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LARGE-TIME BEHAVIOR OF SOLUTIONS TO THE EQUATIONS OF ONE-DIMENSIONAL NONLINEAR THERMOVISCOELASTICITY

By

L. HSIAO AND T. LUO

Academia Sinica, Institute of Mathematics, Beijing 100080, China

1. Introduction. This paper is concerned with the investigation of large-time behavior of globally defined smooth solutions of the initial-boundary value problem for the system in one-dimensional nonlinear thermoviscoelasticity, namely,

$$\begin{cases} u_t - v_x = 0, \\ v_t - \sigma_x = 0, \\ [e + \frac{1}{2}v^2]_t - [\sigma v]_x + q_x = 0, \end{cases} \tag{1.1}$$

which is the referential (Lagrangian) description of the balance laws of mass, momentum and energy for one-dimensional materials with reference density $\rho_0 = 1$ and is supplemented with the second law of thermodynamics expressed through the Clausius Duhem inequality

$$\eta_t + \left(\frac{q}{\theta}\right)_x \geq 0 \tag{1.2}$$

where $u, v, e, \sigma, \eta, \theta$, and q denote specific volume (deformation gradient), velocity, internal energy, stress, specific entropy, temperature, and heat flux, respectively, while e, σ, η, and q are given by so-called constitutive relations for the thermoviscoelastic materials to be considered. The quantities u, θ, and e may only take positive values.

We consider here a body with reference configuration the interval $[0, 1]$ whose endpoints are stress-free and thermally insulated, that is,

$$\begin{cases} \sigma(0,t) = \sigma(1,t) = 0, \\ q(0,t) = q(1,t) = 0, \end{cases} \quad t \geq 0, \tag{1.3}$$

and we prescribe the initial values of u, v, and θ as follows:

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad \theta(x,0) = \theta_0(x), \quad 0 \leq x \leq 1. \tag{1.4}$$

For the material of ideal gas, in which the constitutive relations take the form

$$e = c\theta, \qquad \sigma = -R\frac{\theta}{u} + \mu\frac{v_x}{u}, \qquad q = -K\frac{\theta_x}{u} \tag{1.5}$$

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where c, R, μ, and K are positive constants, it has been proved in [NA] that the solution (u^*, v^*, θ^*) to the problem (1.1), (1.3), and (1.4) satisfies

$$u^*(x,t) \geq c^* \log(1+t), \quad c^* > 0.$$

On the other hand, however, totally different phenomena may occur on large-time behavior of solutions for other kinds of constitutive relations to be considered. In the case of isothermal viscoelasticity (i.e., $\theta \equiv$ constant), the solution may approach a unique state exponentially fast as shown by Greenberg and MacCamy in [GM], or phase transition may take place as discovered by Andrews and Ball in [AB] with nonmonotone pressure, who prove that the large-time behavior of strain is described by a Yang measure whose support is confined in the set of zeroes of pressure. Our goal here is to extend the analysis to the nonisothermal case—thermoviscoelastic materials.

For simplicity, we consider in the present paper the kind of solid-like materials with the following constitutive relations:

$$e = C_V\theta, \qquad \sigma = -f(u)\theta + \hat{\mu}(u)v_x, \qquad q = -k\frac{\theta_x}{u}, \tag{1.6}$$

where C_V and k are positive constants, and $f(u)$ is twice continuously differentiable for $u > 0$ such that

$$\begin{aligned} f(u) &\geq 0, \quad 0 < u < \tilde{u}, \\ f(u) &\leq 0, \quad \widetilde{U} < u < +\infty, \end{aligned} \tag{1.7}$$

for some fixed $0 < \tilde{u} \leq \widetilde{U} < +\infty$, and the viscosity $\hat{\mu}(u)u$ is uniformly positive, that is,

$$\hat{\mu}(u)u \geq \mu_0 > 0, \quad 0 < u < +\infty. \tag{1.8}$$

REMARK 1.0. It is known that for rubber a good model for pressure is of the form

$$\hat{p}(u,\theta) = -\gamma\theta\left(u - \frac{1}{u^2}\right), \quad \gamma \text{ is a positive constant,}$$

namely, $f(u) = -\gamma(u - \frac{1}{u^2})$, which satisfies (1.7) with $\tilde{u} = \widetilde{U} = 1$.

We turn to assumptions on initial data now. Without loss of generality, by superimposing a trivial rigid motion, we normalize the initial velocity so that

$$\int_0^1 v_0(x)\,dx = 0. \tag{1.9}$$

Furthermore, we assume that the initial data are compatible with the boundary conditions (1.3).

The global existence of (1.1), (1.3), and (1.4), under the assumptions of (1.6)–(1.9), can be established by the approach in [DH] and [DA] where the solid-like material with more general constitutive relations than (1.6) and (1.7) is concerned. Namely, assume $u_0(x), u_0'(x), v_0(x), v_0'(x), v_0''(x), \theta_0(x), \theta_0'(x), \theta_0''(x)$ are in $C^\alpha[0,1]$ for some $0 < \alpha < 1$ and $u_0(x) > 0$, $\theta_0(x) > 0$, $0 \leq x \leq 1$; under the assumptions (1.6)–(1.9), there exists a unique solution $\{u(x,t), v(x,t), \theta(x,t)\}$ on $[0,1] \times [0,\infty)$ such that for every $T > 0$, the functions $u, u_x, u_t, u_{xt}, v, v_x, v_t, v_{xx}, \theta, \theta_x, \theta_t, \theta_{xx}$ are all in $C^{\alpha,\frac{\alpha}{2}}(Q_T)$ and $u_{tt}, v_{xt}, \theta_{xt}$ are in $L^2(Q_T)$, $Q_T = [0,1] \times [0,T]$. Moreover, $\theta(x,t) > 0$, $0 < \overline{u} < u(x,t) < \overline{U}$, for

$0 \le x \le 1, 0 \le t < +\infty$, where $\overline{u}$ and $\overline{U}$ are positive constants depending only on the initial data but not on T, and $0 < \overline{u} \le \tilde{u} \le \widetilde{U} \le \overline{U}$.

The following results on large-time behavior of solutions have been established in the present paper.

THEOREM 1.1. Assume that (1.6)–(1.9) are satisfied. Let $\{u(x,t), v(x,t), \theta(x,t)\}$, $(x,t) \in [0,1] \times [0,\infty)$, be the globally defined smooth solution of the problem (1.1)–(1.4). Then

I.

$$\|\hat{p}(u,\theta)(\cdot,t)\|_{L^1[0,1]} = \|f(u)\theta(\cdot,t)\|_{L^1[0,1]} \to 0 \quad \text{as } t \to +\infty,$$
$$\|f(u)(\cdot,t)\|_{L^2[0,1]} \to 0 \quad \text{as } t \to +\infty,$$
$$\|v(\cdot,t)\|_{L^2[0,1]} \to 0 \quad \text{as } t \to +\infty,$$

and

$$\overline{\theta}(t) \stackrel{\text{def}}{=} \int_0^1 \theta(x,t)\,dx \to \frac{E_1}{C_V} \quad \text{as } t \to +\infty$$

where $E_1 = \int_0^1 [C_V\theta_0 + \frac{1}{2}v_0^2]_{(x)}\,dx$.

II. There exists a family of probability measure $\{\nu_x\}_{x\in[0,1]}$ on $\mathbb{R}$ (depending measurably on x) with $\operatorname{supp}\nu_x \subset K = \{z : f(z) = 0\}$ such that if $\Phi \in C(\mathbb{R})$ and

$$g_\Phi(x) \stackrel{\text{def}}{=} \langle \nu_x, \Phi\rangle \quad \text{a.e.},$$

then $\Phi(u(\cdot,t)) \stackrel{*}{\rightharpoonup} g_\Phi(\cdot)$ in $L^\infty[0,1]$ as $t \to +\infty$.

REMARK 1.2. Theorem 1.1 extends the phase transition results in [AB] to nonisothermal cases.

COROLLARY 1.3. Suppose the equation $f(z) = 0$ possesses only one root $z = z_1$. Then

$$u(\cdot,t) \to z_1 \quad \text{strongly in } L^q(0,1) \quad \text{as } t \to \infty$$

for all q, $1 \le q < +\infty$, provided the conditions (1.6)–(1.9) hold.

COROLLARY 1.4. Suppose the equation $f(z) = 0$ has exactly m roots, $z_1, z_2, \dots, z_m$, $m > 1$. Then there exist nonnegative functions $\mu_i \in L^\infty[0,1]$, $1 \le i \le m$, such that

$$\Phi(u(\cdot,t)) \stackrel{*}{\rightharpoonup} \sum_{i=1}^m \Phi(z_i)\mu_i(\cdot) \quad \text{in } L^\infty[0,1], \quad \text{as } t \to +\infty,$$

for any $\Phi \in C(\mathbb{R})$.

Furthermore, $\sum_{i=1}^m \mu_i(x) = 1$, a.e.

If $f(u)$ is strictly monotone decreasing, namely,

$$f'(u) < 0 \quad \text{for } u \in [\overline{u}, \overline{U}], \tag{1.10}$$

it follows from (1.7) that there exists a unique $\hat{u} \in [\tilde{u}, \widetilde{U}]$ such that $f(\hat{u}) = 0$. We have further results then in the next theorem.

THEOREM 1.5. Assume that (1.6)–(1.10) hold. Then there are positive constants $\beta, \widehat{T}$, and A, independent of t, such that

$$\|u(\cdot,t)-\hat{u}\|_{H^1(0,1)}+\|v(\cdot,t)\|_{H^1(0,1)}+\left\|\theta(\cdot,t)-\frac{E_1}{C_V}\right\|_{H^1(0,1)}\le Ae^{-\beta t}\quad \text{for } t\ge \widehat{T}.$$

Theorem 1.5 generalizes the results obtained in [GM] which discusses the case of isothermal viscoelasticity.

Section 2 and Section 3 are devoted to proving Theorem 1.1 and Theorem 1.5, respectively.

2. The proof of Theorem 1.1. From now on, $\{u(x,t), v(x,t), \theta(x,t)\}$ will denote the solution described in the global existence theorem.

It is known from [DA] that

$$0<\overline{u}\le u(x,t)\le \overline{U},\quad \theta(x,t)>0,\quad x\in[0,1],\quad t\in[0,+\infty) \tag{2.1}$$

where $\overline{u}$ and $\overline{U}$ are positive constants, independent of t, such that $0<\overline{u}\le\tilde{u}<\widetilde{U}\le\overline{U}$. (2.1) and (1.8) yield

$$0<\mu_1\le\hat{\mu}(u(x,t))\le\mu_2,\quad x\in[0,1],\quad t\in[0,+\infty) \tag{2.2}$$

where μ_1 and μ_2 are positive constants, independent of t.

In the sequel, Λ will denote a generic constant, independent of t.

Integrating (1.1) over $[0,1]\times[0,t]$ and using the boundary condition (1.3) we obtain the conservation laws of total momentum and energy:

$$\int_0^1 v(x,t)\,dx=\int_0^1 v_0(x)\,dx=0,\quad 0\le t<+\infty, \tag{2.3}$$

$$\int_0^1\left[C_V\theta+\tfrac{1}{2}v^2\right](x,t)\,dx=\int_0^1\left[C_V\theta_0+\tfrac{1}{2}v_0^2\right](x)\,dx=E_1. \tag{2.4}$$

LEMMA 2.1.

$$\int_0^t\int_0^1\left[\frac{\mu_1 v_x^2}{\theta}+\frac{k\theta_x^2}{\theta^2}\right](x,\tau)\,dx\,d\tau\le\Lambda,\quad t\in[0,+\infty). \tag{2.5}$$

Proof. Substituting σ from (1.6), we may write $(1.1)_2$ in the form

$$v_t+[f(u)\theta]_x=[\hat{\mu}(u)v_x]_x \tag{2.6}$$

while combining $(1.1)_3$ with $(1.1)_2$ and using (1.6) we obtain

$$C_V\theta_t+f(u)\theta v_x-\hat{\mu}(u)v_x^2-k\theta_{xx}=0. \tag{2.7}$$

Multiplying (2.7) by θ^{-1} and integrating over $[0,1]\times[0,t]$, with the help of (1.3) and $(1.1)_1$, one obtains

$$\begin{aligned}&\int_0^t\int_0^1\left[\frac{\hat{\mu}(u)v_x^2}{\theta}+\frac{k\theta_x^2}{\theta^2}\right](x,\tau)\,dx\,d\tau\\ &\quad=C_V\left[\int_0^1\log\theta(x,t)\,dx-\int_0^1\log\theta(x,0)\,dx\right]+\int_0^1 G(u)(x,t)\,dx-\int_0^1 G(u)(x,0)\,dx\end{aligned}$$

where $G(u) = \int_{\bar{u}}^{u} f(\xi)d\xi$. This, with (2.1), (2.2), and the inequality $\log\theta \le \theta - 1$, for $\theta > 0$, implies (2.5).

Due to (2.3) and the mean value theorem, there exists a $y(t) \in [0,1]$ for every $t \ge 0$ such that

$$v(y(t),t) = 0. \tag{2.8}$$

Thus

$$|v(x,t)| = \left|\int_{y(t)}^{x} v_x(\xi,t)d\xi\right| \le \left[\int_0^1 \theta(x,t)\,dx\right]^{1/2} \left[\int_0^1 \frac{v_x^2}{\theta}(x,t)\,dx\right]^{1/2}, \tag{2.9}$$

which, combined with (2.4) and (2.5), yields

$$\int_0^t \max_{[0,1]} v^2(\cdot,\tau)\,d\tau \le \Lambda, \quad t \in [0,+\infty). \tag{2.10}$$

LEMMA 2.2.

$$\int_0^1 (v^4+\theta^2)(x,t)\,dx + \int_0^t \int_0^1 [\theta_x^2 + v^2 v_x^2]\,dx\,d\tau \le \Lambda, \quad t \in [0,+\infty). \tag{2.11}$$

Proof. Multiply $(1.1)_3$ with $(C_V\theta + \frac{v^2}{2})$ and integrate over $[0,1]\times[0,t]$. With the help of (1.3), (1.6), (2.1), (2.2), and Young's inequality, we arrive at

$$\begin{aligned}\frac{1}{2}\int_0^1 \left[C_V\theta + \frac{v^2}{2}\right]^2 (x,t)\,dx + \mu_1 \int_0^t\int_0^1 v^2 v_x^2(x,\tau)\,dx\,d\tau + \frac{C_V k}{2}\int_0^t\int_0^1 \theta_x^2(x,\tau)\,dx\,d\tau \\ \le \Lambda + \Lambda\int_0^t \max_{[0,1]} v^2(\cdot,\tau)\int_0^1 \theta^2(x,\tau)\,dx\,d\tau + \Lambda\int_0^t\int_0^1 v^2 v_x^2(x,\tau)\,dx\,d\tau.\end{aligned} \tag{2.12}$$

To estimate the term $\int_0^t\int_0^1 v^2 v_x^2\,dx\,d\tau$, we multiply $(1.1)_2$ by v^3, integrate the resulting equation over $[0,1]\times[0,t]$, and use the boundary conditions (1.3), (2.1), (2.2) and the Cauchy inequality. It then follows that

$$\begin{aligned}\frac{1}{4}\int_0^1 v^4(x,t)\,dx + 2\mu_1\int_0^t\int_0^1 v^2 v_x^2(x,\tau)\,dx\,d\tau \\ \le \Lambda + \Lambda\int_0^t \max_{[0,1]} v^2(\cdot,\tau)\int_0^1 \theta^2(x,\tau)\,dx\,d\tau.\end{aligned} \tag{2.13}$$

By using the Cauchy inequality with the term $(C_V\theta + \frac{v^2}{2})^2$ in (2.12), we obtain

$$\begin{aligned}\int_0^1 \theta^2(x,t)\,dx + \int_0^t\int_0^1 v^2 v_x^2(x,\tau)\,dx\,d\tau + \int_0^t\int_0^1 \theta_x^2(x,\tau)\,dx\,d\tau \\ \le \Lambda + \Lambda\int_0^t\int_0^1 v^2 v_x^2(x,\tau)\,dx\,d\tau + \Lambda\int_0^t \max_{[0,1]} v^2(\cdot,\tau)\int_0^1 \theta^2(\cdot,\tau)\,dx\,d\tau \\ + \Lambda\int_0^1 v^4(x,t)\,dx.\end{aligned} \tag{2.14}$$

Multiplying (2.13) with a suitably large positive constant, and combining with (2.14), we get

$$\int_0^1 \theta^2(x,t)\,dx + \int_0^1 v^4(x,t)\,dx + \int_0^t\int_0^1 v^2 v_x^2(x,\tau)\,dx\,d\tau + \int_0^t\int_0^1 \theta_x^2(x,\tau)\,dx\,d\tau$$
$$\leq \Lambda + \Lambda \int_0^t \max_{[0,1]} v^2(\cdot,\tau) \int_0^1 \theta^2(x,\tau)\,dx\,d\tau. \quad (2.15)$$

Applying Gronwall's inequality to (2.15) and using (2.10), one arrives at (2.11).

LEMMA 2.3.

$$\int_0^t\int_0^1 v_x^2(x,\tau)\,dx\,d\tau \leq \Lambda. \quad (2.16)$$

Proof. Multiplying $(1.1)_2$ by v and integrating over $[0,1]\times[0,t]$, it follows, with the help of (1.3), (2.1), and (2.2), that

$$\frac{1}{2}\int_0^1 v^2(x,t)\,dx + \mu_1 \int_0^t\int_0^1 v_x^2(x,\tau)\,dx\,d\tau \leq \Lambda + \int_0^t\int_0^1 f(u)\theta v_x\,dx\,d\tau. \quad (2.17)$$

Due to $(1.1)_1$ and (2.1),

$$\begin{aligned}
&\int_0^t\int_0^1 f(u)\theta v_x\,dx\,d\tau \\
&\quad = \int_0^t\int_0^1 f(u)(\theta-\overline{\theta})v_x(x,\tau)\,dx\,d\tau + \int_0^t\int_0^1 f(u)\overline{\theta}v_x(x,\tau)\,dx\,d\tau \\
&\quad \leq \frac{\mu_1}{4}\int_0^t\int_0^1 v_x^2(x,\tau)\,dx\,d\tau + \Lambda\int_0^t\int_0^1 (\theta-\overline{\theta})^2(x,\tau)\,dx\,d\tau \\
&\qquad + \int_0^t\int_0^1 f(u)\overline{\theta}u_t(x,\tau)\,dx\,d\tau
\end{aligned} \quad (2.18)$$

where $\overline{\theta}(t) = \int_0^1 \theta(x,t)\,dx$, $t\in[0,+\infty)$.

By the mean value theorem, there exists a $z(t)\in[0,1]$ such that $\overline{\theta}(t) = \theta(z(t),t)$. Thus

$$|\theta-\overline{\theta}|(x,\tau) = \left|\int_{z(\tau)}^x \theta_x(\xi,\tau)\,d\xi\right| \leq \left[\int_0^1 \theta_x^2(x,\tau)\,dx\right]^{1/2}, \quad \tau\in[0,+\infty). \quad (2.19)$$

Integrating $(1.1)_3$ over $[0,1]$ and using (1.3) and (1.6), we arrive at

$$C_V\left(\int_0^1 \theta(x,\tau)\,dx\right)_t = -\left(\int_0^1 \frac{1}{2}v^2(x,\tau)\,dx\right)_t, \quad \tau\in[0,+\infty). \quad (2.20)$$

Namely,

$$\overline{\theta}_t(\tau) = -\frac{1}{C_V}\left[\int_0^1 \frac{1}{2}v^2(x,\tau)\,dx\right]_t, \quad \tau\in[0,+\infty), \quad (2.21)$$

which, together with (2.4), implies

$$\overline{\theta}(\tau) = \frac{1}{C_V}\left[E_1 - \int_0^1 \frac{1}{2}v^2(x,\tau)\,dx\right], \quad t \in [0,+\infty). \tag{2.22}$$

In view of $(1.1)_1$, (2.1), (2.4), (2.10), (2.21), and (2.22), it follows, upon integrating by parts and using the Cauchy inequality, that

$$\begin{aligned}
&\int_0^t \int_0^1 [f(u)\overline{\theta}u_t](x,\tau)\,dx\,d\tau \\
&\quad\le \Lambda - \int_0^t \int_0^1 \left\{ f'(u)uu_t \left[\frac{1}{C_V}\left(E_1 - \int_0^1 \frac{1}{2}v^2(s,\tau)\,ds\right)\right]\right\}(x,\tau)\,dx\,d\tau \\
&\qquad + \int_0^t \int_0^1 \left\{ f(u)u \left[\frac{1}{C_V}\left(\int_0^1 \frac{1}{2}v^2(x,\tau)\,ds\right)\right]_\tau \right\}(x,\tau)\,dx\,d\tau \\
&\quad = \Lambda - \frac{E_1}{C_V}\left[\int_0^1 \lambda(u)(x,t)\,dx - \int_0^1 \lambda(u)(x,0)\,dx\right] \\
&\qquad + \int_0^t \int_0^1 \left[f'(u)uv_x \left(\int_0^1 \frac{1}{2}v^2(s,\tau)\,ds\right)\right](x,\tau)\,dx\,d\tau \\
&\qquad + \frac{1}{C_V}\left\{\int_0^1 \left[f(u)u\left(\int_0^1 \frac{1}{2}v^2(s,t)\,ds\right)\right](x,t)\,dx \right. \\
&\qquad \left. - \int_0^1 \left[f(u)u\left(\int_0^1 \frac{1}{2}v^2(s,0)\,ds\right)\right](x,0)\,dx\right\} \\
&\qquad - \frac{1}{C_V}\int_0^t \int_0^1 \left\{[f'(u)u + f(u)]v_x \left(\int_0^1 \frac{1}{2}v^2(s,\tau)\,ds\right)\right\}(x,\tau)\,dx\,d\tau \\
&\quad \le \Lambda + \frac{\mu_1}{4}\int_0^t \int_0^1 v_x^2(x,\tau)\,dx\,d\tau + \Lambda \int_0^t \int_0^1 v^4(x,\tau)\,dx\,d\tau + \frac{1}{4}\int_0^1 v^2(x,t)\,dx \\
&\quad \le \Lambda + \frac{\mu_1}{4}\int_0^t \int_0^1 v_x^2(x,\tau)\,dx\,d\tau + \frac{1}{4}\int_0^1 v^2(x,\tau)\,dx
\end{aligned} \tag{2.23}$$

where

$$\lambda(u)(x,t) = \int_{\overline{u}}^{u(x,t)} f'(\xi)\xi\,d\xi.$$

(2.16) then follows from (2.15), (2.17)–(2.19), and (2.23).

LEMMA 2.4.

$$\int_0^t \int_0^1 [f(u)\theta]^2(x,\tau)\,dx\,d\tau \le \Lambda. \tag{2.24}$$

Proof. By integrating $(1.1)_2$ over $[0,x]$ for any $x \in [0,1]$ and using the boundary condition (1.3), it follows that

$$[f(u)\theta](x,t) = [\hat{\mu}(u)v_x](x,t) - \left(\int_0^x v(y,t)\,dy\right)_t, \quad x \in [0,1],\ t \in [0,+\infty). \tag{2.25}$$

Multiplying (2.25) by $f(u)\theta$ and integrating it over $[0,1]\times[0,t]$, we arrive at

$$\begin{aligned}\int_0^t\int_0^1 &[f(u)\theta]^2(x,\tau)\,dx\,d\tau \\ &= \int_0^t\int_0^1 [\hat{\mu}(u)v_x f(u)\theta](x,\tau)\,dx\,d\tau \\ &\quad - \int_0^t\int_0^1 \left\{\left[\int_0^x v(y,\tau)\,dy\right]_\tau f(u)(\theta-\overline{\theta})\right\}(x,\tau)\,dx\,d\tau \\ &\quad + \int_0^t\int_0^1 \left\{\left[\int_0^x v(y,\tau)\,dy\right]_\tau f(u)\overline{\theta}\right\}(x,\tau)\,dx\,d\tau.\end{aligned} \tag{2.26}$$

We estimate each term in (2.26) separately.

In view of (2.1), (2.16), and Cauchy's inequality,

$$\begin{aligned}&\left|\int_0^t\int_0^1 [\hat{\mu}(u)v_x f(u)\theta](x,\tau)\,dx\,d\tau\right| \\ &\qquad \le \frac{1}{4}\int_0^t\int_0^1 [f(u)\theta]^2(x,\tau)\,dx\,d\tau + \Lambda, \quad t \in [0,+\infty).\end{aligned} \tag{2.27}$$

By (2.1), (2.11), (2.16), (2.19), (2.25), and the Cauchy inequality,

$$\begin{aligned}&\left|\int_0^t\int_0^1 \left\{\left[\int_0^x v(y,\tau)\,dy\right]_\tau f(u)(\theta-\overline{\theta})\right\}(x,\tau)\,dx\,d\tau\right| \\ &\quad \le \frac{1}{16}\int_0^t\int_0^1 \left[\left(\int_0^x v(y,\tau)\,dy\right)_\tau\right]^2(x,\tau)\,dx\,d\tau + \Lambda\int_0^t\int_0^1 \theta_x^2(x,\tau)\,dx\,d\tau \\ &\quad \le \frac{1}{8}\int_0^t\int_0^1 [f(u)\theta]^2(x,\tau)\,dx\,d\tau + \Lambda\int_0^t\int_0^1 (v_x^2+\theta_x^2)(x,\tau)\,dx\,d\tau \\ &\quad \le \frac{1}{8}\int_0^t\int_0^1 [f(u)\theta]^2(x,\tau)\,dx\,d\tau + \Lambda, \quad t \in [0,+\infty).\end{aligned} \tag{2.28}$$

Integrating by parts and using $(1.1)_1$, (2.4), (2.10), (2.16), (2.21), (2.22), (2.25), Hölder's inequality, and Cauchy's inequality, it follows that

$$\begin{aligned}
&\left|\int_0^t\int_0^1\left\{\left[\int_0^x v(y,\tau)\,dy\right]_\tau f(u)\overline{\theta}\right\}(x,\tau)\,dx\,d\tau\right| \\
&\quad\le \Lambda+\Lambda\int_0^t\int_0^1\left[v_x^2+\int_0^1 v^2(s,\tau)\,ds\right](x,\tau)\,dx\,d\tau \\
&\qquad+\frac{1}{C_V}\left|\int_0^t\int_0^1\left\{\left[\int_0^x v(y,\tau)\,dy\right]f(u)\left[\int_0^1\frac{1}{2}v^2(s,\tau)\,ds\right]_\tau\right\}(x,\tau)\,dx\,d\tau\right| \\
&\quad\le \Lambda+\frac{1}{C_V}\left|\int_0^t\int_0^1\left\{\left[\left(\int_0^x v(y,\tau)\,dy\right)_\tau f(u)\right.\right.\right. \\
&\qquad\qquad\left.\left.+\left(\int_0^x v(y,\tau)\,dy\right)f'(u)v_x\right]\cdot\int_0^1\frac{1}{2}v^2(s,\tau)\,ds\right\}(x,\tau)\,dx \\
&\quad\le \Lambda+\frac{1}{16}\int_0^t\int_0^1\left\{\left[\int_0^x v(y,\tau)\,dy\right]_\tau\right\}^2(x,\tau)\,dx\,d\tau+\Lambda\int_0^t\int_0^1[v_x^2+v^4](x,\tau)\,dx\,d\tau \\
&\quad\le \Lambda+\frac{1}{8}\int_0^t\int_0^1[f(u)\theta]^2(x,\tau)\,dx\,d\tau.
\end{aligned}\tag{2.29}$$

(2.26)–(2.29) then imply (2.24).

LEMMA 2.5.

$$\int_0^1 v^2(x,t)\,dx\to 0\quad\text{as } t\to+\infty.\tag{2.30}$$

Proof. It is clear from (2.10) that

$$\int_0^{+\infty}\int_0^1 v^2(x,t)\,dx\,dt\le\Lambda.\tag{2.31}$$

Namely,

$$\int_0^1 v^2(x,t)\,dx\in L^1([0,+\infty)).\tag{2.32}$$

Multiplying $(1.1)_2$ by v and integrating over $[0,1]$, we obtain, with the help of (1.6) and (1.3),

$$\begin{aligned}
&\left|\int_0^1(vv_t)(x,t)\,dx\right| \\
&\quad=\left|\int[(f(u)\theta-\hat{\mu}(u)v_x)v_x](x,t)\,dx\right| \\
&\quad\le\Lambda\int_0^1[(f(u)\theta)^2+v_x^2](x,t)\,dx,
\end{aligned}$$

which, combined with (2.16) and (2.24), implies

$$\int_0^{+\infty}\left|\frac{d}{dt}\int_0^1 v^2(x,t)\,dx\right|\,dx\le\Lambda.\tag{2.33}$$

(2.31) and (2.33) yield (2.30).

LEMMA 2.6.

$$\bar{\theta}(t) = \int_0^1 \theta(x,t)\,dx \to \frac{E_1}{C_V}, \quad \text{as } t \to +\infty, \tag{2.34}$$

$$\int_0^t \int_0^1 [f(u)]^2(x,\tau)\,dx\,d\tau \le \Lambda, \quad t \in [0,+\infty), \tag{2.35}$$

$$\|f(u)(\cdot,t)\|_{L^2[0,1]} \to 0, \quad \text{as } t \to +\infty, \tag{2.36}$$

$$\|\hat{p}(u,\theta)(\cdot,t)\|_{L^1[0,1]} = \|(f(u)\theta)(\cdot,t)\|_{L^1[0,1]} \to 0, \quad \text{as } t \to +\infty. \tag{2.37}$$

Proof. (2.34) follows from (2.22) and (2.30) directly. It is known from (2.34) that there exists $T_0 > 0$ such that

$$\bar{\theta}(t) \ge \frac{E_1}{2C_V} \quad \text{as } t \ge T_0,$$

which, together with (2.1), (2.11), (2.19), and (2.24), implies

$$\begin{aligned}
&\int_{T_0}^t \int_0^1 [f(u)]^2(x,\tau)\,dx\,d\tau \\
&\quad\le \frac{4C_V^2}{E_1^2} \int_{T_0}^t \int_0^1 [f(u)\bar{\theta}]^2(x,\tau)\,dx\,d\tau \\
&\quad\le \Lambda \int_{T_0}^t \int_0^1 [f(u)\theta]^2(x,\tau)\,dx\,d\tau + \Lambda \int_{T_0}^t \int_0^1 [f(u)(\theta-\bar{\theta})]^2(x,\tau)\,dx\,d\tau \\
&\quad\le \Lambda.
\end{aligned} \tag{2.38}$$

(2.38) and (2.1) now yield (2.35).

To prove (2.36), we make the following estimate by using $(1.1)_1$, (2.1), (2.16), and (2.35):

$$\begin{aligned}
&\int_0^{+\infty} \left| \frac{d}{dt} \int_0^1 [f(u)]^2(x,t)\,dx \right| dt \\
&\qquad\le \Lambda + \Lambda \int_0^{+\infty} \int_0^1 v_x^2(x,t)\,dx\,dt \le \Lambda.
\end{aligned} \tag{2.39}$$

(2.39) and (2.35) imply (2.36) directly.

$$\|(f(u)\theta)(\cdot,t)\|_{L^1[0,1]} \le \left(\int_0^1 [f(u)]^2(x,t)\,dx \right)^{1/2} \cdot \left(\int_0^1 \theta^2(x,t)\,dx \right)^{1/2}.$$

This, combined with (2.11) and (2.36), gives (2.37).

So far, part I of Theorem 1.1 has been established by the above lemmas.

Next we will employ an idea of Andrews and Ball (see [AB]) and the results obtained above to prove part II of Theorem 1.1.

Suppose $\Psi \in L^2[0,1]$ with $\Psi \ge 0$ and $\Phi \in C^2([\underline{u},\overline{U}])$ satisfying

$$\Phi'(z)f(z) \ge 0 \quad \text{for } z \in [\underline{u},\overline{U}] \tag{2.40}$$

where $\underline{u}$ and $\overline{U}$ are the lower and upper bounds of u.

Let

$$\varphi(x,t) := \int_0^x \Psi(y)\frac{\Phi'(u(y,t))}{\hat{\mu}(u(y,t))}dy \quad \text{for } x \in [0,1],\ t \geq 0.$$

Multiplying $(1.2)_2$ with φ and integrating over $[0,1] \times [0,t]$ and using the boundary condition (1.3), we get, with the help of (1.1) and integration by parts,

$$\begin{aligned}
&\int_0^t \int_0^1 \left[(f(u)\theta)\Psi(x)\frac{\Phi'(u)}{\hat{\mu}(u)}\right](x,\tau)\,dx\,d\tau \\
&\quad = \int_0^1 v(x,t)\left[\int_0^x \Psi(y)\frac{\Phi'(u(y,t))}{\hat{\mu}(u(y,t))}\,dy\right]dx - \int_0^1 v_0(x)\left[\int_0^x \Psi(y)\frac{\Phi'(u_0(y))}{\hat{\mu}(u_0(y))}\,dy\right]dx \\
&\qquad - \int_0^t \int_0^1 v(x,\tau)\left[\int_0^x \Psi(y)\left(\frac{\Phi'(u)}{\hat{\mu}(u)}\right)'(y,\tau)v_x(y,\tau)\,dy\right]dx\,d\tau \\
&\qquad + \int_0^1 \Psi(x)\Phi(u(x,t))\,dx - \int_0^1 \Psi(x)\Phi(u_0(x))\,dx
\end{aligned} \tag{2.41}$$

where $'$ denotes the differentiation with respect to u.

To show the existence of the limit of the left-hand side of (2.41) as $t \to +\infty$, we estimate each term on the right-hand side of (2.41).

For the first term, it is easy to see that

$$\begin{aligned}
&\left|\int_0^1 v(x,t)\left[\int_0^x \Psi(y)\frac{\Phi'(u(y,t))}{\hat{\mu}(u(y,t))}\,dy\right]dx\right| \\
&\quad \leq \|v(\cdot,t)\|_{L^2[0,1]} \cdot \|\Psi\|_{L^2[0,1]} \cdot \left\|\frac{\Phi'(u(\cdot,t))}{\hat{\mu}(u(\cdot,t))}\right\|_{L^2[0,1]} \\
&\quad \leq \Lambda\|v(\cdot,t)\|_{L^2[0,1]}
\end{aligned}$$

which tends to zero as $t \to +\infty$, due to (2.30).

The third term can be treated as follows:

$$\begin{aligned}
&\left|\int_0^1 v(x,t)\left[\int_0^x \Psi(y)\left[\frac{\Phi'(u)}{\hat{\mu}(u)}\right]'(y,t)v_x(y,t)\,dy\right]dx\right| \\
&\quad \leq \|v(\cdot,t)\|_{L^2}\|\Psi\|_{L^2} \cdot \left\|\left[\frac{\Phi'(u)}{\hat{\mu}(u)}\right]'(\cdot,t)\right\|_{L^\infty} \cdot \|v_x(\cdot,t)\|_{L^2} \\
&\quad \leq \Lambda[\|v(\cdot,t)\|_{L^2}^2 + \|v_x(\cdot,t)\|_{L^2}^2] \quad \text{for all } t \geq 0.
\end{aligned}$$

Therefore, the limit of the third term as $t \to +\infty$ exists by (2.10), (2.16), and the dominated convergence theorem.

It is obvious that the term $\mu_0 \int_0^1 \Psi(x)\Phi(u(x,t))\,dx$ is uniformly bounded in $t \geq 0$ since $\underline{u} \leq u(x,t) \leq \overline{U}$.

Thus, the above estimates imply that

$$\int_0^t \int_0^1 \left\{[f(u)\theta]\Psi(x)\frac{\Phi'(u)}{\hat{\mu}(u)}\right\}(x,\tau)\,dx\,d\tau \text{ is bounded uniformly in } t \geq 0.$$

This, together with (2.40), (2.2), and $\theta > 0$, yields the existence of

$$\lim_{t\to\infty}\int_0^t\int_0^1 [f(u)\theta\Psi(x)\Phi'(u)](x,\tau)\,dx\,d\tau.$$

Furthermore, the existence of

$$\lim_{t\to+\infty}\int_0^1 \Psi(x)\Phi(u(x,t))\,dx,$$

for all $\Psi \in C^2[0,1]$ with $\Psi \geq 0$, is established since each term in (2.41), apart from $\int_0^1 \Psi(x)\Phi(u(x,t))\,dx$, is either independent of t or tends to a limit as $t \to +\infty$.

Therefore, it follows that

$$\Phi(u(\cdot,t)) \rightharpoonup g_\Phi(\cdot) \quad \text{in } L^2[0,1]$$

as $t \to +\infty$ for some $g_\Phi \in L^2[0,1]$.

In view of $\|\Phi(u(\cdot,t))\|_{L^\infty} \leq \Lambda$, it can be shown that

$$g_\Phi \in L^\infty[0,1]$$

and

$$\Phi(u(\cdot,t)) \overset{*}{\rightharpoonup} g_\Phi(\cdot) \text{ in } L^\infty[0,1]. \tag{2.42}$$

Let $\Phi \in C([\underline{u},\overline{U}])$ be arbitrary and $\Psi \in L^1(0,1)$ now. It is easy to verify that

$$\lim_{t\to\infty}\int_0^1 \Psi(x)\Phi(u(x,t))\,dx$$

exists for all $\Psi \in L^1[0,1]$ and $\Phi \in C([\underline{u},\overline{u}])$, by using the same method in [AB] and the following Lemma 2.7 which can be proved by the same argument as used for Lemma 3.1 in [AB].

LEMMA 2.7. Let $f \in C(\mathbb{R})$ and let $0 < \underline{u} < \overline{U}$. Then the set

$$S = \operatorname{span}\{\Phi \in C^2([\underline{u},\overline{U}]) : \Phi'(z)f(z) \geq 0 \text{ if } z \in [\underline{u},\overline{U}]\}$$

is dense in $C([\underline{u},\overline{U}])$.

Thus, it turns out that (2.42) holds for an arbitrary $\Phi \in C([\underline{u},\overline{U}])$. The existence of probability measures ν_x follows at once from (2.42) and Theorem 5 in Tartar's paper in 1979 ([TA]). To prove that $\operatorname{supp}\nu_x \subset K = \{z : f(z) = 0\}$ a.e., it suffices to show that if Φ is zero on K then $\langle \nu_x, \Phi\rangle = 0$ a.e. But, if Φ is zero on K, then $\Phi(u(\cdot,t)) \to 0$ in measure as $t \to +\infty$ due to (2.36). Therefore, $\Phi(u(\cdot,t)) \to 0$ in $L^\infty[0,1]$ as $t \to \infty$, and hence $\langle \nu_x, \Phi\rangle = 0$, a.e., as required.

Theorem 1.1 has been proved completely now.

Corollaries 1.3 and 1.4 can be proved in the same way as in [AB].

3. The proof of Theorem 1.5. In view of (1.10), (2.1), and the smoothness of $f'(u)$, there exists a constant $b > 0$ such that

$$-f'(u) \geq b > 0 \quad \text{for } u \in [\underline{u}, \overline{U}]. \tag{3.1}$$

LEMMA 3.1. If (1.10) holds, then

$$\int_0^1 u_x^2(x,t)\,dx + \int_0^t \int_0^1 (u_x^2 + \theta u_x^2)(x,\tau)\,dx\,d\tau \leq \Lambda, \quad t \in [0,+\infty). \tag{3.2}$$

Proof. Define $M(u) = \int_{\underline{u}}^u \hat{\mu}(\xi)\,d\xi$ and consider $M(u)$ as a function of x and t. Then we may rewrite $(1.1)_2$ as

$$[v - M(u)_x]_t = [-f(u)\theta]_x.$$

Multiply the above equation by $[v - (Mu)_x]$ and then integrate over $[0,1] \times [0,t]$. We arrive at

$$\begin{aligned} &\frac{1}{2}\int_0^1 [v - M(u)_x]^2(x,t)\,dx + \int_0^t \int_0^1 \{[-f'(u)\theta\hat{\mu}(u)]u_x^2\}(x,\tau)\,dx\,d\tau \\ &\quad = \frac{1}{2}\int_0^1 [v - (Mu)_x]^2(x,0)\,dx - \int_0^t \int_0^1 [f'(u)\theta u_x v](x,\tau)\,dx\,d\tau \\ &\qquad - \int_0^t \int_0^1 [f(u)v\theta_x](x,\tau)\,dx\,d\tau + \int_0^t \int_0^1 [f(u)\hat{\mu}(u)u_x\theta_x](x,\tau)\,dx\,d\tau. \end{aligned} \tag{3.3}$$

(2.2), (2.5), (2.10), (2.11), and (3.1) then yield

$$\begin{aligned} &\frac{1}{2}\int_0^1 [v - M(u)_x]^2(x,t)\,dx + \frac{b\mu_1}{2}\int_0^t \int_0^1 [\theta u_x^2](x,\tau)\,dx\,d\tau \\ &\quad \leq \Lambda + \Lambda \int_0^t \int_0^1 \frac{\theta_x^2}{\theta}(x,\tau)\,dx\,d\tau \\ &\quad \leq \Lambda + \Lambda \int_0^t \int_0^1 \left[\theta_x^2 + \frac{\theta_x^2}{\theta^2}\right](x,\tau)\,dx\,d\tau \leq \Lambda, \end{aligned} \tag{3.4}$$

which, with the help of (2.1) and (2.4), implies

$$\int_0^1 u_x^2(x,t)\,dx + \int_0^t \int_0^1 \theta u_x^2(x,\tau)\,dx\,d\tau \leq \Lambda, \quad t \in [0,+\infty). \tag{3.5}$$

Next, it is known from (2.34) that there exists a $T_0 > 0$ such that

$$\overline{\theta}(t) = \int_0^1 \theta(x,t)\,dx = \theta(z(t),t) \geq \frac{E_1}{2C_V} > 0 \quad \text{for } t \geq T_0. \tag{3.6}$$

Then, it can be shown by the Hölder inequality that

$$\begin{aligned} \theta^{1/2}(x,t) &= \theta^{1/2}(z(t),t) + \int_{z(t)}^x \frac{\theta_x(\xi,t)}{2\theta^{1/2}(\xi,t)}\,d\xi \\ &\geq \left(\frac{E_1}{2C_V}\right)^{1/2}\left[1 - \frac{1}{2}\left(\int_0^1 \frac{\theta_x^2}{\theta^2}(x,t)\,dx\right)^{1/2}\right] \end{aligned}$$

which implies

$$\theta^{1/2}(x,t)+\left[\int_0^1 \frac{\theta_x^2}{\theta^2}(x,t)\,dx\right]^{1/2}\geq \Lambda>0,\quad \text{for } t\geq T_0.$$

By using the Cauchy inequality, it then follows that

$$\theta(x,t)+\int_0^1 \frac{\theta_x^2}{\theta^2}(x,t)\,dx\geq \Lambda>0,\quad t\in[\tau_0,+\infty).$$

Therefore,

$$u_x^2(x,t)\leq \Lambda\left\{\theta u_x^2(x,t)+u_x^2(x,t)\int_0^1 \frac{\theta_x^2}{\theta^2}(x,t)\,dx\right\},\quad t\in[T_0,+\infty).$$

This, combined with (2.5) and (3.5), implies

$$\int_{T_0}^t\int_0^1 u_x^2(x,\tau)\,dx\,d\tau\leq \Lambda,\quad t\in[\tau_0,+\infty). \tag{3.7}$$

Thus, (3.2) follows from (3.5) and (3.7).

LEMMA 3.2. If (1.10) holds, then there exists a $T^*>0$ such that

$$\int_0^1(v_x^2+\theta_x^2)(x,t)\,dx+\int_{T^*}^t\int_0^1[v_{xx}^2+\theta_t^2](x,\tau)\,dx\,d\tau\leq\Lambda\quad\text{for } t\geq T^* \tag{3.8}$$

and

$$\lim_{t\to+\infty}\int_0^1(v_x^2+\theta_x^2)(x,t)\,dx\to 0\quad\text{as } t\to+\infty. \tag{3.9}$$

Proof. Multiplying $(1.1)_2$ by $(-f(u)\theta+\hat{\mu}(u)v_x)_x$ and using $(1.1)_1$, (1.3), (2.1), (2.2), and Cauchy's inequality, we get

$$\begin{aligned}
&\int_T^t\int_0^1[\hat{\mu}(u)v_{xx}]^2(x,\tau)\,dx\,d\tau+\frac{1}{2}\int_0^1(\hat{\mu}(u)v_x^2)(x,t)\,dx\\
&\quad\leq\frac{1}{2}\int_0^1[\hat{\mu}(u)v_x^2](x,T)\,dx+\int_T^t\int_0^1\left[\frac{1}{2}\hat{\mu}'(u)v_x^3\right](x,\tau)\,dx\,d\tau\\
&\qquad+\int_T^t\int_0^1[f(u)\theta v_{xt}](x,\tau)\,dx\,d\tau+\frac{\mu_1^2}{4}\int_T^t\int_0^1 v_{xx}^2(x,\tau)\,dx\,d\tau\\
&\qquad+\Lambda\int_T^t\left[\max_{[0,1]}v_x^2(\cdot,\tau)\int_0^1 u_x^2(x,\tau)\,dx\right]d\tau+\Lambda\int_T^t\int_0^1\theta_x^2(x,\tau)\,dx\,d\tau\\
&\qquad+\Lambda\int_T^t\left[\max_{[0,1]}\theta^2(\cdot,\tau)\int_0^1 u_x^2(x,\tau)\,dx\right]d\tau\quad\text{for any } t\geq T>0.
\end{aligned}\tag{3.10}$$

To estimate the terms in (3.10), we first give an estimate on $\int_0^t\int_0^1 v_x^4(x,\tau)\,dx\,d\tau$, which plays a key role in the following estimates. Due to $W^{1,1}\hookrightarrow L^\infty$, it follows that

$$\begin{aligned}
&\int_T^t\int_0^1 v_x^4(x,\tau)\,dx\,d\tau\leq\int_T^t\max_{[0,1]}v_x^2\left(\int_0^1 v_x^2\,dx\right)d\tau\\
&\quad\leq\Lambda\sup_{\tau\in[T,t]}\int_0^1 v_x^2(x,\tau)\,dx\cdot\left[\int_T^t\int_0^1[v_x^2+v_{xx}^2]\right](x,\tau)\,dx\,d\tau\quad\text{for any } t\geq T>0.
\end{aligned}\tag{3.11}$$

Similarly, it can be shown, with the help of $W^{1,1} \hookrightarrow L^\infty$ and Cauchy's inequality, that

$$\begin{aligned}\int_T^t \max_{[0,1]} v_x^2(\cdot,\tau)\,d\tau \\ \le \Lambda(\delta)\int_T^t\int_0^1 v_x^2\,dx\,d\tau + \delta\int_T^t\int_0^1 v_{xx}^2\,dx\,d\tau \\ \text{for any } t \ge T > 0 \text{ and } \delta > 0.\end{aligned} \tag{3.12}$$

It reads from (2.19), (2.22), and (2.11) that

$$\max_{[0,1]} \theta(\cdot,t) \le \Lambda + \left(\int_0^1 \theta_x^2(x,t)\,dx\right)^{1/2} \quad \text{for } t \ge 0. \tag{3.13}$$

Moreover, (2.4) and (3.2) yield

$$\int_0^1 [\theta^2(x,t) + u_x^2(x,t)]\,dx \le \Lambda \quad \text{for } t \ge 0. \tag{3.14}$$

We turn to estimate the terms in (3.10) by using these inequalities (3.11)–(3.14) in which δ can be chosen suitably.

Using $(1.1)_1$, (2.1), (2.4), (3.12), (3.13), Cauchy's inequality, and integration by parts, we obtain

$$\begin{aligned}&\int_T^t\int_0^1 [f(u)\theta v_{xt}](x,\tau)\,dx\,d\tau \\ &= \int_0^1 [f(u)\theta v_x](x,t)\,dx - \int_0^1 [f(u)\theta v_x](x,T)\,dx \\ &\quad - \int_T^t\int_0^1 v_x[f'(u)\theta v_x + f(u)\theta_t]\,dx\,d\tau \\ &\le \frac{\mu_1}{4}\int_0^1 v_x^2(x,t)\,dx + \Lambda \max_{[0,1]}\theta(x,t)\int_0^1 |f(u)\theta|(x,t)\,dx \\ &\quad + \Lambda\int_0^1 \{[f(u)\theta]^2 + v_x^2\}(x,T)\,dx + \Lambda\int_T^t \max_{[0,1]} v_x^2\left(\int_0^1 \theta\,dx\right)d\tau \\ &\quad + \Lambda\int_T^t\int_0^1 v_x^2\,dx\,d\tau + \frac{C_V}{4}\int_T^t\int_0^1 \theta_t^2\,dx\,d\tau \\ &\le \frac{\mu_1}{4}\int_0^1 v_x^2(x,t)\,dx + \Lambda\int_0^1 [(f(u)\theta)^2 + v_x^2](x,T)\,dx + \Lambda\int_0^1 |f(u)\theta|(x,t)\,dx \\ &\quad + \frac{k}{4}\int_0^1 \theta_x^2(x,t)\,dx + \Lambda\left[\int_0^1 |f(u)\theta|(x,t)\,dx\right]^2 + \Lambda\int_T^t\int_0^1 v_x^2\,dx\,d\tau \\ &\quad + \frac{\mu_1^2}{8}\int_T^t\int_0^1 v_{xx}^2\,dx\,d\tau + \frac{C_V}{4}\int_T^t\int_0^1 \theta_t^2\,dx\,d\tau \quad \text{for any } t \ge T > 0,\end{aligned} \tag{3.15}$$

whereafter, k is the constant in (1.6).

By using (2.1), (3.11), and Cauchy's inequality, we get

$$\begin{aligned}
&\left|\int_T^t \int_0^1 \frac{1}{2}\hat{\mu}'(u) v_x^3 \, dx \, d\tau\right| \\
&\quad \leq \Lambda \int_T^t \int_0^1 v_x^2 \, dx \, d\tau + \Lambda \int_T^t \int_0^1 v_x^4 \, dx \, d\tau \\
&\quad \leq \Lambda \int_T^t \int_0^1 v_x^2 \, dx \, d\tau + \Lambda \sup_{\tau \in [T,t]} \int_0^1 v_x^2(x,\tau) \, dx \cdot \left[\int_T^t \int_0^1 (v_x^2 + v_{xx}^2) \, dx \, d\tau\right], \\
&\qquad\qquad \text{for } t \geq T > 0.
\end{aligned} \tag{3.16}$$

Similarly, the fifth and seventh terms on the right-hand side of (3.10) can be bounded by

$$\Lambda \int_T^t \int_0^1 v_x^2 \, dx \, d\tau + \frac{\mu_1^2}{8} \int_T^t \int_0^1 v_{xx}^2 \, dx \, d\tau$$

and

$$\Lambda \int_T^t \int_0^1 (u_x^2 + \theta_x^2) \, dx \, d\tau$$

by using (3.12) and (3.14), and (3.13)–(3.14), respectively.

This, together with (3.10)–(3.16) and Cauchy's inequality, implies

$$\begin{aligned}
&\frac{\mu_1^2}{2} \int_T^t \int_0^1 v_{xx}^2 \, dx \, d\tau + \frac{\mu_1}{4} \int_0^1 v_x^2(x,t) \, dx \\
&\quad \leq \Lambda \int_0^1 [(f(u)\theta)^2 + v_x^2](x,T) \, dx + \Lambda \left[\int_0^1 |f(u)\theta|(x,t) \, dx + \left(\int_0^1 |f(u)\theta|(x,t) \, dx\right)^2\right] \\
&\qquad + \Lambda \int_T^t \int_0^1 [u_x^2 + v_x^2 + \theta_x^2](x,\tau) \, dx \, d\tau \\
&\qquad + \Lambda \sup_{\tau \in [T,t]} \int_0^1 v_x^2(x,\tau) \, dx \cdot \left[\int_T^t \int_0^1 (v_x^2 + v_{xx}^2) \, dx \, d\tau\right] \\
&\qquad + \frac{k}{4} \int_0^1 \theta_x^2(x,t) \, dx + \frac{C_V}{4} \int_T^t \int_0^1 \theta_t^2(x,\tau) \, dx \, d\tau.
\end{aligned} \tag{3.17}$$

Multiplying (2.7) by θ_t and integrating over $[0,1] \times [T,t]$, we get, with the help of (1.3), (3.11), (3.12), (3.14), and the Cauchy inequality,

$$
\begin{aligned}
&\frac{k}{2}\int_0^1 \theta_x^2(x,t)\,dx + \frac{C_V}{2}\int_T^t\int_0^1 \theta_t^2(x,\tau)\,dx\,d\tau \\
&\quad \le \frac{k}{2}\int_0^1 \theta_x^2(x,T)\,dx + \Lambda\int_T^t \max_{[0,1]} v_x^2(\cdot,\tau)\left(\int_0^1 \theta^2(x,\tau)\,dx\right)d\tau \\
&\qquad + \Lambda\int_T^t\int_0^1 v_x^4(x,\tau)\,dx\,d\tau \\
&\quad \le \frac{k}{2}\int_0^1 \theta_x^2(x,T)\,dx + \frac{\mu_1^2}{4}\int_T^t\int_0^1 v_{xx}^2(x,\tau)\,dx\,d\tau + \Lambda\int_T^t\int_0^1 v_x^2(x,\tau)\,dx\,d\tau \\
&\qquad + \Lambda \sup_{\tau\in[T,t]}\int_0^1 v_x^2(x,\tau)\,dx\left[\int_T^t\int_0^1 (v_x^2+v_{xx}^2)(x,\tau)\,dx\,d\tau\right] \quad \text{for any } t\ge T>0.
\end{aligned}
\tag{3.18}
$$

(3.17) and (3.18) imply

$$
\begin{aligned}
&\int_0^1 (v_x^2+\theta_x^2)(x,t)\,dx + \int_T^t\int_0^1 (v_{xx}^2+\theta_t^2)(x,\tau)\,dx\,d\tau \\
&\quad \le \Lambda\int_0^1 [v_x^2+\theta_x^2+(f(u)\theta)^2](x,T)\,dx + \int_0^1 |f(u)\theta|(x,t)\,dx + \left(\int_0^1 |f(u)\theta|(x,t)\,dx\right)^2 \\
&\qquad + \int_T^t\int_0^1 (u_x^2+v_x^2+\theta_x^2)(x,\tau)\,dx\,d\tau \\
&\qquad + \sup_{\tau\in[T,t]}\int_0^1 v_x^2(x,\tau)\,dx\cdot\left[\int_T^t\int_0^1 (v_x^2+v_{xx}^2)(x,\tau)\,dx\,d\tau\right] \\
&\qquad\qquad \text{for any } t\ge T>0.
\end{aligned}
\tag{3.19}
$$

Due to Lemma 3.1 and the results in Sec. 2, it is known that

$$\int_0^{+\infty}\int_0^1 \{[f(u)\theta]^2+u_x^2+v_x^2+\theta_x^2\}\,dx\,d\tau \le \Lambda$$

and

$$\int_0^1 |f(u)\theta|(x,t)\,dx \to 0 \quad \text{as } t\to+\infty.$$

Therefore, for any $\varepsilon>0$, there exists a $T_1>0$ such that the following holds:

$$\Lambda\int_0^1 \{[f(u)\theta]^2+v_x^2+\theta_x^2\}(x,T_1)\,dx<\varepsilon, \tag{3.20}$$

$$\Lambda\left\{\int_0^1 |f(u)\theta|(x,t)\,dx+\left[\int_0^1 |f(u)\theta|(x,t)\,dx\right]^2\right\}<\varepsilon \quad \text{for any } t\ge T_1, \tag{3.21}$$

$$\Lambda\int_T^t\int_0^1 [u_x^2+v_x^2+\theta_x^2]\,dx\,d\tau<\varepsilon \quad \text{for any } t\ge T_1. \tag{3.22}$$

For convenience, we assume that the constant Λ in (3.19)–(3.22) satisfies $\Lambda\ge 1$.

We choose ε so small that

$$\varepsilon < \frac{1}{10}. \tag{3.23}$$

It follows from (3.20) that

$$\int_0^1 (v_x^2 + \theta_x^2)(x, T_1)\, dx < \frac{\varepsilon}{\Lambda} \le \varepsilon. \tag{3.24}$$

Let us define

$$T_2 = \sup\left\{ t : \sup_{\tau\in[T_1,t]} \int_0^1 (v_x^2 + \theta_x^2)(x,\tau)\, dx \le 5\varepsilon \right\}$$

and show next that $T_2 = +\infty$.

Suppose that $T_2 < +\infty$. By taking $t = T_2$ and $T = T_1$ in (3.19) and using (3.20)–(3.23), it turns out that

$$\int_0^1 (\theta_x^2 + v_x^2)(x, T_2)\, dx + \frac{1}{2}\int_{T_1}^{T_2}\int_0^1 v_{xx}^2\, dx\, d\tau + \int_{T_1}^{T_2}\int_0^1 \theta_t^2\, dx\, d\tau$$
$$\le 3\varepsilon + 5\varepsilon^2 < 3\varepsilon + \frac{\varepsilon}{2} < 4\varepsilon.$$

Namely, $\int_0^1 (\theta_x^2 + v_x^2)(x, T_2)\, dx < 4\varepsilon$, which contradicts the definition of T_2. Then $T_2 = +\infty$.

This implies that

$$\int_0^1 (\theta_x^2 + v_x^2)(x,t)\, dx \le 5\varepsilon \quad \text{for } t \ge T_1$$

which yields, due to the arbitrary smallness of ε, that

$$\int_0^1 (\theta_x^2 + v_x^2)(x,t)\, dx \to 0 \quad \text{as } t \to +\infty. \tag{3.25}$$

(3.8) can be obtained from (3.19), (3.25) and the above arguments. The proof of Lemma 3.2 is finished then.

We prove Theorem 1.5 now. It has been proved that

$$\|v(\cdot,t)\|_{H^1[0,1]} \to 0 \quad \text{as } t \to +\infty.$$

It is known from part I of Theorem 1.1 that

$$\int_0^1 [f(u)]^2(x,t)\, dx \to 0 \quad \text{as } t \to +\infty.$$

On the other hand, due to (3.1) and the mean value theorem, it can be shown that

$$\int_0^1 [f(u)]^2(x,t)\, dx \ge b^2 \int_0^1 (u - \hat{u})^2(x,t)\, dx$$

where $\hat{u}$ is the unique root of $f(u) = 0$ in $[\tilde{u}, \widetilde{U}]$. Thus,

$$\int_0^1 (u - \hat{u})^2(x,t)\, dx \to 0 \quad \text{as } t \to +\infty. \tag{3.26}$$

Furthermore, it follows from Lemma 3.1 that $\int_0^{+\infty}\int_0^1 u_x^2\,dx\,d\tau \le \Lambda < +\infty$, while Lemma 3.2 and $(1.1)_1$ imply

$$\int_{T^*}^{+\infty}\left|\frac{d}{dt}\int_0^1 u_x^2(x,t)\,dx\right|\,dt \le \Lambda\int_{T^*}^{+\infty}\int_0^1 u_x^2\,dx\,dt + \Lambda\int_{T^*}^{+\infty}\int_0^1 v_{xx}^2\,dx\,dt$$
$$\le \Lambda < +\infty.$$

Therefore,

$$\int_0^1 u_x^2(x,t)\,dx \to 0 \quad \text{as } t\to+\infty. \tag{3.27}$$

(3.26) and (3.27) yield

$$\|u(\cdot,t)-\hat u\|_{H^1}^2 \to 0.$$

It is known from part I of Theorem 1.1 that

$$\bar\theta(t) = \int_0^1 \theta(x,t)\,dx \to \frac{E_1}{C_V} \quad \text{as } t\to+\infty.$$

This, combined with (3.9), implies

$$\left\|\theta(\cdot,t)-\frac{E_1}{c_V}\right\|_{L^2[0,1]} \le \left[\int_0^1 \theta_x^2(x,t)\,dx\right]^{1/2} + \left\|\bar\theta(t)-\frac{E}{C_V}\right\|_{L^2} \to 0 \quad \text{as } t\to+\infty \tag{3.28}$$

and furthermore,

$$\left\|\theta(\cdot,t)-\frac{E_1}{c_V}\right\|_{H^1} \to 0 \quad \text{as } t\to+\infty.$$

So far, we have proved that all of $(u-\hat u), v$, and $(\theta - \frac{E_1}{C_V})$ become small in the H^1-norm for large t. Thus, the arguments similar to those in [OK] can be used to obtain the exponential convergence of $\{u,v,\theta\}$ to the constant state $\{\hat u, 0, \frac{E_1}{C_V}\}$ as $t\to+\infty$. We omit the details.

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