NONLINEAR DIFFUSIVE PHENOMENA OF ENTROPY WEAK SOLUTIONS FOR A SYSTEM OF QUASILINEAR HYPERBOLIC CONSERVATION LAWS WITH DAMPING

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Abstract. We consider a system of isentropic flow through porous media and show that nonlinear diffusive phenomena occur even if there are shocks in the flow.

1. Introduction. We study the following hyperbolic conservation laws with damping:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha u, \quad \alpha > 0, \quad p'(v) < 0 \text{ for } v > 0, \end{cases} \quad (1.1)$$

which can be viewed as isentropic Euler equations in Lagrangian coordinates with friction term, $-\alpha u$, for the momentum equation and can be used to model the compressible flow through porous media where u is the velocity, $v > 0$ is the specific volume, and $p(v)$ is the pressure.

The well-known porous media equation can be obtained by approximating the second equation with Darcy's law:

$$\begin{cases} v_t = -\frac{1}{\alpha} p(v)_{xx}, \\ p(v)_x = -\alpha u. \end{cases} \quad (1.2)$$

The diffusion effect created by the damping is studied for solutions of (1.1) without shock waves in [HLI1, HLI2], which shows that the solutions of (1.1) tend to those of (1.2) as the time t tends to infinity. Namely, nonlinear diffusive phenomena in the solutions of (1.1) are discovered for solutions of (1.1) without shock waves. This result has been established even for adiabatic flow (see [HLU]). In the case when shock waves may develop in the solution of (1.1) with rough data, it is expected in [HLI1] that (1.2) would still model the time-asymptotic behavior of (1.1).

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The purpose of this paper is to prove this expectation, which means that Darcy's law may be obtained, in fact, from the more complete equations (1.1) time-asymptotically even in the presence of shocks.

For simplicity, we consider (1.1) with Riemann data, namely

$$(u(x, 0), v(x, 0)) = \begin{cases} (u_-, v_-) & \text{for } x < 0, \\ (u_+, v_+) & \text{for } x > 0, \end{cases} \quad (1.3)$$

where $v_{\mp} > 0$, and we take a typical form of the state function $p(v)$ to discuss, instead of a general one, namely $p(v) = av^{-\gamma}$ with $a > 0$ and $1 < \gamma < 3$, which is the state function for polytropic gas. Without loss of generality, this paper only deals with the case in which the two states (u_-, v_-) and (u_+, v_+) are connected by a forward shock curve in the phase plane and the entropy condition is satisfied as well, namely

$$\begin{cases} u_+ - u_- = -(v_+ - v_-) \sqrt{\frac{p(v_+) - p(v_-)}{v_- - v_+}}, \\ v_+ > v_-. \end{cases} \quad (1.4)$$

For the system (1.1) without the damping term, i.e., $\alpha = 0$, a Riemann problem has been studied very well (see [CH], [DD], [LI], [SM], etc.). However, for the system (1.1) with the damping term, the Riemann problem is much more complicated since there is no self-similar solution anymore and has not been studied so well yet in the literature although the problem is significant in the qualitative theory of inhomogeneous hyperbolic systems and in the applications as well.

It is known from [HT1, HT2] that if the initial shock is suitably weak, then the problem of (1.1), (1.3) with (1.4) has a unique entropy weak solution globally in time that is piecewise smooth and contains a forward shock $x = x_2(t)$ ($0 \leq t < +\infty$), satisfying the entropy condition, namely,

$$\begin{cases} \frac{dx_2(t)}{dt} = \sqrt{\frac{p(v_R) - p(v_L)}{v_L - v_R}}, & x_2(0) = 0, \\ u_R - u_L = -(v_R - v_L) \cdot \sqrt{\frac{p(v_R) - p(v_L)}{v_L - v_R}}, \\ v_R > v_L, \end{cases} \quad (1.5)$$

where $(u_R, v_R) = (u, v)(x_2(t) + 0, t)$, $(u_L, v_L) = (u, v)(x_2(t) - 0, t)$. Moreover, it is proved in [HT1, HT2] that

(i) the solution (u, v) is smooth in the domains Ω_1 and Ω_2 up to the boundary respectively, where

$$\begin{aligned} \Omega_1 &=: \{(x, t) : x < x_2(t), t \geq 0\}, \\ \Omega_2 &=: \{(x, t) : x > x_2(t), t \geq 0\}; \end{aligned}$$

(ii) in the domain Ω_2 , (u, v) is defined as

$$(u, v) = \{u^+(x, t), v^+(x, t)\} = \{u_+ e^{-t}, v_+\}, \quad (1.6)_1$$

and in the domain I_1 , (u, v) is defined as

$$(u, v) = \{u^-(x, t), v^-(x, t)\} = \{u_- e^{-t}, v_-\}, \quad (1.6)_2$$

where $I_1 =: \{(x, t) : x \leq x_1(t), t \geq 0\}$ and $x_1(t)$ is the backward characteristic satisfying

$$\frac{dx_1(t)}{dt} = \lambda(v_-) = -\sqrt{\gamma}(v_-)^{-\frac{\gamma+1}{2}}, \quad x_1(0) = 0; \quad (1.7)$$

(iii) along the shock curve $x = x_2(t)$,

$$(v_+ - v_-)e^{-A_1 t} \leq v_+ - v \leq (v_+ - v_-)e^{-A_2 t}, \quad t \geq 0, \quad (1.8)$$

where $v = v(x_2(t) - 0, t)$, A_i is a positive constant ($i = 1, 2$), and

(iv) $v_- \leq v(x, t) \leq v_+$, for $x \in R$, $t \geq 0$.

On the other hand, it is well known that (1.2) has a unique similarity solution $v^*(\eta)$, $\eta = \frac{x}{\sqrt{t+1}}$, satisfying $v^*(\pm\infty) = v_{\pm}$ and $v^*(\eta)$ is strictly monotone for $\eta \in R$ (see [DP]).

The main result obtained in the present paper says that there exists an $\bar{x}_0 \in R$ such that if $\delta = |u_+ - u_-| + |v_+ - v_-|$ is suitably small, then

$$\begin{aligned} \sup_{x \in R} |u(x, t) - u^*(x + \bar{x}_0, t)| + \sup_{x \in R} |v(x, t) - v^*(x + \bar{x}_0, t)| \\ = O(1)(t+1)^{-1/2}, \quad \text{as } t \rightarrow +\infty \end{aligned} \quad (1.9)$$

where

$$v^*(x + \bar{x}_0, t) = v^*\left(\frac{x + \bar{x}_0}{\sqrt{t+1}}\right), \quad u^*(x + \bar{x}_0, t) = \frac{-1}{\alpha} p(v^*(x + \bar{x}_0, t))_x.$$

This shows that nonlinear diffusive phenomena also occur for entropy weak solutions of (1.1).

2. Estimates along the shock curve $x = x_2(t)$. We deduce certain decay estimates along the shock curve $x = x_2(t)$ in this section, which plays a key role for establishing the main result in the next section.

Consider the system (1.1) with $a = 1$ and $\alpha = 1$, for convenience, namely

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -u, \quad p(v) = v^{-\gamma} \quad (1 < \gamma < 3). \end{cases} \quad (2.1)$$

It is easy to observe that the characteristic speed of (2.1) takes the form

$$\lambda = -\sqrt{\gamma}v^{-\frac{\gamma+1}{2}} \quad \text{and} \quad \mu = \sqrt{\gamma}v^{-\frac{\gamma+1}{2}}$$

respectively, corresponding to the backward and forward family.

Introduce Riemann invariants

$$\begin{aligned} r &= u - \frac{2\sqrt{\gamma}}{\gamma-1} v^{\frac{1-\gamma}{2}}, \\ s &= u + \frac{2\sqrt{\gamma}}{\gamma-1} v^{\frac{1-\gamma}{2}}. \end{aligned} \quad (2.2)$$

Thus

$$\begin{aligned} u &= \frac{s+r}{2}, \\ v &= \left[\frac{\gamma-1}{4\sqrt{\gamma}} (s-r) \right]^{\frac{2}{1-\gamma}}, \end{aligned} \quad (2.3)$$

and the system (2.1) can be written as

$$\begin{aligned} r_t + \lambda r_x &= -\frac{1}{2}(r + s), \\ s_t + \mu s_x &= -\frac{1}{2}(r + s), \end{aligned} \quad (2.4)$$

which is equivalent to (2.1) wherever the solution is smooth.

By the notation

$$\begin{aligned} ' &= \frac{\partial}{\partial t} + \lambda \frac{\partial}{\partial x}, \\ \cdot &= \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial x}, \end{aligned} \quad (2.5)$$

Eqs. (2.4) become

$$\begin{aligned} r' &= -\frac{1}{2}(r + s), \\ s' &= -\frac{1}{2}(r + s). \end{aligned} \quad (2.6)$$

It is known from [HT1, HT2] that, along the shock curve $x = x_2(t)$ ($t \geq 0$),

$$\begin{aligned} &\left[\sqrt{\gamma} v^{-\frac{1+\gamma}{2}} + \sqrt{\frac{v^{-\gamma} - v_+^{-\gamma}}{v_+ - v}} \right] \left\{ 1 + \frac{[\gamma v^{-(1+\gamma)} \cdot (v_+ - v) + (v^{-\gamma} - v_+^{-\gamma})] v^{\frac{1+\gamma}{2}}}{2\sqrt{\gamma(v_+ - v)(v^{-\gamma} - v_+^{-\gamma})}} \right\} r_x \\ &= 2\sqrt{(v_+ - v)(v^{-\gamma} - v_+^{-\gamma})} - \left[\sqrt{\gamma} v^{-\frac{1+\gamma}{2}} - \sqrt{\frac{v^{-\gamma} - v_+^{-\gamma}}{v_+ - v}} \right] \\ &\quad \cdot \left\{ -1 + \frac{[\gamma v^{-(1+\gamma)} \cdot (v_+ - v) + (v^{-\gamma} - v_+^{-\gamma})] v^{\frac{1+\gamma}{2}}}{2\sqrt{\gamma(v_+ - v)(v^{-\gamma} - v_+^{-\gamma})}} \right\} \cdot s_x, \end{aligned} \quad (2.7)$$

where r_x , s_x , and v take values at $(x_2(t) - 0, t)$, and it is shown in [HT1, HT2] that

$$|s_x(x_2(t) - 0, t)| \leq C; \quad (2.8)$$

hereafter C denotes a generic constant, independent of t .

Due to (1.8), it follows that, along $x = x_2(t)$,

$$v_+ - \delta e^{-A_2 t} \leq v(x_2(t) - 0, t) \leq v_+ + \delta e^{-A_1 t}. \quad (2.9)$$

Conditions (2.7)–(2.9), with the help of the mean value theorem used on $(v^{-\gamma} - v_+^{-\gamma})$, imply that

$$|r_x(x_2(t) - 0, t)| \leq C e^{-\beta_1 t}, \quad t \geq 0 \text{ for some } \beta_1 > 0. \quad (2.10)$$

To get the decay estimates of s_x along $x = x_2(t)$, we need the following lemma.

LEMMA 2.1. If $\delta = |u_+ - u_-| + |v_+ - v_-|$ is suitably small, then

$$|s_{xx}(x, t)| \leq C \quad \text{for } x < x_2(t), \quad t \geq 0. \quad (2.11)$$

Particularly,

$$|s_{xx}(x_2(t) - 0, t)| \leq C, \quad t \geq 0.$$

Proof. The existence of all of the derivatives r_{xx} , s_{xx} , r_{xxx} , r_{xxt} , s_{xxx} , s_{xxt} , etc., in the domain $\Omega_1 = \{(x, t) : x \leq x_2(t), t \geq 0\}$ can be obtained, based on (2.4), by using difference quotients and then taking limits.

Let

$$y = \mu^{1/2} s_x + \frac{g}{2} \quad \text{for } (x, t) \in \Omega_1, \quad (2.12)$$

where

$$g = \frac{4\gamma^{\frac{1}{4}}}{3-\gamma} v^{\frac{3-\gamma}{4}}. \quad (2.13)$$

With the help of (2.4), it is easy to find that

$$\begin{aligned} y_t + \sqrt{\gamma} \cdot v^{-\frac{\gamma+1}{2}} y_x \\ = -f \left(y - \frac{1}{2}g \right) \left(y - \frac{\gamma-1}{\gamma+1}g \right) \quad \text{for } (x, t) \in \Omega_1 \end{aligned} \quad (2.14)$$

where $f = \frac{\gamma+1}{3-\gamma} g^{-1}$.

Differentiating (2.14) with respect to x , we arrive, by virtue of (2.2), (2.3), (2.4), and (2.6), at

$$\begin{aligned} (y_x)' - \frac{(\gamma+1)}{4} v^{-1} \cdot v' \cdot y_x + \frac{(\gamma+1)}{4} v^{-1} s_x y_x \\ = f_1(x, t) - f \left[2y - \frac{(3\gamma-1)}{2(\gamma+1)}g \right] y_x + f_2(x, t) \quad \text{for } (x, t) \in \Omega_1, \end{aligned} \quad (2.15)$$

where

$$\begin{aligned} f_1(x, t) &= -f_x \left[y^2 - \frac{(2\gamma-1)}{2(\gamma+1)}gy + \frac{\gamma-1}{2(\gamma+1)}g^2 \right] (x, t), \\ f_2(x, t) &= -f \left[-\frac{(3\gamma-1)}{2(\gamma+1)}g_x \cdot y + \frac{(\gamma-1)}{(\gamma+1)}g \cdot g_x \right] (x, t). \end{aligned}$$

Then, multiplying (2.15) by $v^{-\frac{\gamma+1}{4}}$, we get

$$\begin{aligned} (v^{-\frac{\gamma+1}{4}} y_x)' \\ = - \left\{ f \left[2y - \frac{(3\gamma-1)}{2(\gamma+1)}g \right] + \frac{(\gamma+1)}{4} v^{-1} s_x \right\} \cdot (v^{-\frac{\gamma+1}{4}} \cdot y_x) + f_3(x, t), \quad (x, t) \in \Omega_1, \end{aligned} \quad (2.16)$$

where

$$f_3(x, t) = v^{-\frac{\gamma+1}{4}} [f_1(x, t) + f_2(x, t)].$$

Let

$$b = f \left[2y - \frac{(3\gamma-1)}{2(\gamma+1)}g \right] + \frac{(\gamma+1)}{4} \cdot v^{-1} s_x, \quad (x, t) \in \Omega_1.$$

By the definition of y , f , and g , it is easy to see that

$$b(x, t) = \frac{1}{2} + \frac{3(\gamma+1)}{4} (v^{-1} s_x)(x, t), \quad (x, t) \in \Omega_1.$$

It is known from [HT1, HT2] that $|s_x|$ is small if δ is small. This implies that

$$b(x, t) \geq \frac{1}{4} \quad \text{for } (x, t) \in \Omega_1 \quad (2.17)$$

provided δ is suitably small.

We also know from [HT1, HT2] that

$$|f_3(x, t)| \leq C \quad \text{for } (x, t) \in \Omega_1. \quad (2.18)$$

For any fixed $(x, t) \in \Omega_1$, let $x_2(\tau; x, t)$ ($0 \leq \tau \leq t$) denote the forward characteristic passing through (x, t) such that $x_2(t; t, x) = x$ and $x_2(0; x, t) = \beta$. Let

$$k(\tau) = v^{-\frac{\gamma+1}{4}} y_x(x_2(\tau; x, t), \tau)$$

where $(x, t) \in \Omega_1$ is fixed. We still use the same notations b and f_3 to denote the corresponding values taken along $x_2(\tau; x, t)$ respectively, namely,

$$\begin{aligned} b(\tau) &= b(x_2(\tau; x, t), \tau), \\ f_3(\tau) &= f_3(x_2(\tau; x, t), \tau). \end{aligned}$$

Then, (2.16) implies that

$$\frac{dk(\tau)}{d\tau} = -b(\tau)k(\tau) + f_3(\tau), \quad 0 \leq \tau \leq t, \quad (2.19)$$

where it is known from (2.18) that

$$|f_3(\tau)| \leq C, \quad 0 \leq \tau \leq t.$$

Multiply (2.19) by $\exp \left[\int_0^\tau b(\xi) d\xi \right]$ and integrate the resulting equation over $[0, t]$. It turns out from (2.17)–(2.19) that

$$|k(\tau)| \leq C, \quad 0 \leq \tau \leq t. \quad (2.20)$$

Particularly,

$$|k(t)| \leq C.$$

Due to the arbitrariness of $(x, t) \in \Omega_1$, we obtain

$$\left| v^{-\frac{\gamma+1}{4}} y_x(x, t) \right| \leq C \quad \text{for } (x, t) \in \Omega_1.$$

Lemma 2.1 follows then by virtue of the facts: $v_- \leq v(x, t) \leq v_+$, the uniform boundedness of u_x and v_x in Ω_1 (see [HT1, HT2]), and the definition of y .

The decay estimate of s_x along the shock curve $x = x_2(t)$ is given in the next lemma.

LEMMA 2.2. If δ is suitably small, then it follows that

$$|s_x(x_2(t) - 0, t)| \leq C e^{-\beta_2 t}, \quad t \geq 0, \quad \text{for some } \beta_2 > 0.$$

Proof. Let

$$y_1(x_2(t), t) =: [\mu(v_+)]^{1/2} \cdot \frac{2v_+}{3-\gamma} - \left(\mu^{1/2} s_x + \frac{g}{2} \right) (x_2(t) - 0, t), \quad t \geq 0$$

and

$$Y(t) =: y_1(x_2(t), t),$$

where $\mu(v_+)$ is the value of $\mu(v)$ at $v = v_+$.

It is easy to verify that

$$\begin{aligned} \frac{dY(t)}{dt} = & f \left[\mu(v_+)^{1/2} \frac{2v_+}{3-\gamma} - \frac{1}{2}g - Y(t) \right] \left[\mu(v_+)^{1/2} \frac{2v_+}{3-\gamma} - \frac{\gamma-1}{\gamma+1}g - Y(t) \right] \\ & + y_{1x}(x_2(t) - 0, t) \left[\frac{dx_2(t)}{dt} - \mu(v(x_2(t) - 0, t)) \right], \end{aligned} \quad (2.21)$$

where g and f take values at $v = v(x_2(t) - 0, t)$, defined by (2.13) and (2.14) respectively.

We estimate next each term on the right-hand side of (2.21).

Define

$$k_1(t) =: \mu(v_+)^{1/2} \frac{2v_+}{3-\gamma} - \frac{1}{2}g(x_2(t) - 0, t).$$

It follows from (1.8) and (2.13) that

$$|k_1(t)| \leq Ce^{-\beta_3 t}, \quad t \geq 0, \quad \text{for some } \beta_3 > 0. \quad (2.22)$$

Let

$$\begin{aligned} k_2(t) &=: \mu(v_+)^{1/2} \frac{2v_+}{3-\gamma} - \frac{\gamma-1}{\gamma+1}g - Y(t) \\ &= \left(\mu^{1/2}s_x + \frac{3-\gamma}{2(\gamma+1)}g \right) (x_2(t) - 0, t). \end{aligned}$$

Since $|s_x|$ is small when δ is small (see [HT1, HT2]), we are able to choose a positive constant $a_1 > 0$ such that

$$k_2(t) \geq a_1, \quad t \geq 0. \quad (2.23)$$

Define

$$k_3(t) =: y_{1x}(x_2(t) - 0, t) \left[\frac{dx_2(t)}{dt} - \mu(v(x_2(t) - 0, t)) \right].$$

Due to (1.5), (1.8), Lemma 2.1, and the uniform boundedness of $(u_x, v_x)(x_2(t) - 0, t)$ for $t \geq 0$ (see [HT1, HT2]), it can be shown that

$$|k_3(t)| \leq Ce^{-\beta_4 t}, \quad t \geq 0, \quad \text{for some } \beta_4 > 0. \quad (2.24)$$

In terms of $k_1(t)$, $k_2(t)$, and $k_3(t)$, (2.21) can be written as

$$\frac{dY(t)}{dt} = -fk_2(t)Y(t) + fk_1(t)k_2(t) + k_3(t), \quad t \geq 0. \quad (2.25)$$

Conditions (2.22)–(2.24), with the definition of f together, yield

$$fk_2(t) \geq d_0 > 0 \quad (t \geq 0), \quad \text{for some positive constant } d_0 > 0 \quad (2.26)$$

and

$$|fk_1(t)k_2(t) + k_3(t)| \leq Ce^{-\beta_5 t} \quad (t \geq 0) \quad \text{for some positive constant } \beta_5 > 0. \quad (2.27)$$

Multiplying (2.25) by $\exp \left[\int_0^t (fk_2)(\xi) d\xi \right]$ and integrating it with respect to t , we arrive, with the help of (2.25)–(2.27), at

$$|Y(t)| \leq Ce^{-\beta_6 t} \quad (t \geq 0) \quad \text{for some positive constant } \beta_6 > 0. \quad (2.28)$$

It is clear from (1.8) that

$$\left| \mu(v_+)^{1/2} \frac{2v_+}{3-\gamma} - \frac{g}{2}(x_2(t) - 0, t) \right| \leq Ce^{-\beta_7 t} \quad (t \geq 0)$$

for some positive constant $\beta_7 > 0$. (2.29)

Lemma 2.2 follows then from (2.28), (2.29), and the definition of Y .

Differentiating (2.7) with respect to t along $x = x_2(t)$, and using (2.4), we can find a relation between r_{xx} and s_{xx} . This, combined with Lemma 2.1 and the uniform boundedness of r_x , s_x , r_t , and s_t , yields that

$$|r_{xx}(x_2(t) - 0, t)| \leq C, \quad t \geq 0. \quad (2.30)$$

It follows from (1.5), (1.6)₁, (1.8), and (2.3)₁ that

$$|u(x_2(t) - 0, t) - u(x_2(t) + 0, t)| \leq Ce^{-\beta' t}, \quad t \geq 0, \quad (2.31)$$

and

$$\left| \frac{r+s}{2}(x_2(t) - 0, t) \right| \leq Ce^{-\beta' t}, \quad t \geq 0 \quad (2.32)$$

for some positive constant $\beta' > 0$.

Conditions (2.32), (2.4), (2.10), and Lemma 2.2 yield

$$|r_t(x_2(t) - 0, t)| + |s_t(x_2(t) - 0, t)| \leq Ce^{-\beta'_1 t}, \quad t \geq 0, \quad \text{for some positive } \beta'_1 > 0. \quad (2.33)$$

Hence

$$(|u_t| + |u_x| + |v_t| + |v_x|)(x_2(t) - 0, t) \leq Ce^{-\beta'_2(t)}, \quad t \geq 0, \quad \text{for some } \beta'_2 > 0. \quad (2.34)$$

Similarly, it can be obtained that

$$(|u_{tt}| + |u_{tx}| + |u_{xx}| + |v_{tt}| + |v_{tx}| + |v_{xx}|)(x_2(t) - 0, t) \leq C, \quad t \geq 0. \quad (2.35)$$

Differentiating (2.16) with respect to x , and using (2.17) and other established estimates, it can be shown by a similar approach to that used for Lemma 2.1 that

$$|S_{xxx}(x, t)| \leq C \quad \text{for } (x, t) \in \Omega_1.$$

Let

$$y_2(x_2(t) - 0, t) =: \left(v^{-\frac{\gamma+1}{4}} y_x \right) (x_2(t) - 0, t)$$

and

$$Z(t) =: y_2(x_2(t) - 0, t),$$

where y is defined in (2.12).

It is not difficult to get, from (2.16), that

$$\begin{aligned} \frac{dZ(t)}{dt} &= -b(x_2(t) - 0, t) \cdot Z(t) + f_3(x_2(t) - 0, t) \\ &\quad + y_{2x}(x_2(t) - 0, t)[\dot{x}_2(t) - \mu(v(x_2(t) - 0, t))], \end{aligned}$$

where the functions b and f_3 are defined in the proof of Lemma 2.1. By a similar argument to that used in Lemma 2.2, we obtain

$$|Z(t)| \leq Ce^{-\beta'_3 t}, \quad t \geq 0 \quad (2.36)$$

for some positive constant β'_3 .

It then turns out that

$$|s_{xx}(x_2(t) - 0, t)| \leq Ce^{-\beta'_4 t}, \quad t \geq 0 \quad \text{for some positive constant } \beta'_4 > 0. \quad (2.37)$$

Due to (2.37), the relation between r_{xx} and s_{xx} along $x = x_2(t)$, mentioned before, and other obtained estimates, it can be shown that

$$|r_{xx}(x_2(t) - 0, t)| \leq Ce^{-\beta'_5 t}, \quad t \geq 0, \quad \text{for some positive constant } \beta'_5 > 0. \quad (2.38)$$

Thus, it is possible to get the exponential decay rates for other second-order partial derivatives of r and s along $x = x_2(t)$, with the help of (2.4).

On the other hand, along the backward characteristic curve $x = x_1(t)$, it follows that

$$\begin{aligned} \frac{dx_1(t)}{dt} &= -\sqrt{\gamma}v_-^{-\frac{\gamma+1}{2}}, \quad x_1(0) = 0, \\ v(x_1(t), t) &\equiv v_-, \quad u(x_1(t), t) \equiv u_-e^{-t}. \end{aligned} \quad (2.39)$$

Moreover, it is easy to see from (1.6) and the higher smoothness of (u, v) in the domain I_1 that all of $v_t, v_x, v_{xx}, v_{xxx}$, etc. equal to zero along $x = x_1(t)$ while all of $u_t, u_x, u_{xx}, u_{xxx}$, etc. decay exponentially fast as $t \rightarrow +\infty$ along $x = x_1(t)$.

We now collect in the following lemma all of the estimates obtained so far.

LEMMA 2.3. Along the shock curve $x = x_2(t)$, it follows that

$$\begin{aligned} &|u(x_2(t) - 0, t)| + |v(x_2(t) - 0, t) - v_+| \\ &+ (|u_x| + |u_t| + |u_{xx}| + |u_{xt}| + |u_{tt}| + |v_x| + |v_t| + |v_{xx}| + |v_{xt}| + |v_{tt}|)(x_2(t) - 0, t) \\ &\leq Ce^{-\beta^* t}, \quad t \geq 0, \quad \text{for some positive constant } \beta^* > 0. \end{aligned} \quad (2.40)$$

A similar estimate along $x = x_1(t)$ can be expressed if we use v_- and $x_1(t)$ instead of v_+ and $x_2(t) - 0$ in (2.40) respectively.

3. Energy estimates and the main results. We compare the solution (u, v) of (1.1), (1.3) with the similarity solution v^* of (1.2) and establish the main theorem in this section.

It is known (see [DP]) that there exists a unique similarity solution $v^*(\eta)$, $\eta = \frac{x}{\sqrt{t+1}}$, satisfying $v^*(\mp\infty) = v_{\mp}$, for the equation (1.2), namely $v^*(\eta)$ satisfies

$$\begin{cases} [-p(v^*(\eta))]_{\eta\eta} + \frac{1}{2}\eta v^*_{\eta} = 0, \\ v^*(\mp\infty) = v_{\mp}. \end{cases} \quad (3.1)$$

Moreover, it has been shown that $v^*_{\eta}(\eta) > 0$ for $\eta \in \mathbb{R}$ if $v_- < v_+$, which implies

$$v_- < v^*(\eta) < v_+. \quad (3.2)$$

The following estimates on $v^*(\eta) = v^*\left(\frac{x}{\sqrt{t+1}}\right)$ are needed in our analysis (see [HLI1, HLI2] for details).

LEMMA 3.1. For the similarity solution $v^*(\eta)$, $\eta = \frac{x}{\sqrt{t+1}}$, it follows that

$$|v_{\eta\eta\eta}^*(\eta)| + |v_{\eta\eta}^*(\eta)| + |v_{\eta}^*(\eta)| + \begin{cases} |v^*(\eta) - v_+| & \text{if } \eta \geq 0 \\ |v^*(\eta) - v_-| & \text{if } \eta \leq 0 \end{cases} \leq C|v_+ - v_-|e^{-C\eta^2}, \quad (3.3)$$

$$\int_{-\infty}^{+\infty} |v_x^*(x, t)|^2 dx \leq C(v_+ - v_-)^2(t+1)^{-1/2}, \quad (3.4)$$

$$\int_{-\infty}^{\infty} [|v_t^*(x, t)|^2 + |v_{xx}^*(x, t)|^2] dx \leq C(v_+ - v_-)^2(t+1)^{-3/2}, \quad (3.5)$$

$$\int_{-\infty}^{\infty} |v_{xt}^*(x, t)|^2 dx \leq C(v_+ - v_-)^2(t+1)^{-5/2}, \quad (3.6)$$

$$\int_{-\infty}^{\infty} |v_{xxt}^*(x, t)|^2 dx \leq C(v_+ - v_-)^2(t+1)^{-7/2}, \quad (3.7)$$

$$\int_{-\infty}^{\infty} |v_{xtt}^*(x, t)|^2 dx \leq C(v_+ - v_-)^2(t+1)^{-9/2}, \quad (3.8)$$

where $v^*(x, t) = v^*\left(\frac{x}{\sqrt{t+1}}\right)$.

It follows from (1.5), (1.6)₁, and (1.8) that

$$D_1 t \leq x_2(t) \leq D_2 t, \quad t \geq 0 \quad (3.9)$$

for some positive constants D_1 and D_2 .

Take any smooth function $m_0(x)$ with

$$\text{supp } m_0(x) \subset [-1, 1] \quad (3.10)$$

and

$$\int_{-\infty}^{\infty} m_0(x) dx = \int_{-1}^1 m_0(x) dx = 1. \quad (3.11)$$

Due to (3.9) and (1.7), it is possible to choose $T_1 > 0$ such that

$$[-1, 1] \subset [x_1(T_1), x_2(T_1)]. \quad (3.12)$$

Define

$$\begin{aligned} \nu(x_0, t) = & \left[u(x_2(t) - 0, t) - u^*\left(\frac{x_2(t) + x_0}{\sqrt{t+1}}\right) \right] + \left[u^*\left(\frac{x_1(t) + x_0}{\sqrt{t+1}}\right) - u_- e^{-t} \right] \\ & + \left[v(x_2(t) - 0, t) - v^*\left(\frac{x_2(t) + x_0}{\sqrt{t+1}}\right) \right] \dot{x}_2(t) \\ & + \left[v^*\left(\frac{x_1(t) + x_0}{\sqrt{t+1}}\right) - v_- \right] \dot{x}_1(t) \end{aligned}$$

where $u^*\left(\frac{x+x_0}{\sqrt{t+1}}\right) = -\left[p\left(v^*\left(\frac{x+x_0}{\sqrt{t+1}}\right)\right)\right]_x$.

It can be shown that $F(x_0) =: -\int_{T_1}^{+\infty} \nu(x_0, t) dt$ is well defined for $x_0 \in R$. In fact, (3.3), (3.9), and Lemma 2.3 yield

$$\left| u(x_2(t) - 0, t) - u^*\left(\frac{x_2(t) + x_0}{\sqrt{t+1}}\right) \right| \leq C(x_0)e^{-\beta_8 t}, \quad t \geq 0, \quad (3.13)$$

for some positive constant $\beta_8 > 0$ and $C(x_0) > 0$, depending only on x_0 . Similarly,

$$\begin{aligned} \left| v(x_2(t) - 0, t) - v^* \left(\frac{x_2(t) + x_0}{\sqrt{t+1}} \right) \right| &\leq |v(x_2(t) - 0, t) - v_+| \\ &\quad + \left| v^* \left(\frac{x_2(t) + x_0}{\sqrt{t+1}} \right) - v_+ \right| \\ &\leq C(x_0)e^{-\beta_9 t}, \quad t \geq 0, \quad \text{for some } \beta_9 > 0, \end{aligned} \quad (3.14)$$

$$\left| u^* \left(\frac{x_1(t) + x_0}{\sqrt{t+1}} \right) - u_- e^{-t} \right| \leq C(x_0)e^{-\beta_{10} t}, \quad t \geq 0, \quad \text{for some } \beta_{10} > 0, \quad (3.15)$$

$$\left| v^* \left(\frac{x_1(t) + x_0}{\sqrt{t+1}} \right) - v_- \right| \leq C(x_0)e^{-\beta_{11} t}, \quad t \geq 0, \quad \text{for some } \beta_{11} > 0. \quad (3.16)$$

From (3.13)–(3.16), we know that $F(x_0)$ is well defined in $x_0 \in R$.

Let

$$G(x_0) = \int_{x_1(T_1)}^{x_2(T_1)} \left[v(x, T_1) - v^* \left(\frac{x + x_0}{\sqrt{T_1}} \right) \right] dx - F(x_0).$$

It can be verified, by using (1.5), (1.6), (1.8), and the facts of $v^*(\pm\infty) = v_{\pm}$, $v_- < v_+$, and $u^*(\pm\infty) = 0$ that

$$\lim_{x_0 \rightarrow -\infty} G(x_0) = +\infty \quad \text{and} \quad \lim_{x_0 \rightarrow +\infty} G(x_0) = -\infty. \quad (3.17)$$

Thus, we are able to choose $\bar{x}_0 \in R$ such that

$$G(\bar{x}_0) = 0. \quad (3.18)$$

Let

$$m(x, t) = m_0(x)h(t) \quad \text{for any } (x, t) \in \Omega_3 =: \{x_1(t) \leq x < x_2(t), \quad t \geq T_1\}, \quad (3.19)$$

where

$$h(t) = - \int_t^{+\infty} \nu(\bar{x}_0, \xi) d\xi \quad (3.20)$$

and let

$$\begin{aligned} w(x, t) &= v(x, t) - v^* \left(\frac{x + \bar{x}_0}{\sqrt{t+1}} \right) - m(x, t), \\ z(x, t) &= u(x, t) - u^* \left(\frac{x + \bar{x}_0}{\sqrt{t+1}} \right) - \int_{-\infty}^x m_t(\xi, t) d\xi \end{aligned}$$

where $(x, t) \in \Omega_3$.

Thus, (1.1) and (1.2) imply that

$$\begin{cases} w_t = z_x, \\ z_t + [p(v^* + m + w) - p(v^*)]_x + z - p(v^*)_{xt} = -[h_t(t) + h_{tt}(t)] \int_{-\infty}^x m_0(\xi) d\xi \end{cases} \quad (3.21)$$

in Ω_3 .

Introduce

$$\varphi(x, t) = \int_{x_1(t)}^x w(\xi, t) d\xi, \quad (x, t) \in \Omega_3. \quad (3.22)$$

It follows then that

$$\varphi_x(x, t) = w(x, t)$$

and

$$\varphi_t(x, t) = z(x, t) - z(x_1(t), t) - w(x_1(t), t)\dot{x}_1(t).$$

Define $\theta(t) =: z(x_1(t), t) + w(x_1(t), t)\dot{x}_1(t)$. It turns out that

$$\begin{aligned} \varphi_{tt} + [p(v^* + m + \varphi_x) - p(v^*)]_x + \varphi_t - p(v^*)_{xt} \\ = -(h_t(t) + h_{tt}(t)) \int_{-\infty}^x m_0(\xi) d\xi + (\theta_t(t) - \theta(t)). \end{aligned} \quad (3.23)$$

Denote the right-hand side of (3.23) by $\Lambda(x, t)$, namely

$$\Lambda(x, t) =: -(h_t(t) + h_{tt}(t)) \int_{-\infty}^x m_0(\xi) d\xi + (\theta_t(t) - \theta(t)).$$

In view of Lemma 2.3, Lemma 3.1, (3.10), (3.12), and the uniform boundedness of $\dot{x}_i(t)$ and $\ddot{x}_i(t)$ ($i = 1, 2$), it can be shown that

$$|\Lambda(x, t)| + |\Lambda_t(x, t)| + |\Lambda_x(x, t)| \leq C e^{-\beta_{12}t}, \quad \beta_{12} > 0. \quad (3.24)$$

Denote $\varphi(x_2(t) - 0, t)$ by $\psi(t)$. We claim that

$$\psi(t) \equiv 0 \quad \text{for } t \geq T_1. \quad (3.25)$$

In fact, it follows from (3.10)–(3.12) and (3.18) that

$$\psi(T_1) = 0. \quad (3.26)$$

Furthermore, due to (3.10)–(3.12), (3.19)–(3.22), and (1.6), it can be shown that

$$\frac{d\psi(t)}{dt} = 0, \quad t \geq T_1. \quad (3.27)$$

This, together with (3.26), implies (3.25).

The main result in the present paper is that

THEOREM 3.2. For the solution (u, v) of (1.1), (1.3) with (1.4), it follows that

$$\begin{aligned} \sup_{x \in R} \{|u(x, t) - u^*(x + \bar{x}_0, t)| + |v(x, t) - v^*(x + \bar{x}_0, t)|\} \\ \leq C(1 + t)^{-1/2}, \quad t \geq T_1, \end{aligned} \quad (3.28)$$

where $v^*(x + \bar{x}_0, t) = v^*\left(\frac{x + \bar{x}_0}{\sqrt{t+1}}\right)$, $u^*(x + \bar{x}_0, t) = -p(v^*(x + \bar{x}_0, t))_x$.

Proof. It follows from (3.3), (3.9), and (1.6) that

$$\begin{aligned} \sup_{x > x_2(t)} |v^*(x + \bar{x}_0, t) - v(x, t)| &\leq C|v_+ - v_-| e^{-C(\frac{P_1 t + \xi_0}{\sqrt{t+1}})^2} \\ &\leq C(1 + t)^{-1/2}, \quad t \geq T_1. \end{aligned} \quad (3.29)$$

Similarly,

$$\begin{aligned} & \sup_{x > x_2(t)} |u(x, t) - u^*(x + \bar{x}_0, t)| \\ & \quad + \sup_{x \leq x_1(t)} \{|v(x, t) - v^*(x + \bar{x}_0, t)| + |u(x, t) - u^*(x + \bar{x}_0, t)|\} \\ & \leq C(1+t)^{-1/2}, \quad t \geq T_1. \end{aligned} \quad (3.30)$$

Thus, to prove (3.28) it is sufficient to show that

$$\sup_{x_1(t) < x < x_2(t)} \{|u(x, t) - u^*(x + \bar{x}_0, t)| + |v(x, t) - v^*(x + \bar{x}_0, t)|\} \leq C(1+t)^{-1/2}, \quad t \geq T_1. \quad (3.31)$$

By virtue of (3.16) and the fact that

$$|h(t)| \leq Ce^{-\beta_{13}t}, \quad \text{for some positive constant } \beta_{13} > 0, \quad (3.32)$$

we only need to show that

$$\sup_{x_1(t) < x < x_2(t)} [|\varphi_x(x, t)| + |\varphi_t(x, t)|] \leq C(1+t)^{-1/2}, \quad t \geq T_1. \quad (3.33)$$

We prove (3.33) next.

Multiplying (3.23) by φ and φ_t respectively, then integrating the resulting equations on the region $\{(x, \tau) | x_1(\tau) \leq x \leq x_2(\tau), T_1 \leq \tau \leq t\}$, integrating by parts and using (3.25) we arrive at

$$\begin{aligned} & \int_{x_1(t)}^{x_2(t)} \left(\varphi_t \cdot \varphi + \frac{1}{2} \varphi^2 \right) (x, t) dx \\ & \quad + \int_{T_1}^t \int_{x_1(\tau)}^{x_2(\tau)} [p(v^*) - p(v^* + \varphi_x)] \varphi_x(x, \tau) dx d\tau \\ & = \int_{T_1}^t \int_{x_1(\tau)}^{x_2(\tau)} \varphi_t^2(x, \tau) dx d\tau \\ & \quad + \int_{T_1}^t \int_{x_1(\tau)}^{x_2(\tau)} [p(v^* + m + \varphi_x) - p(v^* + \varphi_x)] \varphi_x(x, \tau) dx d\tau \\ & \quad - \int_{T_1}^t \int_{x_1(\tau)}^{x_2(\tau)} p(v^*)_t \varphi_x(x, \tau) dx d\tau + \int_{T_1}^t \int_{x_1(\tau)}^{x_2(\tau)} \wedge(x, \tau) \varphi(x, \tau) dx d\tau \\ & \quad + \int_{x_1(T_1)}^{x_2(T_1)} \left[\varphi_t \varphi + \frac{1}{2} \varphi^2 \right] (x, T_1) dx, \quad t \geq T_1, \end{aligned} \quad (3.34)$$

and

$$\begin{aligned}
& \int_{x_1(t)}^{x_2(t)} \frac{1}{2} \varphi_t^2(x, t) dx + \int_{T_1}^t \int_{x_1(\tau)}^{x_2(\tau)} [p(v^* + m) - p(v^* + m + \varphi_x)] \varphi_{xt}(x, \tau) dx d\tau \\
& \quad + \int_{T_1}^t \int_{x_1(\tau)}^{x_2(\tau)} \varphi_t^2(x, \tau) dx d\tau \\
& = \int_{x_1(T_1)}^{x_2(T_1)} \frac{1}{2} \varphi_t^2(x, T_1) dx + \int_{T_1}^t \int_{x_1(\tau)}^{x_2(\tau)} [p(v^*) - p(v^* + m)]_x \varphi_t(x, \tau) dx d\tau \\
& \quad + \int_{T_1}^t \int_{x_1(\tau)}^{x_2(\tau)} p(v^*)_{xt} \varphi_t(x, \tau) dx d\tau \\
& \quad + \int_{T_1}^t \left[\frac{1}{2} \varphi_t^2(x_2(\tau) - 0, \tau) \dot{x}_2(\tau) - \frac{1}{2} \varphi_t^2(x_1(\tau), \tau) \dot{x}_1(\tau) \right] d\tau \\
& \quad + \int_{T_1}^t \int_{x_1(\tau)}^{x_2(\tau)} \Lambda(x, \tau) \varphi_t(x, \tau) dx d\tau, \quad t \geq T_1.
\end{aligned} \tag{3.35}$$

Let

$$q(x, t) =: \int_0^{\varphi_x} [p(v^* + m) - p(v^* + m + y)] dy.$$

It can be easily verified that

$$\begin{aligned}
q_t &= [p(v^* + m) - p(v^* + m + \varphi_x)] \varphi_{xt} \\
& \quad + \int_0^{\varphi_x} \{p'(v^* + m) - p'(v^* + m + y)\} (v_t^* + m_t) dy.
\end{aligned}$$

It is known that

$$|m(x, t)| \leq C\delta, \quad |\varphi_x(x, t)| \leq C\delta \quad \text{for } (x, t) \in \Omega_3 \tag{3.36}$$

which implies

$$v^* + m + \varphi_x \geq v_- - 2C\delta \geq \frac{v_-}{2} > 0 \quad \text{for small } \delta > 0. \tag{3.37}$$

Since $p' < 0$, we have

$$C_2 \varphi_x^2 \leq q \leq C_1 \varphi_x^2 \quad \text{for some } C_1 \geq C_2 > 0. \tag{3.38}$$

It follows from Lemma 2.3 and (3.3) that

$$\begin{aligned}
& |\varphi_t(x_2(t) - 0, t)| + |\varphi_t(x_1(t), t)| \\
& \leq C e^{-\beta_{14} t}, \quad t \geq T_1, \quad \text{for some } \beta_{14} > 0.
\end{aligned} \tag{3.39}$$

From (3.9) and (3.22), it can be shown that

$$|\varphi(x, t)| \leq C\delta t, \quad |\varphi_t(x, t)| \leq C, \quad t \geq T_1, \quad x_1(t) \leq x \leq x_2(t). \tag{3.40}$$

Turning to the second term on the left-hand side of (3.35) and using integration by parts, it reads

$$\begin{aligned}
 & \int_{T_1}^t \int_{x_1(\tau)}^{x_2(\tau)} [p(v^* + m) - p(v^* + m + \varphi_x)] \varphi_{xt}(x, \tau) dx d\tau \\
 &= \int_{T_1}^t \int_{x_1(\tau)}^{x_2(\tau)} q_t dx d\tau \\
 &\quad - \int_{T_1}^t \int_{x_1(\tau)}^{x_2(\tau)} \left\{ \int_0^{\varphi_x} [p'(v^* + m) - p'(v^* + m + y)] dy \right\} (v_t^* + m_t)(x, \tau) dx d\tau \\
 &= \int_{x_1(t)}^{x_2(t)} q(x, t) dx - \int_{x_1(T_1)}^{x_2(T_1)} q(x, T_1) dx \\
 &\quad + \int_{T_1}^t [q(x_1(\tau), \tau) \dot{x}_1(\tau) - q(x_2(\tau) - o, \tau) \dot{x}_2(\tau)] d\tau \\
 &\quad - \int_{T_1}^t \int_{x_1(\tau)}^{x_2(\tau)} \left\{ \int_0^{\varphi_x} [p'(v^* + m) - p'(v^* + m + y)] dy (v_t^* + m_t) \right\} (x, \tau) dx d\tau.
 \end{aligned} \tag{3.41}$$

In view of (3.38) and (3.13)–(3.16), it follows that

$$|q(x_1(\tau), \tau)| + |q(x_2(\tau) - o, \tau)| \leq C\delta^2 e^{-\beta_{15}\tau}, \quad \tau \geq T_1, \quad \text{for some } \beta_{15} > 0. \tag{3.42}$$

Due to (3.3), (3.19), and (3.20), the following holds:

$$\begin{aligned}
 & \left| \int_{T_1}^t \int_{x_1(\tau)}^{x_2(\tau)} \left\{ \int_0^{\varphi_x} [p'(v^* + m) - p'(v^* + m + y)] dy (v_t^* + m_t) \right\} (x, \tau) dx d\tau \right| \\
 & \leq C\delta \int_{T_1}^t \int_{x_1(\tau)}^{x_2(\tau)} \varphi_x^2(x, \tau) dx d\tau.
 \end{aligned} \tag{3.43}$$

Conditions (3.24) and (3.34)–(3.43), with the help of Cauchy's inequality, yield that

$$\int_{x_1(t)}^{x_2(t)} [\varphi^2 + \varphi_x^2 + \varphi_t^2](x, t) dx + \int_{T_1}^t \int_{x_1(\tau)}^{x_2(\tau)} (\varphi_t^2 + \varphi_x^2)(x, \tau) dx d\tau \leq C, \tag{3.44}$$

provided δ is suitably small.

Differentiate (3.23) with respect to t and multiply it by φ_{tt} . Differentiate (3.23) with respect to x and multiply it by φ_{xt} . Using a treatment similar to that used above to these two equalities respectively and taking account of (3.3), (3.9), Lemma 3.1, and Lemma 2.3, it turns out that

$$\int_{x_1(t)}^{x_2(t)} [\varphi_{tt}^2 + \varphi_{xt}^2 + \varphi_{xx}^2](x, t) dx + \int_{T_1}^t \int_{x_1(\tau)}^{x_2(\tau)} (\varphi_{tt}^2 + \varphi_{xt}^2)(x, \tau) dx d\tau \leq C. \tag{3.45}$$

Define

$$A(t) =: \int_{x_1(t)}^{x_2(t)} (\varphi_t^2 + \varphi_x^2)(x, t) dx.$$

It can be shown by (3.44), (3.45), and Lemma 2.1 that

$$A(t) \in L^1(T_1, +\infty) \tag{3.46}$$

and

$$\int_{T_1}^{+\infty} \left| \frac{dA(t)}{dt} \right| dt \leq C. \quad (3.47)$$

Conditions (3.46) and (3.47) imply that

$$A(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (3.48)$$

It is known that

$$|\varphi_x^2(x, t)| \leq 2 \left(\int_{x_1(t)}^{x_2(t)} \varphi_x^2 dx \right)^{1/2} \cdot \left(\int_{x_1(t)}^{x_2(t)} \varphi_{xx}^2 dx \right)^{1/2} + |\varphi_x^2(x_1(t), t)|. \quad (3.49)$$

Thus, (3.45), (3.48), and (3.49), combined with (3.16) and the fact that $m(x_1(t), t) \equiv 0$ for $t > T_1$, yield

$$\sup_{x \in [x_1(t), x_2(t))} |\varphi_x(x, t)| \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Similarly, it can be proved that

$$\sup_{x \in [x_1(t), x_2(t))} |\varphi_t(x, t)| \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

We now turn to the estimates of decay rates. Instead of integrating (3.23) $\cdot \varphi_t$ over $\{(x, \tau) \mid x_1(\tau) \leq x \leq x_2(\tau), T_1 \leq \tau \leq t\}$ to obtain (3.35) we integrate over $\{(x, \tau) \mid x_1(\tau) \leq x \leq x_2(\tau), t \leq \tau \leq T, t \geq T_1\}$ and use a similar approach as in (3.35)–(3.43) to obtain that

$$\begin{aligned} & \int_{x_1(T)}^{x_2(T)} \left(\frac{1}{2} \varphi_t^2 + q \right) (x, T) dx \\ & \leq \int_{x_1(t)}^{x_2(t)} \left(\frac{1}{2} \varphi_t^2 + q \right) (x, t) dx \\ & \quad + C \int_t^T \int_{x_1(\tau)}^{x_2(\tau)} \{ \varphi_x^2(|v_t^*| + |m_t|) + p(v^*)_{xt}^2 + [(p(v^*) - p(v^* + m))_x]^2 \} (x, \tau) dx d\tau \\ & \quad + C \int_t^T \tau e^{-\beta\tau} d\tau, \quad \text{for some } \beta > 0. \end{aligned}$$

We then integrate the above for $T_1 \leq t \leq T$ to get

$$\begin{aligned} & (T - T_1) \int_{x_1(T)}^{x_2(T)} \left(\frac{1}{2} \varphi_t^2 + q \right) (x, T) dx \\ & \leq \int_{T_1}^T \int_{x_1(t)}^{x_2(t)} \left(\frac{1}{2} \varphi_t^2 + q \right) (x, t) dx dt + C \int_{T_1}^T \left(\int_t^T l(\tau) d\tau \right) dt \\ & \quad + C \int_{T_1}^T \left(\int_t^T \tau e^{-\beta\tau} d\tau \right) dt \end{aligned} \quad (3.50)$$

where

$$l(\tau) = \int_{x_1(\tau)}^{x_2(\tau)} \{ \varphi_x^2(|v_t^*| + |m_t|) + p(v^*)_{xt}^2 + [(p(v^*) - p(v^* + m))_x]^2 \} (x, \tau) dx.$$

Integrating by parts gives

$$\int_{T_1}^T \left(\int_t^T l(\tau) d\tau \right) dt = \int_{T_1}^T (t - T_1) l(t) dt. \quad (3.51)$$

From (3.3)–(3.6), the definition of m , and (3.44), noting that $v_t^* = -\frac{1}{2}\eta v_\eta^*(t+1)^{-1}$, we can show

$$\int_{T_1}^T (t - T_1) l(t) dt \leq C. \quad (3.52)$$

Similarly to (3.51), we have

$$\int_{T_1}^T \left(\int_t^T \tau e^{-\beta\tau} d\tau \right) dt = \int_{T_1}^T t(t - T_1) e^{-\beta t} dt \leq C. \quad (3.53)$$

Conditions (3.44) and (3.50)–(3.53) imply that

$$\int_{x_1(T)}^{x_2(T)} \left(\frac{1}{2} \varphi_t^2 + q \right) (x, T) dx \leq \frac{C}{T - T_1}. \quad (3.54)$$

With a similar approach as above we can arrive at

$$\int_{x_1(\tau)}^{x_2(\tau)} [\varphi_{tx}^2 + \varphi_{xx}^2] (x, T) dx \leq \frac{C}{T - T_1}. \quad (3.55)$$

From the arbitrariness of T , and (3.49), it can be shown that

$$\sup_{x \in [x_1(t), x_2(t))} [|\varphi_x(x, t)| + |\varphi_t(x, t)|] \leq C(1+t)^{-1/2}, \quad (3.56)$$

which completes the proof of (3.33). Theorem 3.2 then follows.

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