}{dx} = \frac{\sinh 2xu_0 - 2xu_0}{2 \sinh^2 xu_0} \quad (11)$$

$$> \frac{(2xu_0)^3}{3! 2 \sinh^2 xu_0} > \frac{8(xu_0)^3}{3e^{2xu_0}} > 0, \quad x, u_0 > 0. \quad (12)$$

Thus, since $f(1) = 0$,

$$f(x) = \int_1^x \frac{df}{d\xi} d\xi > \int_1^x \frac{8u_0^3}{3} \xi^3 e^{-2\xi u_0} d\xi, \quad x > 1, \quad u_0 > 0, \quad (13)$$

$$\begin{aligned} &= \frac{1}{6u_0} [e^{-2u_0} (8u_0^3 + 12u_0^2 + 12u_0 + 6) \\ &\quad - e^{-2xu_0} (8x^3u_0^3 + 12x^2u_0^2 + 12xu_0 + 6)] \end{aligned} \quad (14)$$

by [2] 2.322.3. Furthermore, the latter expression is strictly positive since the integrand in Eq. (13) is strictly positive for $\xi > 1, u_0 > 0$. $\square$

THEOREM 2. For $q \rightarrow 1^-$ the following relations hold:

$$\begin{aligned} \sum_{n=1}^{\infty} n^k q^n &\sim \frac{k!}{(1-q)^{k+1}}, \quad k \geq 0, \\ &\sim -\ln(1-q), \quad k = -1, \\ &\sim \zeta(-k), \quad k < -1. \end{aligned} \quad (15)$$

The first two parts of the theorem were derived in [1]; the last part reduces to the definition of the Riemann ζ -function.

COROLLARY. For $u_0 \rightarrow 0^+$, $m > 0$ the following relations hold:

$$\begin{aligned} \sum_{n=1}^{\infty} n^k e^{-nmu_0} &\sim \frac{1}{u_0^{k+1}} \frac{k!}{m^{k+1}}, \quad k \geq 0, \\ &\sim -\ln(mu_0), \quad k = -1, \\ &\sim \zeta(-k), \quad k < -1. \end{aligned} \quad (16)$$

Now Eq. (7) can be proven.

THEOREM 3.

$$\lim_{u_0 \rightarrow 0} \sum_{n=1}^{\infty} a_n(u_0) > 0.1. \quad (17)$$

Proof. The definition of $a_n(u_0)$ leads to $a_1(u_0) = 0$.

For $u_0 > 0, n > 1$ the following inequalities hold:

(i)

$$nu_0 \sinh u_0 > nu_0^2; \quad (18)$$

(ii)

$$\frac{1}{\sinh^2 nu_0 \sinh(n+1)u_0} > \frac{1}{\left(\frac{e^{2nu_0}}{4}\right)\left(\frac{e^{(n+1)u_0}}{2}\right)} = \frac{8}{(e^{3u_0})^n e^{u_0}}; \quad (19)$$

(iii)

$$f(n) = n \coth nu_0 - \coth u_0 \quad (20)$$

satisfies the inequality of Theorem 1 for $x = n$;

(iv)

$$\sinh u_0 \sinh nu_0 > nu_0^2. \quad (21)$$

Combining those four inequalities leads to

$$\begin{aligned} \sum_{n=1}^{\infty} a_n(u_0) &> \frac{8e^{-3u_0}u_0^3}{6} (8u_0^3 + 12u_0^2 + 12u_0 + 6) \sum_{n=1}^{\infty} n^2 (e^{-3u_0})^n \\ &\quad - \frac{32}{3} \frac{u_0^6}{e^{u_0}} \sum_{n=1}^{\infty} n^5 (e^{-5u_0})^n - \frac{16u_0^5}{e^{u_0}} \sum_{n=1}^{\infty} n^4 (e^{-5u_0})^n \\ &\quad - 16 \frac{u_0^4}{e^{u_0}} \sum_{n=1}^{\infty} n^3 (e^{-5u_0})^n - \frac{8u_0^3}{e^{u_0}} \sum_{n=1}^{\infty} n^2 (e^{-5u_0})^n. \end{aligned} \quad (22)$$

Taking the limit on both sides and using Theorem 2 leads to

$$\lim_{u_0 \rightarrow 0} \sum_{n=1}^{\infty} a_n(u_0) \geq \frac{8.2!}{3^3} - \frac{32.5!}{3.5^6} - \frac{16.4!}{5^5} - \frac{16.3!}{5^4} - \frac{8.2!}{5^3} > 0.1. \quad (23)$$

Eqs. (7) and (8) imply that the limit function is not continuous. A well-known theorem states that a uniformly convergent series of continuous functions over some interval J has a continuous limit function over J [3]. Thus, the series I^* is not uniformly convergent in $[0, \alpha]$, $\alpha \in \mathbb{R}^+$.

4. Simplification of the series. In what follows frequent recourse will be taken to series that are easy to sum and that are termwise greater or smaller than the series at stake without changing the fundamental asymptotic behaviour in the areas of interest. In this context the following inequalities will frequently be used.

THEOREM 4. If $x > 0$ then

(i)

$$\cosh x < e^x; \quad (24)$$

(ii)

$$0 < \frac{x}{1+x} \frac{e^x}{2} < \sinh x < xe^x. \quad (25)$$

Proof.

(i) Trivial, since $e^{-x} < e^x$ for $x > 0$.

(ii) (a) Trivial.

(b) Let

$$f(x) \doteq (1+x)(e^x - e^{-x}) - xe^x. \quad (26)$$

Then

$$f'(x) = e^x + xe^{-x} > 0 \quad (27)$$

for $x > 0$ and $f(0) = 0 \Rightarrow f(x) > 0$ for $x > 0$.

(c) Let

$$f(x) \doteq 2xe^x - e^x + e^{-x}. \quad (28)$$

Then

$$f'(x) = 2xe^x + 2\sinh x > 0 \quad (29)$$

for $x > 0$ and $f(0) = 0 \Rightarrow f(x) > 0$ for $x > 0$. $\square$

A first simplification of Eq. (1) is performed by splitting the series of the sum into a sum of two series. This is allowed if both resulting series converge ([3]):

$$\lim_{u_0 \rightarrow 0} I^* = \lim_{u_0 \rightarrow 0} \left\{ u_0 \sinh^2 u_0 \sum_{n=1}^{\infty} \frac{n^2 \cosh nu_0}{\sinh^2 nu_0 \sinh(n+1)u_0} - u_0 \sinh u_0 \cosh u_0 \sum_{n=1}^{\infty} \frac{n}{\sinh nu_0 \sinh(n+1)u_0} \right\}. \quad (30)$$

This is satisfied, since the terms in both series decay exponentially. For example, for the second series, one obtains, using Theorem 4:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n}{\sinh nu_0 \sinh(n+1)u_0} &< \sum_{n=1}^{\infty} \frac{n}{\sinh^2 nu_0} \\ &< \frac{1}{u_0^2} \sum_{n=1}^{\infty} \frac{4(e^{-2u_0})^n (1+nu_0)^2}{n}, \quad u_0 > 0, \end{aligned} \quad (31)$$

which is a convergent series. Using the linear properties of the limit operator combined with the asymptotic behaviour of $\sinh x$ and $\cosh x$ for $x \rightarrow 0$ one arrives at

$$\lim_{u_0 \rightarrow 0} I^* = \lim_{u_0 \rightarrow 0} \sum_{n=1}^{\infty} nu_0^2 \frac{nu_0 \cosh nu_0 - \sinh nu_0}{\sinh^2 nu_0 \sinh(n+1)u_0}. \quad (32)$$

The last step, performed in Eq. (4) is a very delicate one. What really happens in this step can best be expressed by

$$\lim_{u_0 \rightarrow 0} \sum_{n=1}^{\infty} nu_0^2 \frac{nu_0 \cosh nu_0 - \sinh nu_0}{\sinh^2 nu_0 \sinh(n+1)u_0} = \lim_{u_0 \rightarrow 0} \sum_{n=1}^{\infty} nu_0^2 \frac{nu_0 \cosh nu_0 - \sinh nu_0}{\sinh^3 nu_0}. \quad (33)$$

That this is not trivial is shown by the following theorem.

THEOREM 5.

$$\begin{aligned} \lim_{u_0 \rightarrow 0} \sum_{n=1}^{\infty} f[(n+1)u_0] &= \lim_{u_0 \rightarrow 0} \sum_{n=1}^{\infty} f(nu_0) \\ &\Leftrightarrow \lim_{u_0 \rightarrow 0} f(u_0) = 0 \end{aligned} \quad (34)$$

assuming the series converge.

Proof.

$$\begin{aligned} \lim_{u_0 \rightarrow 0} \sum_{n=1}^{\infty} f[(n+1)u_0] &= \lim_{u_0 \rightarrow 0} \sum_{n=2}^{\infty} f(nu_0) \\ &= \lim_{u_0 \rightarrow 0} \sum_{n=1}^{\infty} f(nu_0) - \lim_{u_0 \rightarrow 0} f(u_0). \quad \square \end{aligned} \quad (35)$$

This shows that each term in the series must vanish for $u_0 \rightarrow 0$. Unfortunately, the series in Eq. (33) contains terms in nu_0 and $(n+1)u_0$, and thus Theorem 5 is not straightforward applicable.

The proof of Eq. (33) will be performed in a direct way by finding a series that is termwise greater than

$$I \doteq \sum_{n=1}^{\infty} d_n(u_0) \doteq \sum_{n=1}^{\infty} nu_0^2 \frac{nu_0 \cosh nu_0 - \sinh nu_0}{\sinh^2 nu_0} \left[\frac{1}{\sinh nu_0} - \frac{1}{\sinh(n+1)u_0} \right] \quad (36)$$

and for which the limit for $u_0 \rightarrow 0$ tends to 0. A couple of preliminary theorems yield some information on the terms in Eq. (36):

THEOREM 6. Let $f(x) \doteq x \cosh x - \sinh x$; then

$$0 < f(x) < e^x(x^2 - 2x) + 2(e^x - 1) \quad (37)$$

for $x > 0$.

Proof. $f'(x) = x \sinh x > 0$ and $f(0) = 0 \Rightarrow f(x) > 0$ for $x > 0$; furthermore, since $x \sinh x < x^2 e^x$ for $x > 0$ (Theorem 4):

$$f(x) < \int_0^x \zeta^2 e^{\zeta} d\zeta = e^x(x^2 - 2x) + 2(e^x - 1) \quad (38)$$

by [2] 2.322.2.

THEOREM 7. Let $f(x) \doteq \sinh(x+1)u_0 - \sinh nu_0$; then

$$0 < f(x) < u_0 e^{xu_0}(xu_0^2 + 2), \quad x > 0, \quad 0 < u_0 < 1. \quad (39)$$

Proof. The first inequality is trivial. Furthermore,

$$\begin{aligned} f(x) &= \sinh xu_0(\cosh u_0 - 1) + \cosh xu_0 \sinh u_0 \\ &< u_0^2 \sinh xu_0 + 2u_0 \cosh xu_0, \quad 0 < u_0 < 1, \\ &< xu_0^3 e^{xu_0} + 2u_0 e^{xu_0}, \quad xu_0 > 0. \quad \square \end{aligned} \quad (40)$$

THEOREM 8.

$$\frac{1}{\sinh^3 nu_0 \sinh(n+1)u_0} < \frac{16(1+nu_0)^3[1+(n+1)u_0]}{(nu_0)^4 e^{4nu_0}}, \quad n \geq 1, u_0 > 0. \quad (41)$$

Proof. Use Theorem 4, $n+1 > n$, and $\exp(u_0) > 1$. □

Combining Eq. (36) with Theorems 6, 7, and 8 one obtains

$$d_n(u_0) < \frac{16\{e^{nu_0}[(nu_0)^2 - 2nu_0] + 2[e^{nu_0} - 1]\}}{n^3 u_0 e^{3nu_0}} (nu_0^2 + 2)(1+nu_0)^3[1+(n+1)u_0],$$

$$n \geq 1, 0 < u_0 < 1. \quad (42)$$

Since

$$\begin{aligned} 2 + nu_0^2 &< 2 + nu_0 < 2(1 + nu_0), \\ 1 + nu_0 + u_0 &< 2 + nu_0 < 2(1 + nu_0) \end{aligned} \quad (43)$$

for $0 < u_0 < 1$, $n \geq 1$ one gets

$$d_n(u_0) < \frac{64[e^{nu_0}(nu_0)^2 - 2nu_0 e^{nu_0} + 2e^{nu_0} - 2]}{n^3 u_0 e^{3nu_0}} (1 + nu_0)^5,$$

$$n \geq 1, 0 < u_0 < 1. \quad (44)$$

Taking the limit for $u_0 \rightarrow 0$ of the sum of the above expression split into four contributions corresponding to the four terms in square brackets yields, using the Corollary of Theorem 2:

$$\lim_{u_0 \rightarrow 0} \sum_{n=1}^{\infty} \frac{64e^{nu_0}(nu_0)^2(1+nu_0)^5}{n^3 u_0 e^{3nu_0}} = 0, \quad (45)$$

$$\sum_{n=1}^{\infty} \frac{-128e^{nu_0} nu_0(1+nu_0)^5}{n^3 u_0 e^{3nu_0}} = -128\zeta(2) + O(u_0), \quad u_0 \rightarrow 0, \quad (46)$$

$$\sum_{n=1}^{\infty} \frac{128e^{nu_0}(1+nu_0)^5}{n^3 u_0 e^{3nu_0}} = 128 \left(\frac{\zeta(3)}{u_0} + 3\zeta(2) + O(u_0) \right), \quad u_0 \rightarrow 0, \quad (47)$$

$$\sum_{n=1}^{\infty} \frac{-128(1+nu_0)^5}{n^3 u_0 e^{3nu_0}} = -128 \left(\frac{\zeta(3)}{u_0} + 2\zeta(2) + O(u_0) \right), \quad u_0 \rightarrow 0. \quad (48)$$

To get Eqs. (47) and (48), the Corollary of Theorem 2 was extended as follows:

$$\sum_{n=1}^{\infty} \frac{(e^{-mu_0})^n}{n^k} = \zeta(k) - mu_0 \zeta(k-1) + O(u_0^2), \quad k > 2, \quad (49)$$

by using the Taylor expansion of the exponential function. Summing Eqs. (45) to (48) yields

$$\lim_{u_0 \rightarrow 0} \sum_{n=1}^{\infty} d_n(u_0) \leq 0. \quad (50)$$

Since $d_n(u_0) > 0$ for $u_0 > 0$, $n \geq 1$ (cf. Theorem 6 and Eq. (36)), this leads to

$$\lim_{u_0 \rightarrow 0} \sum_{n=1}^{\infty} d_n(u_0) = 0, \quad (51)$$

which was to be proven. Thus, it was legitimate to replace $(n + 1)u_0$ by nu_0 in Eq. (1).

5. Conclusions. A crucial and at first intuitive step in the asymptotic analysis for a given non-uniformly-convergent series was analysed in depth. It was shown that the step is not trivial and a rigorous mathematical proof justifying its application was obtained.

REFERENCES

- [1] G. Dhondt and M. Köhl, *Force between a dielectric cylinder in a constant electric field and a conducting half-space*, Quart. Appl. Math. (to appear)
- [2] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, 1980
- [3] M. D. Greenberg, *Foundations of Applied Mathematics*, Prentice-Hall Inc., Englewood Cliffs, N. J., 1978